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A – P CONGRUENCES ON BAER SEMIGROUPS

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In this paper a coordinatizing Baer semigroup is used to pick out an interesting sublattice of the lattice of congruence relations on a lattice with 0 and 1. These congruences are defined for any lattice with 0 and 1 and have many of the nice properties enjoyed by congruence relations on a relatively complemented lattice.

These results generalize the work of S. Maeda on Rickart (Baer) rings and are related to G. Gratzer and E. T. Schmidt's work on standard ideals.

In [7] M. F. Janowitz shows that lattice theory can be approached by means of Baer semigroups. A *Baer semigroup* is a multiplicative semigroup S with 0 and 1 in which the left and right annihilators, $L(x) = \{y \in S : yx = 0\}$ and $R(x) = \{y \in S : xy = 0\}$, of any $x \in S$ are principal left and right ideals generated by idempotents. For any Baer semigroup S, $\mathscr{L}(S) = \{L(x) : x \in S\}$ and $\mathscr{R}(S) = \{R(x) : x \in S\}$, ordered by set inclusion, are dual isomorphic lattices with 0 and 1. The Baer semigroup S is said to *coordinatize* the lattice L if $\mathscr{L}(S)$ is isomorphic to L. The basic point is Theorem 2.3, p. 1214 of [7], which states: a partially ordered set P with 0 and 1 is a lattice if and only if it can be coordinatized by a Baer semigroup.

It will be convenient to introduce the convention that S will always denote a Baer semigroup and that for any $x \in S$, x^l and x^r will denote idempotent generators of $L(x)$ and $R(x)$ respectively. Also the letters $e, f, g,$ and h shall always denote idempotents of S.

Some background material is presented in § 1. In § 2, $A - P$ congruences are defined and it is shown that every $A - P$ congruence ρ on S induces a lattice congruence Θ_ρ on $\mathscr{L}(S)$ such that $\mathscr{L}(S)/\Theta_\rho \cong \mathscr{L}(S/\rho)$. In § 3 congruences which arise in this manner are characterised as the set of all equivalence relations on $\mathscr{L}(S)$ which are compatible with a certain set of maps on $\mathscr{L}(S)$. These congruences are called compatible with S. They are standard congruences and are thus determined by their kernels.

The ideals of $\mathscr{L}(S)$ which are kernels of congruences compatible with S are characterised in § 4. In § 5 it is shown that a principal ideal, $[(0), Se]$, is the kernel of a congruence compatible with S if and only if e is central in S. In § 6 this is applied to complete Baer semigroups to show that, in this case, the congruence compatible with S form a Stone lattice.

1. Preliminaries. We shall let $L(M) = \{y \in S : yx = 0$ for all $x \in M\}$ and $R(M) = \{y \in S : xy = 0$ for all $x \in M\}$ for any set $M \subseteqq S$. The following is a summary of results found on pp. 85–86 of [8].

LEMMA 1.1. *Let $x, y \in S$.*

(i) *$xS \subseteqq yS$ implies $L(y) \subseteqq L(x)$; $Sx \subseteqq Sy$ implies $R(y) \subseteqq R(x)$.*

(ii) *$Sx \subseteqq LR(x)$; $xS \subseteqq RL(x)$.*

(iii) *$L(x) = LRL(x)$; $R(x) = RLR(x)$.*

(iv) *$Sx \in \mathscr{L}(S)$ if and only if $Sx = LR(x)$; $xS \in \mathscr{R}(S)$ if and only if $xS = RL(x)$.*

(v) *The mappings $eS \to L(eS)$ and $Sf \to R(Sf)$ are mutually inverse dual isomorphisms between $\mathscr{R}(S)$ and $\mathscr{L}(S)$.*

(vi) *Let $Se, Sf \in \mathscr{L}(S)$ and $Sh = L(ef^r)$. Then $he = (he)^2$, $Se \cap Sf = She \in \mathscr{L}(S)$, and $Se \vee Sf = L(e^rS \cap f^rS)$.*

(vii) *Let $eS, fS \in \mathscr{R}(S)$ and $gS = R(f^le)$. Then $eg = (eg)^2$, $eS \cap fS = egS \in \mathscr{R}(S)$, and $eS \vee fS = R(Se^l \cap Sf^l)$.*

Note that the meet operation in $\mathscr{L}(S)$ and $\mathscr{R}(S)$ is set intersection and that the trivial ideals, S and (0), are the largest and smallest elements of both $\mathscr{L}(S)$ and $\mathscr{R}(S)$.

We shall be interested in a class of isotone maps introduced by Croisot in [2].

DEFINITION 1.2. Let P be a partially ordered set. An isotone map ϕ of P into itself is called *residuated* if there exists an isotone map ϕ^+ of P into P such that for any $p \in P$, $p\phi^+\phi \leqq p \leqq p\phi\phi^+$. In this case ϕ^+ is called a *residual* map.

Clearly ϕ^+ is uniquely determined by ϕ and conversly. The pair (ϕ, ϕ^+) sets up a Galois connection between P and its dual. Thus we can combine results from [2], [3], and [11] to get.

LEMMA 1.3. *Let P be a partially ordered set and ϕ and ψ maps of P into itself.*

(i) *If ϕ and ψ are residuated then $\phi\psi$ is residuated and $(\phi\psi)^+ = \psi^+\phi^+$.*

(ii) *If ϕ is residuated then $\phi = \phi\phi^+\phi$ and $\phi^+ = \phi^+\phi\phi^+$.*

(iii) *Let ϕ be residuated and $\{x_\alpha\}$ be any family of elements of P. If $\bigvee_\alpha x_\alpha$ exists then $\bigvee_\alpha (x_\alpha\phi)$ exists and $\bigvee_\alpha (x_\alpha\phi) = (\bigvee_\alpha x_\alpha)\phi$. Dually if $\bigwedge_\alpha x_\alpha$ exists then $\bigwedge_\alpha (x_\alpha\phi^+)$ exists and $\bigwedge_\alpha (x_\alpha\phi^+) = (\bigwedge_\alpha x_\alpha)\phi^+$.*

(iv) *A necessary and sufficient condition that ϕ be residuated is that for any $x \in L$, $\{z : z\phi \leqq x\}$ has a largest element x^*. In this case ϕ^+ is given by $x\phi^+ = x^*$.*

According to Lemma 1.3 (i) the set of residuated maps forms a semigroup for any partially ordered set P. We shall denote the semigroup of residuated maps on P by $S(P)$. In [7], Theorem 2.3, p. 1214, it is shown that P is a lattice if and only if $S(P)$ is a Baer semigroup. In this case $S(P)$ coordinatizes P.

In [8], pp. 93, 94, it is shown that any Baer semigroup S can be represented as a semigroup of residuated maps on $\mathscr{L}(S)$. We shall be interested in the maps introduced to achieve this.

LEMMA 1.4. *For any* $x \in S$ *define* $\phi_x : \mathscr{L}(S) \to \mathscr{L}(S)$ *by* $Se\phi_x = LR(ex)$.

(i) ϕ_x *is residuated with residual* ϕ_x^+ *given by* $Se\phi_x^+ = L(xe^r)$.

(ii) *If* $LR(y) = Se$ *then* $Se\phi_x = LR(yx)$.

(iii) *Let* $S_0 = \{\phi_x : x \in S\}$. *Then* S_0 *is a Baer semigroup which coordinatizes* $\mathscr{L}(S)$.

