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CLOSED AND IMAGE-CLOSED RELATIONS

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If X and Y are topological spaces, a relation $T \subseteqq X \times Y$ is upper semi-continuous at the point x of the domain $D(T)$ of T if for each neighborhood V of $T(x)$, there is a neighborhood U of x such that $T(U) \subseteqq V$. Results so far published about such relations usually require that they be closed (as subsets of the product space) or image-closed ($T(x)$ is closed in Y for each $x \in X$). Given any relation T, it seems natural to consider the associated relations T' and $\bar{T}$, where T' is defined by $T'(x) = \overline{T(x)}$ and $\bar{T}$ is the closure of T in the product space. In particular, it is pertinent to ask under what conditions the upper semi-continuity of T implies that of T' or $\bar{T}$, or that $T' = \bar{T}$. As might be expected, the answers to these questions take the form of restrictions on Y, and, indeed, serve to characterize regularity, normality, and compactness.

Other relation-theoretic characterizations have been given previously. In [6], Engelking characterizes regularity and compactness (in two ways), and in [10], Michael characterizes normality, collectionwise normality, perfect normality, and paracompactness. Ceder [1] characterizes $\mathfrak{m}$-compactness.

Terminology in this paper will follow Kelley [9]; in particular, regular and normal spaces need not be T_1. The following well known fact will be used: T is upper semi-continuous (hereinafter abbreviated usc) on $D(T)$ if and only if the inverse under T of each closed subset of Y is closed in $D(T)$. A relation $T \subseteqq X \times Y$ will be said to be *on* X *into* Y if and only if $D(T) = X$.

Statement of results. These are arranged so that for $n = 1, 2, 3, 4$, result $(2n)$ is in the nature of a converse of result $(2n-1)$, thus yielding the promised characterizations of regularity, normality, and various types of compactness.

(1) *If Y is regular and $T \subseteqq X \times Y$ is usc at $x \in D(T)$, then* $T'(x) = \bar{T}(x)$.

Regularity of Y does not imply the upper semi-continuity of T' or $\bar{T}$ for usc $T \subset X \times Y$ (see (6a) and (6b) below).

The statement of the next result, a converse of (1), and of several others will be expedited by a definition: Let Δ be a directed set and $p \notin \Delta$. Define a topology for $X = \Delta \cup \{p\}$ by letting each point of Δ be isolated and taking as a base at p all sets of the form $S \cup \{p\}$ where

S is a final segment in Δ. When equipped with this topology, X will be called the *net-space* of Δ. It is clear that each net-space has at most one accumulation point and therefore a rather simple structure.

(2) *If for each net-space X and usc T on X into Y, $T' = \bar{T}$, then Y is regular.*

(3) *If Y is regular and $T \subseteqq X \times Y$ is usc and image-closed, then T is closed in $D(T) \times Y$.*

Under certain circumstances the hypothesis of regularity can be relaxed. A Fréchet space is one in which the closure of any subset A is the set of all limits of sequences in A. Clearly, any first countable space is Fréchet, but the converse is not true (see [7]).

(3′) *Let X and Y be such that $X \times Y$ is a Fréchet space (e.g., X and Y first countable). If Y is Hausdorff and $T \subset X \times Y$ is usc and image-closed, then T is closed in $D(T) \times Y$.*

(4) *If for each net-space X and usc image-closed relation T on X into Y, T is closed, then either* (a) *Y is regular, or* (b) *every closed nonregular subspace of Y fails to be R_0.*[1]

The authors have been unable to remove the possibility (b) from the conclusion of this result. It is clear, however, that for R_0 (hence for T_1)-spaces, (3) and (4) characterize regularity.

(5) *If Y is normal and $T \subseteqq X \times Y$ is usc at $x \in D(T)$, then both T' and $\bar{T}$ are usc at x.*

From (5) it is clear that if Y is normal and $D(T)$ is closed, the upper semi-continuity of $T \subseteqq X \times Y$ implies that of $\bar{T}$. That this need not be the case if $D(T)$ is not closed is shown by the following

EXAMPLE. Let X and Y be the reals with usual topology and $f\colon Y \to X$ be defined by $f(y) = y^{-1} \sin y$ for $y \neq 0$, $f(0) = 1$. Then $T = f^{-1} \backslash \{(0, y) \mid y \in Y\}$ is usc on $D(T)$. However, $\bar{T}$ is not usc at $0 \in D(\bar{T})$ since $V = \cup \{(n\pi - 1, n\pi + 1) \mid n \text{ an integer}\}$ is a neighborhood of $\bar{T}(0)$, but there is no neighborhood U of 0 such that $T(U) \subseteqq V$.