(iv) *The map* $x \to \phi_x$ *is a homomorphism, with kernel* $\{0\}$, *of* S *into* S_0.

We shall now develop an unpublished result due to D. J. Foulis and M. F. Janowitz.

DEFINITION 1.5. A semigroup S is a *complete Baer semigroup* if for any subset M of S there exist idempotents e, f such that $L(M) = Se$ and $R(M) = fS$.

In proving Lemma 2.3 of [7] the crucial observation was [7] Lemma 2.1, p. 1213, where it is shown that for any lattice L and any $a \in L$ there are idempotent residuated maps θ_a and ψ_a given by:

$$x\theta_a = \begin{cases} x & x \leqq a \\ a & \text{otherwise} \end{cases} \qquad x\psi_a = \begin{cases} 0 & x \leqq a \\ x \vee a & \text{otherwise.} \end{cases}$$

THEOREM 1.6. *Let* P *be a partially ordered set with* 0 *and* 1. *Then the following conditions are equivalent.*

(i) *P is a complete lattice.*

(ii) *$S(P)$ is a complete Baer semigroup.*

(iii) *P can be coordinatized by a complete Baer semigroup.*

Proof. (i) $\Rightarrow$ (ii) Let P be a complete lattice and $M \subseteq S(P)$ with $m = \bigvee \{1\phi : \phi \in M\}$ and $n = \bigwedge \{0\phi^+ : \phi \in M\}$. It is easily verified that $L(M) = S(P)\theta_n$ and $R(M) = \psi_m S(P)$.

(ii) $\Rightarrow$ (iii) follows from [7], Theorem 2.3.

(iii) $\Rightarrow$ (i) Let S be a complete Baer semigroup coordinatizing P and $\mathscr{P}(S)$ the complete lattice of all subsets of S. Define α and β

mapping $\mathscr{P}(S)$ into $\mathscr{P}(S)$ by $M\alpha = L(M)$ and $M\beta = R(M)$. Clearly (α, β) sets up a Galois connection of $\mathscr{P}(S)$ with itself. Since S is a complete Baer semigroup $\mathscr{L}(S)$ is the set of Galois closed objects of (α, β). Thus $\mathscr{L}(S)$ is a complete lattice.

We conclude this section with some relatively well known facts about lattice congruences. An equivalence relation Θ on a lattice is a *lattice congruence* if $a\Theta b$ and $c\Theta d$ imply $(a \vee c)\Theta(b \vee d)$ and $(a \wedge c)\Theta(b \wedge d)$. We shall sometimes write $a \equiv b(\Theta)$ in place of $a\Theta b$. With respect to the order $\Theta \leqq \Theta'$ if and only if $a\Theta b$ implies $a\Theta' b$, the set of all lattice congruences on a lattice L is a complete lattice, denoted by $\Theta(L)$, with meet and join given as follows:

THEOREM 1.7. *Let L be a lattice and Γ a subset of $\Theta(L)$.*
(i) *$a \equiv b(\bigwedge \Gamma)$ if and only if $a\gamma b$ for all $\gamma \in \Gamma$.*
(ii) *$a \equiv b(\bigvee \Gamma)$ if and only if there exist finite sequences $a_0, a_1, \cdots, a_n$ of elements of L and $\gamma_1, \cdots, \gamma_n$ of elements of Γ, such that $a = a_0$, $a_n = b$, and $a_{i-1}\ \gamma_i a_i$ for $i = 1, \cdots, n$.*

The largest element ι of $\Theta(L)$ is given by $a\iota b$ for all $a, b \in L$ and the smallest element ω is given by $a\omega b$ if and only if $a = b$.
In [4] it is shown that $\Theta(L)$ is distributive. In fact we have:

THEOREM 1.8. *Let L be a lattice. The $\Theta(L)$ is a distributive lattice such that for any family $\{\Theta_\alpha\} \subseteqq \Theta(L)$*

$$(\bigvee_\alpha \Theta_\alpha) \wedge \Psi = \bigvee_\alpha (\Theta_\alpha \wedge \Psi)$$

for any $\Psi \in \Theta(L)$.

Thus by Theorem 15, p. 147, of [1] we have:

THEOREM 1.9. *For any lattice $L, \Theta(L)$ is pseudo-complemented.*

Finally we mention that if $\Theta \in \Theta(L)$ then $a\Theta b$ if and only if $x\Theta y$ for all $x, y \in [a \wedge b, a \vee b]$.

2. **A − P congruences.** In [10] S. Maeda defines annihilator preserving homomorphisms for rings. We shall take the same definition for semigroups with 0.

DEFINITION 2.1. A homomorphism ϕ of a semigroup S with 0 is called an *annihilator preserving* $(A - P)$ *homomorphism* if for any $x \in S$, $R(x)\phi = R(x\phi) \cap S\phi$ and $L(x)\phi = L(x\phi) \cap S\phi$. A congruence relation ρ on a semigroup S is called an *$A - P$ congruence* if the natural homomorphism induced by ρ is an $A - P$ homomorphism.

For any congruence ρ on a semigroup S and any $x \in S$ let x/ρ denote the equivalence class of S/ρ containing x. Similarly for any set $A \subseteq S$, let $A/\rho = \{x/\rho \in S/\rho : x \in A\}$. If S has a 0 then $R(x)/\rho \subseteq R(x/\rho)$ and $L(x)/\rho \subseteq L(x/\rho)$. Thus a congruence ρ is an $A - P$ congruence if and only if $R(x/\rho) \subseteq R(x)/\rho$ and $L(x/\rho) \subseteq L(x)/\rho$. Note that we are using L and R to denote the left and right annihilators both in S and in S/ρ.

THEOREM 2.2. *Let ρ be an $A - P$ congruence on a semigroup S. If e and f are idempotents of S such that $Se = L(x)$ and $fS = R(y)$ for some $x, y \in S$, then $(S/\rho)(e/\rho) = L(x/\rho)$ and $(f/\rho)(S/\rho) = R(y/\rho)$. Thus if S is a Baer semigroup so is S/ρ.*

Proof. Since ρ is an $A - P$ congruence $L(x/\rho) = L(x)/\rho$. Thus $L(x) = Se$ gives $L(x/\rho) = L(x)/\rho = (Se)/\rho = (S/\rho)(e/\rho)$. Similarly $R(x) = fS$ gives $R(x/\rho) = (f/\rho)(S/\rho)$.

We now use an $A - P$ congruence ρ on S to induce a homomorphism of $\mathscr{L}(S)$ onto $\mathscr{L}(S/\rho)$.

THEOREM 2.3. *Let ρ be an $A - P$ congruence on S. Then $\theta_\rho : \mathscr{L}(S) \to \mathscr{L}(S/\rho)$ by $L(x)\theta_\rho = L(x/\rho)$ is a lattice homomorphism of $\mathscr{L}(S)$ onto $\mathscr{L}(S/\rho)$.*

Proof. Let $Se, Sf \in \mathscr{L}(S)$ and note that, by Theorem 2.2,

$$Se\theta_\rho = (S/\rho)(e/\rho) \quad \text{and} \quad Sf\theta_\rho = (S/\rho)(f/\rho)\ .$$

Clearly θ_ρ is well defined since if $L(x) = L(y)$ then

$$L(x/\rho) = L(x)/\rho = L(y)/\rho = L(y/\rho)\ .$$

By Lemma 1.1 (vi), $She = Se \cap Sf$ where $Sh = L(ef^r)$. Applying Theorem 2.2 gives $(f^r/\rho)(S/\rho) = R(f/\rho)$ and $(S/\rho)(h/\rho) = L((e/\rho)(f^r/\rho))$. Thus applying Lemma 1.1 (vi) to S/ρ yields

$$(S/\rho)(e/\rho) \cap (S/\rho)(f/\rho) = (S/\rho)(h/\rho)(e/\rho) = (S/\rho)(he/\rho)\ .$$

Therefore, $Se\theta_\rho \cap Sf\theta_\rho = (Se \cap Sf)\theta_\rho$. By a dual argument $\theta^*_\rho \mathscr{R}(S) \to \mathscr{R}(S/\rho)$ by $R(x)\theta^*_\rho = R(x/\rho)$ is also a meet homomorphism.

By Lemma 1.1 (vi) $Se \vee Sf = L(R(e) \cap R(f))$. Let $gS = R(e) \cap R(f)$ so that $Se \vee Sf = Sg^l$. Since θ^*_ρ is a meet homomorphism,

$$R(e/\rho) \cap R(f/\rho) = (g/\rho)(S/\rho)\ .$$

Noting that $L(g/\rho) = (S/\rho)(g^l/\rho)$ and applying Lemma 1.1 (vi) to S/ρ gives

$$(S/\rho)(e/\rho) \vee (S/\rho)(f/\rho) = (S/\rho)(g^l/\rho) .$$

Thus $Se\theta_\rho \vee Sf\theta_\rho = (Se \vee Sf)\theta_\rho$ and θ_ρ is a lattice homomorphism. Clearly θ_ρ is onto.

For any $A - P$ congruence ρ on S let Θ_ρ denote the lattice congruence $\theta_\rho \circ \theta_\rho^{-1}$ induced on $\mathscr{L}(S)$ by θ_ρ.

COROLLARY 2.4. $\mathscr{L}(S)/\Theta_\rho \cong \mathscr{L}(S/\rho)$.

3. Compatible congruences. In this section we shall characterise lattice congruence which are induced by an $A - P$ congruence on a coordinatizing Baer semigroup in the manner given in Theorem 2.3. Since $L \cong \mathscr{L}(S)$ for any Baer semigroup S coordinatizing L, we shall lose no generality by considering only lattices of the form $\mathscr{L}(S)$.

The residuated maps ϕ_x, $x \in S$, defined in Lemma 1.4, play a central role in the theory of Bear semigroups. We shall be interested in equivalence relations on $\mathscr{L}(S)$ which are compatible with ϕ_x and ϕ_x^+, considered as unary operations on $\mathscr{L}(S)$.

DEFINITION 3.1. An equivalence relation E on $\mathscr{L}(S)$ is called *compatible with* S if for any $x \in S$,

$$SeESf \Rightarrow (Se\phi_x)E(Sf\phi_x) \quad \text{and} \quad (Se\phi_x^+)E(Sf\phi_x^+) .$$

By [7] Lemma 3.1 and 3.2, pp. 1214–1215, $Se \cap Sf = Se \cap S\phi_f = Se\phi_f^+\phi_f$. Dually $Se \vee Sf = Se\phi_{f^r}\phi_{f^r}^+$. Thus we have:

LEMMA 3.3. *Any equivalence relation compatible with S is a lattice congruence.*

We now consider an $A - P$ congruence ρ on S and Θ_ρ, the lattice congruence induced on $\mathscr{L}(S)$ by ρ as in Theorem 2.3.

THEOREM 3.4. *Let ρ be an $A - P$ congruence on S. Then Θ_ρ is compatible with S.*

Proof. Since $Se\Theta_\rho Sf$ if and only if $(S/\rho)(e/\rho) = (S/\rho)(f/\rho)$, $Se\Theta_\rho Sf$ implies $(e/\rho) = (e/\rho)(f/\rho)$ and $(f/\rho) = (f/\rho)(e/\rho)$. Note that for any $y \in S$, $LR(y)\theta_\rho = LR(y/\rho)$. If $Se\Theta_\rho Sf$ we have

$$\begin{aligned}(Se\phi_x)\theta_\rho &= LR((ex)/\rho) = LR((e/\rho)(x/\rho))\\ &= LR((e/\rho)(f/\rho)(x/\rho)) \subseteqq LR((f/\rho)(x/\rho)) = (Sf\phi_x)\theta_\rho\ .\end{aligned}$$

By symmetry $(Sf\phi_x)\theta_\rho = (Se\phi_x)\theta_\rho$ ie. $Se\phi_x\Theta_\rho Sf\phi_x$.

Now $R(e/\rho) = R((e/\rho)(f/\rho)) \supseteqq R(f/\rho) = R((f/\rho)(e/\rho)) \supseteqq R(e/\rho)$. Thus $(e^r/\rho)(S/\rho) = (f^r/\rho)(S/\rho)$. But

$$(Se\phi_x^+)\theta_\rho = L(xe^r)\theta_\rho = L((xe^r)/\rho) = L((x/\rho)(e^r/\rho))$$

and similarly $(Sf\phi_x^+)\theta_\rho = L((x/\rho)(f^r/\rho))$. Clearly $y/\rho \in L((x/\rho)(e^r/\rho))$ if and only if $(y/\rho)(x/\rho) \in L((e^r/\rho)(S/\rho)) = L((f^r/\rho)(S/\rho))$. Thus we have $(Se\phi_x^+)\theta_\rho = (Sf\phi_x^+)\theta_\rho$. Therefore, $Se\phi_x^+\Theta_\rho Sf\phi_x^+$ and Θ_ρ is compatible with S.

By the following theorem every congruence compatible with S is determined by its kernel in a very nice way.

THEOREM 3.5. *Let Θ be a congruence compatible with S. Then the following are equivalent.*

(i) $Se\Theta Sf$.

(ii) $Se\phi_{f^r} \vee Sf\phi_{e^r} \in ker\ \Theta$.

(iii) *There is an $Sg \in ker\ \Theta$ such that $Se \vee Sf = Se \vee Sg = Sf \vee Sg$.*

Proof (i) $\Rightarrow$ (ii) Since Θ is compatible with $S, Se\Theta Sf$ gives $Se\phi_{f^r}\Theta Sf\phi_{f^r} = (0)$, i.e., $Se\phi_{f^r} \in \ker \Theta$. By symmetry $Sf\phi_{e^r} \in \ker \Theta$ so we have (ii).