(6a) *If for each net-space X and usc relation T on X into Y, T' is usc, then Y is normal.*

[1] A space is R_0 if and only if point closures partition it. (Davis [**4**].)

(6b) *If Y is Hausdorff and for each net-space X and usc relation T on X into Y, $\bar{T}$ is usc, then Y is normal.*

If Y is infinite and equipped with the co-finite topology,[2] then for every X and usc T on X into Y, $\bar{T}$ is usc; hence the Hausdorff hypothesis in (6b) cannot be weakened even to T_1. Thus (5) and (6a) characterize normality, while (5) and (6b) characterize normality in the class of Hausdorff spaces.

Recall that for any infinite cardinal $\mathfrak{m}$ (defined as an initial ordinal) a topological space Y is called $\mathfrak{m}$-*compact* if and only if each open cover of power $\leqq \mathfrak{m}$ has a finite subcover. Compact spaces are precisely those which are $\mathfrak{m}$-compact for each $\mathfrak{m}$. $\aleph_0$-compact spaces are the countably compact spaces. $\mathfrak{m}$-compact spaces have been characterized in terms of the behavior of usc relations on them by Ceder [1]. A space X is said to have *local weight* $\mathfrak{m}$ if and only if $\mathfrak{m}$ is the least cardinal such that each point of X has a basis of neighborhoods of power $\leqq \mathfrak{m}$. First countable spaces are those of local weight $\leqq \aleph_0$.

(7) *If Y is compact and $T \subseteqq X \times Y$ is closed, then T is usc on $D(T)$.*

This result is well known and was apparently first noticed by Choquet [3].

(7$\mathfrak{m}$) *If X has local weight $\mathfrak{m}$, Y is $\mathfrak{m}$-compact and $T \subseteqq X \times Y$ is closed, then T is usc on $D(T)$.*

(7$\aleph_0$) *If X is first countable, Y is countably compact and $T \subseteqq X \times Y$ is closed. Then T is usc on $D(T)$.*

The corresponding results (7′), (7$\mathfrak{m}$′) and (7$\aleph_0'$) about functions, in which the hypotheses on X and Y are the same and the conclusion is that every function $f: X \to Y$ with closed graph is continuous, are immediate corollaries. The net-space of an ordinal α will be denoted by X_α.

(8) *Let Y be T_1. If for each net-space X every closed T on X into Y is usc, then Y is compact.*

(8$\mathfrak{m}$) *Let Y be T_1. If for each ordinal $\alpha \leqq \mathfrak{m}$, every closed T on X_α into Y is usc, then Y is $\mathfrak{m}$-compact.*

[2] i.e., the topology generated by the complements of finite sets.

($8\aleph_0$) *Let* Y *be* T_1. *If every closed* T *on the sequence space* $X_{\aleph_0}$ *into* Y *is usc, then* Y *is countably compact.*

These results are immediate consequence of the corresponding statements (8′), (8m′) and ($8\aleph_0'$) in which it is hypothesized that each function f from X ($X_{\mathfrak{m}}$, $X_{\aleph_0}$) into Y with closed graph is continuous. If Y is the set of natural numbers with the initial segments as a basis for the topology, then Y is T_0 but not T_1, no function into Y has closed graph, and Y is not countably compact. Hence the T_1 hypothesis in (8′), (8m′) and ($8\aleph_0'$) cannot be relaxed even to T_0. Clearly compactness (m-compactness, countable compactness) in T_1 spaces is characterized by (7) and (8) ((7m) and (8m), ($7\aleph_0$) and ($8\aleph_0$)) as well as by their corresponding function results.

The hypothesis of first countability on X in ($7\aleph_0$) can be relaxed if the hypothesis on Y is strengthened.

(9) *If* X *is a Hausdorff Fréchet space,* Y *sequentially compact, and* $T \subseteqq X \times Y$ *closed, then* T *is usc on* $D(T)$.

The corresponding function result (9′) is again an immediate corollary. One might hope for a converse to (9) patterned after ($8\aleph_0$), but the existence of compact, nonsequentially compact spaces (such as βN) makes the hope a vain one in view of (7).

Proofs of results. It will be convenient to give these in a somewhat different order from that of the statements.

Proof of (1). It is clear that for any relation, $T'(x) \leqq \bar{T}(x)$. Suppose, therefore, that $y \in \bar{T}(x) \backslash T'(x)$. Since Y is regular and $T'(x)$ is closed, there is a closed neighborhood N of $T'(x)$ not containing y. Since T is usc at x, there is an open neighborhood U of x such that $T(U) \subseteqq N$. Then $U \times (Y \backslash N)$ is a neighborhood of (x, y) not intersecting T, whence $(x, y) \notin \bar{T}$ or $y \notin \bar{T}(x)$.