(ii) $\Rightarrow$ (iii) Let $Sg = Se\phi_{f^r} \vee Sf\phi_{e^r} \in \ker \Theta$ and claim $Se \vee Sg = Sf \vee Sg$, i.e.,

$$LR(e) \vee LR(ef^r) \vee LR(fe^r) = LR(f) \vee LR(ef^r) \vee LR(fe^r)\ .$$

By Lemma 1.1 (v) this is equivalent to

$$R(e) \cap R(ef^r) \cap R(fe^r) = R(f) \cap R(ef^r) \cap R(fe^r)\ .$$

Let $x \in R(e) \cap R(ef^r) \cap R(fe^r)$. Then $x = e^rx$ and $fx = fe^rx = 0$ so $x \in R(f) \cap R(ef^r) \cap R(fe^r)$. By symmetry

$$R(e) \cap R(ef^r) \cap R(fe^r) = R(f) \cap R(ef^r) \cap R(fe^r)$$

so we have $Se \vee Sg = Sf \vee Sg$. To show that $Se \vee Sf = Se \vee Sg = Sf \vee Sg$ we need only show that $Sg \subseteqq Se \vee Sf$. This is equivalent to $R(e) \cap R(f) \subseteqq R(ef^r) \cap R(fe^r)$. But if $x \in R(e) \cap R(f)$ then $x = e^rx = f^rx$, so $ef^rx = ex = 0$ and $fe^rx = fx = 0$, i.e., $x \in R(ef^r) \cap R(fe^r)$.

(iii) $\Rightarrow$ (i) If $Se \vee Sg = Sf \vee Sg$ and $Sg\Theta(0)$ then $Se\Theta Se \vee Sg = Sf \vee Sg\Theta Sf$.

A congruence Θ on a lattice is called *standard* if there is an ideal S such that $a\Theta b$ if and only if $a \vee b = (a \wedge b) \vee s$ for some $s \in S$.

COROLLARY 3.6. *Any congruence Θ compatible with S is a standard congruence.*

Proof. Since Θ is a lattice congruence $Se\Theta Sf$ if and only if $(Se \vee Sf)\Theta(Se \cap Sf)$. By Theorem 3.5 this is equivalent to

$$(Se \vee Sf) \vee (Se \cap Sf) = Se \vee Sf = (Se \cap Sf) \vee Sg$$

for some $Sg \in \ker \Theta$.

Thus by Lemma 7, p. 36, of [5] we have:

COROLLARY 3.7. *Compatible congruences are permutable.*

By Theorem 3.5 every congruence compatible with S is determined by its kernel. Since, by Theorem 3.4, Θ_ρ is compatible with S for any $A - P$ congruence ρ on S we know that Θ_ρ is determined by its kernel. By the following lemma, Θ_ρ is also uniquely determined by $\ker \rho$.

LEMMA 3.8. *Let ρ be an $A - P$ congruence on S. Then $x \in \ker \rho$ if and only if $LR(x) \in \ker \Theta_\rho$.*

Proof. Let $x \in \ker \rho$. Then $x\rho 0 \Rightarrow xy\rho 0$ for any $y \in S$. Thus $R(x/\rho) = S/\rho$ so that $LR(x/\rho) = L(S/\rho) = (0/\rho)$, i.e., $LR(x) \in \ker \Theta_\rho$. If we let $LR(x) \in \ker \Theta_\rho$ then $LR(x/\rho) = (0/\rho)$. Thus $R(x/\rho) = RLR(x/\rho) = R(0/\rho) = S/\rho$ which gives $x/\rho = 0/\rho$ and we have $x \in \ker \rho$.

That Θ should be determined completely by $\ker \rho$ is unexpected since an $A - P$ congruence need not be determined by its kernel. For clearly the congruence ω given by $x\omega y$ if and only if $x = y$ is an $A - P$ congruence with kernel $\{0\}$ as is the congruence ρ_0 given by $x\rho_0 y$ if and only if $\phi_x = \phi_y$. Clearly ρ_0 is not generally equal to ω. It turns out that ρ_0 is the largest $A - P$ congruence with kernel $\{0\}$. Our next project shall be to start with a congruence Θ compatible with S and determine the existance of an $A - P$ congruence λ on S, such that $\Theta = \Theta_\lambda$. By lemma 3.8 we shall have to construct λ so that $\ker \lambda = \{x \in S : LR(x) \in \ker \Theta\}$.

For any congruence Θ on $\mathscr{L}(S)$ let Se/Θ denote the equivalence class of $\mathscr{L}(S)/\Theta$ containing Se.

LEMMA 3.9. *Let Θ be a congruence compatible with S. For each $x \in S$ define Φ_x ; $\mathscr{L}(S)/\Theta \to \mathscr{L}(S)/\Theta$ by $(Se/\Theta)\Phi_x = (Se\phi_x)/\Theta$. Then Φ_x is residuated with residual Φ_x^+ given by $(Se/\Theta)\Phi_x^+ = (Se\phi_x^+)/\Theta$.*

Proof. Clearly Φ_x and Φ_x^+ are well defined since Θ is compatible with S. We shall use Lemma 1.3 (iv), i.e., we shall show that the inverse image of a principal ideal is principal. Let $Sf/\Theta \in [(0)/\Theta, Se/\Theta]\Phi_x^{-1}$. Then $(Sf/\Theta)\Phi_x = (Sf\phi_x)/\Theta = (Sf\phi_x)/\Theta \cap Se/\Theta = (Sf\phi_x \cap Se)/\Theta$. This gives $Sf\phi_x\Theta(Sf\phi_x \cap Se)$ so by compatibility with S,

$$Sf \subseteqq Sf\phi_x\phi_x^+\Theta(Sf\phi_x \cap Se)\phi_x^+ \subseteqq Se\phi_x^+ .$$

Thus in $\mathscr{L}(S)/\Theta$, $Sf/\Theta \subseteqq (Sf\phi_x\phi_x^+)/\Theta = (Sf\phi_x \cap Se)\phi_x^+/\Theta \subseteqq (Se/\Theta)\Phi_x^+$, i.e., $[(0)/\Theta, Se/\Theta)]\Phi_x^{-1} \subseteqq [(0)/\Theta, (Se/\Theta)\Phi_x^+]$. Now let $Sf/\Theta \subseteqq (Se/\Theta)\Phi_x^+$. Then

$$Sf/\Theta = Sf/\Theta \cap [(Se/\Theta)\Phi_x^+] = Sf/\Theta \cap (Se\phi_x^+)/\Theta = (Sf \cap Se\phi_x^+)/\Theta$$

i.e., $Sf\Theta(Sf \cap Se\phi_x^+)$. By compatibility with S

$$Sf\phi_x\Theta[(Sf \cap Se\phi_x^+)\phi_x] \subseteqq Se\phi_x^+\phi_x \subseteqq Se .$$

Hence $(Sf/\Theta)\Phi_x = (Sf\phi_x)/\Theta = (Sf \cap Se\phi_x^+)\phi_x/\Theta \subseteqq Se/\Theta$. Therefore,

$$[(0)/\Theta, Se/\Theta]\Phi_x^{-1} = [(0)/\Theta, (Sf/\Theta)\Phi_x^+]$$

and by Lemma 1.3 (iv), Φ_x is residuated with residual Φ_x^+.

For any equivalence relation E on $\mathscr{L}(S)$ we can define a left congruence λ_E on S by taking $x\lambda_E y$ if and only if $(Se\phi_x)E(Se\phi_y)$ for all $Se \in \mathscr{L}(S)$. Similarly, $x\rho_E y$ if and only if $(Se\phi_x^+)E(Se\phi_x^+)$ for all $Se \in \mathscr{L}(S)$, defines a right congruence on S.

LEMMA 3.10. *If Θ is a congruence compatible with S then $\lambda_\Theta = \rho_\Theta$. Thus λ_Θ is a congruence on S.*

Proof. By definition $x\lambda_\Theta y$ if and only if $\Phi_x = \Phi_y$. But $\Phi_x = \Phi_y$ if and only if $\Phi_x^+ = \Phi_y^+$ which is equivalent to $x\rho_\Theta y$.

THEOREM 3.11. *Let Θ be a congruence compatible with S. Then λ_Θ is an $A - P$ congruence on S.*

Proof. We know that λ_Θ is an $A - P$ congruence if and only if $L(y/\lambda_\Theta) \subseteqq L(y)/\lambda_\Theta$ and $R(y/\lambda_\Theta) \subseteqq R(y)/\lambda_\Theta$ for all $y \in S$. We shall start with $x/\lambda_\Theta \in L(y/\lambda_\Theta)$ and show that $x/\lambda_\Theta = xe/\lambda_\Theta$ where $Se = L(y)$. This, of course, is equivalent to $\Phi_x = \Phi_{xe}$.

Let $x/\lambda_\Theta \in L(y/\lambda_\Theta)$ so that $xy \in \ker \lambda_\Theta$. Thus $Sf\phi_{xy}\Theta Sf\phi_0 = (0)$ for all $Sf \in \mathscr{L}(S)$. In particular, $S\phi_{xy}\Theta(0)$ so for any $Sf \in \mathscr{L}(S)$,

$$Sf\phi_x \subseteqq S\phi_x \subseteqq S\phi_x\phi_y\phi_y^+ = (S\phi_{xy}\phi_y^+)\Theta(0)\phi_y^+ = L(y) = Se\ .$$

Thus $Sf\phi_x = (Sf\phi_x \cap S\phi_{xy}\phi_x^+)\Theta(Se \cap Sf\phi_x)$. Now if $Sg \subseteqq Se$ then $g = ge$ so $Sg\phi_e = LR(ge) = LR(g) = Sg$. Thus applying ϕ_e to both sides of the above gives $Sf\phi_{xe} = Sf\phi_x\phi_e\Theta(Se \cap Sf\phi_x)\phi_e = Se \cap Sf\phi_x$. By transitivity $Sf\phi_{xe}\Theta Sf\phi_x$ and so $xe\lambda_\theta x$. Since $xe \in L(y)$ this gives $x/\lambda_\theta \in L(y)/\lambda_\theta$.

The argument to show $R(y/\lambda_\theta) = R(y)/\lambda_\theta$ is exactly dual to the above but will be included. We have $x/\lambda_\theta \in R(y/\lambda_\theta)$ if and only if $yx \in \ker \lambda_\theta$. By Lemma 3.10 and the definition of ρ_θ this is equivalent to $Sf\phi_{yx}^+\Theta Sf\phi_0^+ = L(0) = S$ for all $Sf \in \mathscr{L}(S)$. In particular $(0)\phi_{yx}^+\Theta S$. By Lemma 1.3 (i) $\phi_{yx}^+ = \phi_x^+\phi_y^+$ so for any $Sf \in \mathscr{L}(S)$ we have

$$Sf\phi_x^+ \supseteqq (0)\phi_x^+ \supseteqq (0)\phi_x^+\phi_y^+\phi_y = (0)\phi_{yx}^+\phi_y\Theta S\phi_y\ .$$

Thus $Sf\phi_x^+ = (Sf\phi_x^+ \vee (0)\phi_{yx}^+\phi_y)\Theta(S\phi_y \vee Sf\phi_x^+)$. Let $eS = R(y)$ and note that $S\phi_y = LR(y) = L(e)$. Now $L(e) \subseteqq Sg$ implies $eS = RL(e) \supseteqq R(g) = g^rS$ so $g^r = eg^r$. Thus $Sg\phi_e^+ = L(eg^r) = L(g^r) = LR(g) = Sg$. Since $L(e) = S\phi_y \subseteqq S\phi_y \vee Sf\phi_x^+$, applying ϕ_e^+ to both sides of the above gives; $Sf\phi_{ex}^+ = Sf\phi_x^+\phi_e^+\Theta(S\phi_y \vee Sf\phi_x^+)\phi_e^+ = S\phi_y \vee Sf\phi_x^+$. By transitivity $Sf\phi_x^+\Theta Sf\phi_{ex}^+$ so by Lemma 3.10 $x\lambda_\theta ex$. Since $ex \in R(y)$ this gives $x/\lambda_\theta \in R(y)/\lambda_\theta$. Thus λ_θ is an $A - P$ congruence.

By Theorem 3.11 every congruence Θ compatible with S gives rise to an $A - P$ congruence λ_θ on S.

LEMMA 3.12. *Let Θ be a congruence compatible with S. Then $x \in \ker \lambda_\theta$ if and only if $LR(x) \in \ker \Theta$.*

Proof. Let $x \in \ker \lambda_\theta$, i.e., $Se\phi_x\Theta Se\phi_0 = (0)$ for all $Se \in \mathscr{L}(S)$. Taking $Se = S$ gives $LR(x) \in \ker \Theta$. Let $LR(x) \in \ker \Theta$. The for any $Se \in \mathscr{L}(S)$ $Se\phi_x \subseteqq S\phi_x = LR(x)\Theta(0) = Se\phi_0$ so $x \in \ker \lambda_\theta$.

THEOREM 3.13. *Let Θ be a congruence compatible with S and $\rho = \lambda_\theta$. Then $\Theta_\rho = \Theta$.*

Proof. By Theorem 3.11 ρ is an $A - P$ congruence so by Theorem 3.4 Θ_ρ is compatible with S. By Lemma 3.8 and 3.12 $\ker \Theta_\rho = \ker \Theta$. Thus by Theorem 3.5 $\Theta_\rho = \Theta$.

We now show that λ_θ is the largest $A - P$ congruence which induces Θ.

COROLLARY 3.14. *Let Θ be a congruence compatible with S. If ρ is an $A - P$ congruence on S such that* $\ker \rho = \ker \lambda_\theta$, then $\rho \leqq \lambda_\theta$.

Proof. Let $x\rho y$. Then for any $Se \in \mathscr{L}(S)$, $ex\rho ey$ so

$$(Se\phi_x)\theta_\rho = LR((ex)/\rho) = LR((ey)/\rho) = (Se\phi_y)\theta_\rho .$$

Thus $Se\phi_x\Theta_\rho Se\phi_y$ and since $\Theta_\rho = \Theta$ this gives $x\lambda_\theta y$.

4. Compatible ideals. In this section ideals which are kernels of congruences compatible with S are characterised. Clearly if Θ is a congruence compatible with S and $J = \ker\Theta$ then $J\phi_x \subseteqq J$ for all $x \in S$. Since ϕ_x preserves join (Lemma 1.3 (iii)) the following is clear.

LEMMA 4.1. *Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteqq J$ for all $x \in S$. Define a relation R on $\mathscr{L}(S)$ by $Se\ R\ Sf$ if and only if there is an $Sg \in J$ such that $Se \vee Sg = Sf \vee Sg$. Then R is an equivalence relation and $Se\,R\,Sf \Rightarrow (Se\phi_x)\,R\,(Sf\phi_x)$ for all $x \in S$.*

In order to find an additional condition on J which will assure that the relation R defined in Lemma 4.1 is compatible with S, it will be valuable to look at certain residuated maps on the lattice $I(\mathscr{L}(S))$ of all ideals of $\mathscr{L}(S)$.