Proof of (3). For all $x \in D(T)$, $T(x) = \overline{T(x)} = T'(x)$ by hypothesis and $T'(x) = \bar{T}(x)$ by (1).

Proof of (3′). Suppose there exist $x \in D(T)$ and $y \in Y$ such that $(x, y) \in \bar{T} \backslash T$. Since $X \times Y$ is Fréchet, there is a sequence $\{(x_n, y_n)\}$ in T converging to (x, y). Since $y \notin T(x)$, a closed set, and $\{y_n\} \to y$, there is an integer k such that if $n > k$, $y_n \notin T(x)$. Thus $K = \{y_n \mid n > k\} \cup \{y\}$ and $T(x)$ are disjoint, and because Y is Hausdorff, K is closed. Since T is usc, $T^{-1}(K)$ is closed in $D(T)$. But for $n > k$, $x_n \in T^{-1}(K)$ and $\{x_n\} \to x$, whence $x \in T^{-1}(K)$. Thus $T(x) \cap K \neq \varnothing$, a contradiction.

Proof of (5). Let E be either $T'(x)$ or $\bar{T}(x)$, and let V be a neighborhood of E. Since E is closed and Y is normal, there is a closed neighborhood N of E contained in V. Since T is usc at x and N is a neighborhood of $T(x)$, there is an open neighborhood U of x such that $T(U) \subseteqq N$. If $\bar{T}(U) \not\subseteqq N$, there are $z \in U$ and $y \in Y \backslash N$ such that $(z, y) \in \bar{T}$. But $U \times (Y \backslash N)$ is a neighborhood of (z, y) not intersecting T. Hence $T'(U) \subseteqq \bar{T}(U) \subseteqq N \subseteqq V$, and both T' and T are usc at x.

Proof of (6a). If Y is not normal, there exist a closed $F \subset Y$ and a neighborhood W of F which contains no closed neighborhood of F. Direct the neighborhood system Δ of F by $\subseteqq$, and let $X = \Delta \cup \{p\}$ be the net-space of Δ. Define T on X by $T(V) = V$ for all $V \in \Delta$, and $T(p) = F$. T is usc at p (and hence on X) since for any neighborhood V_0 of $T(p) = F$, $U = \{V \in \Delta \mid V \subseteqq V_0\} \cup \{p\}$ is a neighborhood of p, and $T(U) \subset V_0$. T', however, is not usc at p since for each $V \in \Delta$, $T'(V) = \bar{V}$ is a closed neighborhood of F and hence is not contained in the neighborhood W of $T(p) = F$.

Proof of (6b). Suppose Y is not normal. We will construct a net space X and usc T on X into Y such that T is not usc.

Case 1. Y is regular. By (6a) there is a net-space X and usc T on X into Y such that T' is not usc. By (1), $T' = \bar{T}$, and the construction is accomplished.

Case 2. Y is not regular. There exist a closed $F \subset Y$ and $p \in Y \backslash F$ such that the closure of every neighborhood of p intersects F. Let Δ be the family of all neighborhoods of p which do not intersect F, direct Δ by $\subseteqq$, and let $X = \Delta \cup \{p\}$ be the net-space of Δ. Then T defined on X by $T(x) = x$ is usc.

We now show that $\bar{T}(p) = p$: Let $p \neq q \in Y$. Since Y is Hausdorff, there exist $V_0 \in \Delta$ and a neighborhood W of q such that $W \cap V_0 = \varnothing$. Then $U = \{V \in \Delta \mid V \subseteqq V_0\} \cup \{p\}$ is a neighborhood of p in X, hence $U \times W$ is a neighborhood of (p, q) in $X \times Y$. If $(V, y) \in U \times W$, then $y \notin V = T(V)$ since $y \in W$ and $V \cap W \subseteqq V_0 \cap W = \varnothing$. Hence $(V, y) \notin T$, i.e., $(U \times W) \cap T = \varnothing$, whence $(p, q) \notin \bar{T}$, or $q \notin \bar{T}(p)$.

$\bar{T}$ is not usc at p since if $V \in \Delta$, $\bar{T}(V) \supset \bar{V}$ and is therefore not contained in the neighborhood $Y \backslash F$ of $p = \bar{T}(p)$.

Proof of (4). Assuming the proposition not true, there is a closed nonregular subspace Z of Y which is R_0. The existence of a net space X and a nonclosed, image-closed, usc relation T on X into Z will be demonstrated. Since Z is closed, T, regarded as a relation on X into Y will have the same properties and provide the desired contradiction.