LEMMA 4.2. *For each $x \in S$ let $\hat{\phi}_x : I(\mathscr{L}(S) \to I(\mathscr{L}(S))$ be given by $I\hat{\phi}_x = \{Se \in \mathscr{L}(S) : Se \subseteqq Sf\phi_x$ for some $Sf \in I\}$. Then $\hat{\phi}_x$ is residuated with residual $\hat{\phi}_x^+$ given by $I\hat{\phi}_x^+ = \{Se \in \mathscr{L}(S) : Se \subseteqq Sf\phi_x^+$ for some $Sf \in I\}$.*

Proof. Clearly $I\hat{\phi}_x$ and $I\hat{\phi}_x^+$ are ideals. Also $\hat{\phi}_x$ and $\hat{\phi}_x^+$ are clearly isotone. Now since $Sf \subseteqq Sf\phi_x\phi_x^+$, $Sf \in I$ implies $Sf \in I\hat{\phi}_x\hat{\phi}_x^+$. Thus $I \subseteqq I\hat{\phi}_x\hat{\phi}_x^+$. Similarly $Sf \in I\hat{\phi}_x^+\hat{\phi}$ implies $Sf \subseteqq Sg\phi_x^+\phi_x$ for some $Sg \in I$. Thus $Sf \subseteqq Sg\phi_x^+\phi_x \subseteqq Sg \in I$ so we have $I\hat{\phi}_x^+\hat{\phi}_x \subseteqq I$.

We will make use of the residuated maps $\hat{\phi}_x$ to characterise ideals which are kernels of congruences compatible with S.

LEMMA 4.3. *Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteqq J$. Then for any $I \in I(\mathscr{L}(S))$, $I\hat{\phi}_x^+ \vee J \subseteqq (I \vee J)\hat{\phi}_x^+$.*

Proof. Recall that by Lemma 1.3 (iv), for any residuated map ϕ on a lattice L and any $a, b \in L$, $a\phi \leqq b$ if and only if $a \leqq b\phi^+$. Now $(I\hat{\phi}_x^+ \vee J)\hat{\phi}_x = I\hat{\phi}_x^+\hat{\phi}_x \vee J\hat{\phi}_x \subseteqq I \vee J$ since $I\hat{\phi}_x^+\hat{\phi}_x \subseteqq I$ and $J\hat{\phi}_x \subseteqq J$. Thus $I\hat{\phi}_x^+ \vee J \subseteqq (I \vee J)\hat{\phi}_x^+$.

COROLLARY 4.4. *Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteqq J$. Then $J \subseteqq J\phi_x^+$ and for any $I \in I(\mathscr{L}(S))$ we have*

$$I\hat{\phi}_x^+ \vee J \subseteqq I\hat{\phi}_x^+ \vee J\hat{\phi}_x^+ \subseteqq (I \vee J)\hat{\phi}_x^+ \,.$$

The next theorem indicates what all of this has to do with congruences compatible with S.

LEMMA 4.5. *Let Θ be a congruence compatible with S and $J = \ker\Theta$. Then for any $I \in I(\mathscr{L}(S))$, and any $x \in S$,*

$$I\phi_x^+ \vee J = I\phi_x^+ \vee J\phi_x^+ = (I \vee J)\phi_x^+ \,.$$

Proof. By Corollary 4.4 we need only show that $(I \vee J)\phi_x^+ \subseteqq I\hat{\phi}_x^+ \vee J$. Thus let $Se \in I \vee J$. Then there is an $Sf \in I$ and an $Sg \in J$ such that $Se \subseteqq Sf \vee Sg$. Since $Sg\Theta(0)$ we have $Sf \vee Sg\Theta Sf$. Thus, by compatibility with S, $Se\phi_x^+ \subseteqq (Sf \vee Sg)\phi_x^+\Theta Sf\phi_x^+$. By Theorem 3.5 there is an $Sh \in J$ such that $(Sf \vee Sg)\phi_x^+ \vee Sh = Sf\phi_x^+ \vee Sh$. This gives $Se\phi_x^+ \subseteqq (Sf \vee Sg)\phi_x^+ \vee Sh = Sf\phi_x^+ \vee Sh$ so that $Se\phi_x^+ \in I\hat{\phi}_x^+ \vee J$. Thus $(I \vee J)\hat{\phi}_x^+ \subseteqq I\hat{\phi}_x^+ \vee J$.

Without further justification we make the following definition.

DEFINITION 4.6. An *ideal* J of $\mathscr{L}(S)$ is called *compatible with* S if for all $x \in S$, $J\phi_x \subseteqq J$ and, for all $I \in I(\mathscr{L}(S))$, $I\hat{\phi}_x^+ \vee J = (I \vee J)\hat{\phi}_x^+$.

THEOREM 4.7. *An ideal J of $\mathscr{L}(S)$ is compatible with S if and only if it is the kernel of a congruence compatible with S.*

Proof. By Lemma 4.5 the kernel of a congruence compatible with S is an ideal compatible with S. Conversly let J be an ideal compatible with S and define Θ by $Se\Theta Sf$ if and only if there is an $Sg \in J$ such that $Se \vee Sg = Sf \vee Sg$. By Lemma 4.1 Θ is an equivalence relation such that $Se\Theta Sf$ implies $Se\phi_x\Theta Sf\phi_x$ for all $x \in S$. Let $Se \vee Sg = Sf \vee Sg$, $Sg \in J$, i.e., let $Se\Theta Sf$. Note that

$$(Sf \vee Sg)\phi_x^+ \in ([(0), Sf] \vee J)\hat{\phi}_x^+ = [(0), Sf\phi_x^+] \vee J$$

and $(Se \vee Sg)\phi_x^+ \in ([(0), Se] \vee J)\hat{\phi}_x^+ = [(0), Se\phi_x^+] \vee J$. Thus there are $Sh, Sh' \in J$ such that

$$Se\phi_x^+ \subseteqq (Se \vee Sg)\phi_x^+ \subseteqq Se\phi_x^+ \vee Sh$$

and

$$Sf\phi_x^+ \subseteqq (Sf \vee Sg)\phi_x^+ \subseteqq Sf\phi_x^+ \vee Sh' \,.$$

Thus $Se\phi_x^+ \vee Sh = (Se \vee Sg)\phi_x^+ \vee Sh$ and $Sh' \vee (Sf \vee Sg)\phi_x^+ = Sf\phi_x^+ \vee Sh'$. It follows that $Sf\phi_x^+ \vee (Sh \vee Sh') = Se\phi_x^+ \vee (Sh \vee Sh')$ and since $Sh \vee Sh' \in J$ we have $Se\phi_x^+\Theta Sf\phi_x^+$. Thus by Lemma 3.3 Θ is a congruence compatible with S.

Note that in the proof of Theorem 4.7 the only use made of the hypothesis $(I \vee J)\hat{\phi}_x^+ = I\hat{\phi}_x^+ \vee J$ was for I a principal ideal. This observation together with Lemma 4.5 gives.

COROLLARY 4.8. *Let J be an ideal such that $J\phi_x \subseteqq J$ for all $x \in S$. Then J is compatible with S if and only if for any principal ideal $I \in I(\mathscr{L}(S))$ $(I \vee J)\hat{\phi}_x^+ = I\hat{\phi}_x^+ \vee J$ for all $x \in S$.*

By Corollary 4.8 the situation with ideals compatible with S is analogus to that with standard ideals. An ideal J of a lattice L is *standard* if $(I \vee J) \wedge K = (I \wedge K) \vee (J \wedge K)$ for all $I, K \in I(L)$. By Theorem 2, p. 30, of [5] an ideal is standard if and only if the above holds for all principal ideals $I, K \in I(L)$. This similarity is not surprising since by Corollary 3.6, Theorem 4.7, and Theorem 2 of [5] any ideal compatible with S is a standard ideal. In fact the definition of ideal compatible with S is closely related to the definition of standard ideal. To see this we need the following:

LEMMA 4.9. *For any $I \in I(\mathscr{L}(S))$ and any $Se \in \mathscr{L}(S)$,*

$$I \cap [(0), Se] = I\hat{\phi}_e^+\hat{\phi}_e \,.$$

Proof. Clearly $I \cap [(0), Se] = \{Sf \in \mathscr{L}(S) : Sf \subseteqq Sg \cap Se$, for some $Sg \in I\}$. But $Sg \cap Se = Sg\phi_e^+\phi_e$ so $I \cap [(0), Se] = I\hat{\phi}_e^+\hat{\phi}_e$.

For any ideal for which $J\phi_x \subseteqq J$ Corollary 4.4 gives

$$I\hat{\phi}_x^+ \vee J \subseteqq I\hat{\phi}_x^+ \vee J\hat{\phi}_x^+ \subseteqq (I \vee J)\phi_x^+$$

for all $I \in I(L)$. Now $I\hat{\phi}_x^+ \vee J\hat{\phi}_x^+ = (I \vee J)\hat{\phi}_x^+$ implies

$$(I\hat{\phi}_x^+ \vee J\hat{\phi}_x^+)\phi_x = I\hat{\phi}_x^+\hat{\phi}_x \vee J\hat{\phi}_x^+\hat{\phi}_x = (I \vee J)\hat{\phi}_x^+\hat{\phi}_x \,.$$

Taking $x = e$ with $Se \in \mathscr{L}(S)$ and applying Lemma 4.9 this becomes

$$(I \cap [(0), Se]) \vee (J \cap [(0), Se]) = (I \vee J) \cap [(0), Se] \,.$$

Thus if we had required only $I\hat{\phi}_e^+ \vee J\hat{\phi}_e^+ = (I \vee J)\hat{\phi}_e^+$ for all e such that $Se \in \mathscr{L}(S)$ we would have J a standard ideal. However, to define an ideal compatible with S we require the stronger condition that $I\hat{\phi}_x^+ \vee J = (I \vee J)\hat{\phi}_x^+$ and not only for all idempotents x such that $Sx \in \mathscr{L}(S)$ but for all $x \in S$.

5. Compatible elements. An element a of a lattice L is called *standard* if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ for all $x, y \in L$. By Lemma 4, p. 32 of [5] an element is standard if and only if the principal ideal it generates is a standard ideal.

DEFINITION 5.1. An *element* Se of $\mathscr{L}(S)$ is *compatible with* S if $[(0), Se]$ is an ideal compatible with S. Let Θ_{Se} denote the congruence compatible with S having $[(0), Se]$ as kernel.

Note that by Corollary 3.6 every element compatible with S is a standard element of $\mathscr{L}(S)$.

It will be convenient to look at co-kernels of congruences compatible with S.

LEMMA 5.2. *Let Θ be a congruence compatible with S. Then $LR(x) \in \ker \Theta$ if and only if $L(x) \in$ co-ker Θ.*

Proof. Let $Sf = LR(x) \in \ker \Theta$. Then $Sf\phi_x^+ \Theta (0)\phi_x^+ = L(x)$. But $Sf\phi_x^+ = L(xf^r)$ and since $f^r S = R(f) = R(Sf) = RLR(x) = R(x)$ we have $Sf\phi_x^+ = L(0) = S$. Thus $L(x) \in$ co-ker Θ. Conversly let $L(x)\Theta S$ and note that $L(x)\phi_x = LRL(x)\phi_x = LR(x^l)\phi_x = LR(x^l x) = (0)$. Thus

$$(0) = L(x)\phi_x \Theta S\phi_x = LR(x), \quad \text{i.e.,} \quad LR(x) \in \ker \Theta .$$

LEMMA 5.3. *Let Se be compatible with S. Then Se^l is a complement of Se and $[Se^l, S] =$ co-ker Θ_{Se}.*

Proof. *Clearly* $Se \cap Se^l = (0)$. By Lemma 5.2, $Se^l \Theta_{Se} S$ so, by Theorem 3.5, $Se^l \vee Se = S$. Thus we clearly have $[Se^l, S] \subseteq$ co-ker Θ_{Se}. Let $Sf \in$ co-ker Θ_{Se}. Then $Sf \vee Se = S$ and since Se is standard we have $Se^l = Se^l \cap (Sf \vee Se) = (Se^l \cap Sf) \vee (Se^l \cap Se) = Se^l \cap Sf$. Thus $Se^l \subseteq Sf$, i.e., co-ker $\Theta_{Se} = [Se^l, S]$.

We now wish to characterise elements compatible with S.

LEMMA 5.4. *Let Se be compatible with S. Then e is central in S and $eS = RL(e)$.*

Proof. By Lemma 5.2, $Se^l \in$ co-ker Θ_{Se}. Since $Se^l = LR(e^l) = L(e^{lr})$ applying Lemma 5.2 again gives $LR(e^{lr}) \in \ker \Theta_{Se}$, i.e., $LR(e^{lr}) \subseteq Se$. Thus $e^{lr} = e^{lr}e$. But $e^l e = 0$ implies $e \in R(e^l)$ so $e = e^{lr}e$. Thus $e = e^{lr}$ so $eS = R(e^l) = RL(e)$. By Lemma 5.3, $Se^l\phi_x^+ \supseteq Se^l$ and $Se^l\phi_x^+ = L(xe^{lr}) = L(xe)$. Thus $RL(xe) \subseteq R(e^l) = RL(e) = eS$ so $xe = exe$. But $Se\phi_x \subseteq Se$ so $ex = exe = xe$, i.e., e is central in S.

We can use any central idempotent of S to induce an $A - P$ congruence on S as follows :

LEMMA 5.5. *Let e be central in S and define a relation ρ on S*

by $x\rho y$ *if and only if* $xe = ye$. *Then* ρ *is an* $A - P$ *congruence on* S *and* $\ker \rho = Se^l$.

Proof. Clearly ρ is a congruence on S. Let $y/\rho \in L(x/\rho)$. Then $0/\rho = (y/\rho)(x/\rho) = (yx)/\rho$ so $yxe = 0$. But $yxe = (ye)x$ so $ye \in L(x)$. Thus $ye = (ye)e$ gives $y/\rho = (ye)/\rho \in L(x)/\rho$. Similarly $R(x/\rho) \subseteqq R(x)/\rho$. Clearly $x/\rho = 0/\rho$ if and only if $x \in L(e) = Se^l$.

LEMMA 5.6. *If* e *is central in* S *then* Se^l *is compatible with* S.

Proof. Since e is central $x\rho y$ if and only if $xe = ye$ is an $A - P$ congruence with kernel Se^l. By Lemma 3.8, $LR(x) \in \ker \Theta_\rho$ if and only if $x \in Se^l$. But $x \in Se^l$ if and only if $x = xe^l$ if and only if $LR(x) \subseteqq Se^l$. Thus $\ker \Theta_\rho = [(0), Se^l]$ so that Se^l is compatible with S.