There exist closed $F \subset Z$ and $q \in Z \backslash F$ which do not have disjoint neighborhoods. Direct $\Delta = \{(V, W) \mid V$ is a neighborhood of F and W is a neighborhood of $q\}$ by $(V, W) > (V', W')$ if and only if $V \subseteqq V'$ and $W \subseteqq W'$, and let $X = \Delta \cup \{p\}$ be the net-space of Δ. Define T on X into Z by $T((V, W)) = \{p_{V,W}\}^-$, where $p_{V,W} \in V \cap W$, and $T(p) = F$. Then T is image-closed; to show it usc at p, note that characteristic of R_0-spaces is the fact that $x \in O$, open, implies $\{x\}^- \subset O$. Thus if V_0 is a neighborhood of $T(p) = F$, $U = \{(V, W) \mid (V, W) > (V_0, Y) \cup \{p\}$ is a neighborhood of p, and $(V, W) \in U$ implies $p_{V,W} \in V \cap W \subset V_0$, whence $T((V, W)) = \{p_{V,W}\}^- \subset V_0$. But the net $\{((V, W), p_{V,W}) \mid (V, W) \in \Delta\}$ in T converges to $(p, q) \notin T$, and T is not closed.

Proof of (2). Let X be any net-space and T be an image-closed usc relation on X into Y. Then $T = T'$ and, by hypothesis $T' = \bar{T}$. Hence T is closed and the hypothesis of (4) is satisfied. The present result will follow from (4) when it is shown that Y (and hence every subspace of Y) is R_0. If this is not the case, there are points q and r of Y such that $q \in (r)^-$ but $r \notin \{q\}^-$. Let X be the net-space consisting of a sequence $\{x_n\}$ and its limit p, and define T on X into Y by $T(x_n) = \{q, r\}$; $T(p) = \{q\}$. Since every neighborhood of q contains r, T is usc at p. But $r \notin T'(p) = \{q\}^-$, while $r \in \bar{T}(p)$ since the sequence $\{(x_n, r)\}$ in T converges to (p, r). Hence $T' \neq \bar{T}$.

Proof of (7m). If F is a closed subset of Y, $\pi_Y^{-1}(F) \cap T$ is closed in $X \times Y$. Since Y is m-compact, π_X is a closed mapping (Hanai [8]) and so $T^{-1}(F) = \pi_X(\pi_Y^{-1}(F) \cap T)$ is closed in X and therefore in $D(T)$.

Proof of (8m′). If Y is not m-compact, it follows from a lemma attributed to Chittenden [2] (see Ceder [1]), that there is an α-net $\{y_\beta\}_{\beta<\alpha\leqq \mathfrak{m}}$ which has no cluster point. Define a function $f: X_\alpha \to Y$ by $f(\beta) = y_\beta$ if $\beta < \alpha$ and $f(p) = y_0$, where y_0 is an arbitrarily chosen point of Y. f is not continuous at p since $\{\beta\}_{\beta<\alpha}$ converges to p in X_α but $\{f(\beta)\}_{\beta<\alpha} = \{y_\beta\}_{\beta<\alpha}$, having no cluster point, cannot converge to $f(p) = y_0$ in Y.

Suppose $(x, y) \notin f$. If $x = \beta < \alpha$, then $y \neq y_\beta$. Let W be any open neighborhood of y not containing y_0. Then $(x, y) \in \{\beta\} \times W$, which is open and disjoint from f. If, on the other hand, $x = p$, then $y = y_0$. Since y is not a cluster point of $\{y_\beta\}_{\beta<\alpha}$, there is an open neighborhood U of y, not containing y_0, and a $\beta_0 < \alpha$ such that $\beta \geqq \beta_0$ implies $y_\beta \notin U$. Let $N = \{\beta \mid \beta_0 \leqq \beta < \alpha\} \cup \{p\}$. Then $(x, y) \in N \times U$ which is open and disjoint from f.

(8′) follows from (8$\mathfrak{m}'$) since Y is compact if and only if Y is $\mathfrak{m}$-compact for all $\mathfrak{m}$ (Chittenden [2], Ceder [1]).

Proof of (9). Let F be closed in Y. If $x_0 \in \mathrm{cl}_{D(T)} T^{-1}(F)$, there is a sequence $\{x_n\} \subseteq T^{-1}(F)$ converging to x_0 (since subspaces of Fréchet spaces are Fréchet [7]). For each n choose $y_n \in T(x_n) \cap F$ and let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ converging to $y_0 \in Y$. But $y_0 \in F$ and $\{(x_{n_i}, y_{n_i})\}$ is contained in T and converges to (x_0, y_0). Thus, since T is closed, $x_0 \in T^{-1}(F)$.

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