We can now characterise elements compatible with S as follows:

THEOREM 5.7. *Let* $Se \in \mathscr{L}(S)$. *Then* Se *is compatible with* S *if and only if* e *is central in* S.

Proof. Let e be central in S. By Lemma 5.6, Se^l is compatible with S. Now $L(e) = R(e)$ so $Se^l = e^rS$. Thus $e^l = e^re^l = e^r$. By Lemma 5.6, $Se^l = Se^r$ compatible with S gives Se^{rl} compatible with S. But $Se^{rl} = LR(e^{rl}) = LR(e) = Se$. Thus Se is compatible with S. The converse is Lemma 5.4.

Note that Se is compatible with S if and only if Se^l is compatible with S. Thus, by Lemma 5.3, if either Se or Se^l is compatible with S then Se and Se^l are standard elements of $\mathscr{L}(S)$ which are complements. Thus by Theorem 7.3, p. 300, of [6] we have.

THEOREM 5.8. *If either* Se *or* Se^l *is compatible with* S *then*:
(i) *Both* Se *and* Se^l *are compatible with* S.
(ii) *Both* Se *and* Se^l *are central in* $\mathscr{L}(S)$.
(iii) Θ_{Se} *and* Θ_{Se^l} *are complements in* $\Theta(\mathscr{L}(S))$.

COROLLARY 5.9. *Let* $Se \in \mathscr{L}(S)$. *Then if* e *is central in* S, Se *is central in* $\mathscr{L}(S)$.

5. **The lattice of compatible congruences.** From the formula for meet and join in $\Theta(L)$ (see Theorem 1.7) it is clear that both the meet and the join of any family of congruences compatible with S are congruences compatible with S. Thus, applying Theorem 1.8, we have.

THEOREM 6.1. *The lattice $\Theta_S(\mathscr{L}(S))$ of all congruence compatible with S is a subcomplete sublattice of $\Theta(\mathscr{L}(S))$. Thus $\Theta_S(\mathscr{L}(S))$ is an uppercontinuous distributive lattice.*

It follows from [1], Theorem 15, p. 147, that $\Theta_S(\mathscr{L}(S))$ is pseudo-complemented. If $\Theta \in \Theta_S(\mathscr{L}(S))$ we shall use Θ^* to denote the pseudo-complent of Θ in $\Theta(\mathscr{L}(S))$ and Θ' to denote the pseudo-complement of Θ in $\Theta_S(\mathscr{L}(S))$.

In [9], Theorem 4.17 (iii), it is shown that for a complete relatively complemented lattice $L, \Theta(L)$ is a *Stone lattice* in the sense that every pseudo-complement has a complement. The remainder of this section is devoted to showing that for suitable choice of S, $\Theta_S(\mathscr{L}(S))$ is a Stone lattice.

We first look at the left and right annihilators of the kernel of an $A - P$ congruence.

LEMMA 6.2. *Let ρ be an $A - P$ congruence on S and $J = \ker \rho$. Then $L(J) = R(J)$.*

Proof. Let $x \in J$ and $y \in L(J)$. If $z \in J$ then $xyz = 0$. Thus $J \subseteqq R(xy)$ so that $L(J) \supseteqq LR(xy)$. Let $LR(xy) = Sf$ and note that $f \in L(J)$. Since J is an ideal, $xy \in J$, i.e., $xy/\rho = 0/\rho$. Thus

$$f/\rho \in LR(xy)/\rho = LR(xy/\rho) = LR(0/\rho) = (0/\rho)$$

so $f \in J$. But then we have $f \in J \cap L(J)$ so $f = f^2 = 0$. This gives $LR(xy) = (0)$ which implies $xy = 0$. Thus $L(J) \subseteqq R(J)$. By symmetry $R(J) \subseteqq L(J)$ so $R(J) = L(J)$.

Recall that a semigroup S is a complete Baer semigroup if the left and right annihilators of an arbitrary subset of S are principal left and right ideals generated by idempotents. Also (Theorem 1.6) as S ranges over all complete Baer semigroups $\mathscr{L}(S)$ ranges over all complete lattices.

LEMMA 6.3. *Let S be a complete Baer semigroup, Θ a congruence compatible with S, and $Se = \cap$ co-ker Θ. Then Se is compatible with S.*

Proof. Let $J = \ker \lambda_\Theta$. By Lemmas 5.2 and 3.12, $x \in J$ if and only if $L(x) \in$ co-ker Θ. Thus $L(J) \subseteqq Se$ since $L(J) \subseteqq L(x)$ for all $x \in J$. But $Se \subseteqq L(x)$ for all $x \in J$ gives $Se \subseteqq L(J)$. Thus $Se = L(J)$. Now by Lemma 6.2, $L(J) = R(J)$ and since S is a complete Baer semigroup there is an idempotent $f \in S$ such that $fS = R(J)$. Then

$fS = Se$ so $e = fe = f$. Since $Se = eS$ is an ideal we have $ex = exe = xe$ for all $x \in S$. Thus e is central in S so by Theorem 5.7, Se is compatible with S.

We can now characterise the kernel of the pseudo-complement of a congruence compatible with a complete Baer semigroup.

THEOREM 6.4. *Let S be a complete Baer semigroup and Θ a congruence compatible with S. Then* $\ker \Theta^*$ *is a principal ideal generated by an element of $\mathscr{L}(S)$ which is compatible with S.*

Proof. Let $Se = \cap$ co-ker Θ and $J = \ker \lambda_\Theta$. By Lemma 6.3, Se is compatible with S. But $Se = L(J) = R(J)$ and $x \in J$ if and only if $LR(x) \in \ker \Theta$ gives $Se \cap Sf = (0)$ for all $Sf \in \ker \Theta$. Thus ker $\Theta_{Se} \cap \ker \Theta = (0)$ so by Theorem 3.5, $\Theta_{Se} \wedge \Theta = \omega$. By definition of pseudo-complement we have $\Theta_{Se} \leqq \Theta^*$ so $[(0), Se] = \ker \Theta_{Se} \subseteqq \ker \Theta^*$. Now let $Sg \in$ co-ker Θ and $Sf \in \ker \Theta^*$. Then $(Sf \cap Sg)\Theta(Sf \cap S) = Sf$ and $(Sf \cap Sg)\Theta^*(0)$. Since $(0)\Theta^* Sf$ we have $(Sf \cap Sg) \equiv Sf(\Theta \wedge \Theta^*)$. This gives $Sf \cap Sg = Sf$ so $Sf \subseteqq Sg$. Thus $Sf \subseteqq Se$ and $\ker \Theta^* \subseteqq [(0), Se]$. We, therefore, have $\ker \Theta^* = [(0), Se]$ and since Se is compatible with S this completes the proof.

We clearly have $\Theta' \subseteqq \Theta^*$. Since $\ker \Theta^*$ is a principal ideal generated by an element Se compatible with S, it is clear that $\Theta' = \Theta_{Se}$. By Theorem 5.8, Se^l is compatible with S and Θ_{Se^l} is a complement of Θ_{Se} in $\Theta_S(\mathscr{L}(S))$.

THEOREM 6.5. *Let S be a complete Baer semigroup. Then $\Theta_S(\mathscr{L}(S))$ is a Stone lattice.*

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