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SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 3

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We shall establish in this paper some congruence relations with respect to the modulus 3 for some restricted partition functions. The difference between the unrestricted partition function, $p(n)$, and these restricted partition functions which we shall denote by

$${}^{27}_{r}p(n) \quad \textbf{with } r = 3, 6, 12 ,$$

merely lies in the restriction that no number of the forms $27n$, or $27n \pm r$, shall be a part of the partitions which are of relevance in the restricted case. Thus to determine the value of ${}^{27}_{r}p(n)$ one should count all the unrestricted partitions of n excepting those which contain a number of any of the above forms as a part. We shall assume $p(n)$ and ${}^{27}_{r}p(n)$ to be unity when n is zero, and vanishing when the argument is negative. We can now state our theorems.

THEOREM 1. **For almost all values of** n

$${}^{27}_{3}p(n) \equiv {}^{27}_{6}p(n) \equiv {}^{27}_{12}p(n) \equiv 0 \pmod 3 .$$

THEOREM 2. **For all values of** n

$${}^{27}_{3}p(3n) \equiv {}^{27}_{6}p(3n+1) \equiv -{}^{27}_{12}p(3n+2) \pmod 3 .$$

2. Definitions and notations. We shall use m to denote an integer positive zero or negative, but n will stand for a positive or nonnegative integer only.

We define u_r by

$$(1) \qquad u_0 = 1 \quad \text{and} \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n . \sum_{n=0}^{\infty} p(n) x^n,\ r > 0 ,$$

where a_n is defined by the well-known 'pentagonal number' theorem of Euler,

$$(2) \qquad f(x) = \prod_{n=1}^{\infty}(1 - x^n) = \sum_{-\infty}^{+\infty}(-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n ,$$

and $p(n)$ is the number of unrestricted partitions of n given by the expansion,

$$(3) \qquad [f(x)]^{-1} = \left[\prod_{n=1}^{\infty}(1 - x^n)\right]^{-1} = \sum_{n=0}^{\infty} p(n) x^n .$$

We shall use v to denote the pentagonal numbers,

$$v = \frac{1}{2} m(3m+1),\ m = 0, \pm 1, \pm 2, \cdots ; \tag{4}$$

and with each v there corresponds an 'associated' sign, viz., $(-1)^m$. We shall come across sums of the type

$$\sum_v [\mp V(v)]$$

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if v is 1, 2, 12, 15, 35, $\cdots$, that is, when it is of the form $(2m+1)(3m+1)$, and (b) positive if v is 0, 5, 7, 22, 26 $\cdots$, that is, when it is of the form $m(6m+1)$. With the above summation notation we can write,

$$u_r = \sum_v (\mp v^r x^v)/f(x) , \tag{5}$$

$$\sum_v (\mp x^v)/f(x) = 1 . \tag{6}$$

We shall also require the functions U_i, $i = 0, 1, 2$ which are certain linear functions of u_r's, $r = 0, 1, 2$ as given below.

$$\begin{cases} U_0 = - u_2 + u_0 , \\ U_1 = - u_2 - u_1 , \\ U_2 = - u_2 + u_1 . \end{cases} \tag{7}$$

We also need the quadratics $P_i(v)$ in v, $i = 0, 1, 2$ which are obtained by writing $P_i(v)$ for U_i, and v^r for u_r. Thus

$$\begin{cases} P_0(v) = - v^2 + 1 , \\ P_1(v) = - v^2 - v , \\ P_2(v) = - v^2 + v . \end{cases} \tag{8}$$

3. Some lemmas. The truth of the following lemma can be easily verified from the expressions for $P_i(v)$ given in (8).

LEMMA 1.

$$\begin{aligned} P_i(v) &\equiv 1 \pmod 3 , \quad \textit{if } v \equiv i \pmod 3 \\ &\equiv 0 \pmod 3 , \quad \textit{if } v \not\equiv i \pmod 3 . \end{aligned}$$

If we replace the u_r's appearing in the expressions for U_i in (7) by the right hand expressions in (5) we get

$$U_i = \sum_v [\mp P_i(v) x^v]/f(x) ; \tag{9}$$

and then the use of Lemma 1 leads to the next lemma.

LEMMA 2. *$U_i \equiv \sum_{v\equiv i}(\mp x^v)/f(x)$ (mod 3), the summation being extended over all pentagonal numbers $v \equiv i$ (mod 3).*

The truth of the following lemma can be verified without much difficulty by writing $3m+j$, with $j = 0$; -1; and 1 respectively, in place of m in the expression $\frac{1}{2}m(3m+1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered that $\frac{1}{2}(3m-1)(9m-2)$ and $\frac{1}{2}(3m+1)(9m+2)$ represent the same set of numbers.

LEMMA 3. *The solutions of*

$$v \equiv i \pmod 3\,, \quad i = 0, 1, 2$$

are as noted below, (the associated signs are also shown).

i	*solutions*	*sign*
0	$\frac{1}{2}(27m^2+3m)$	$(-1)^m$
1	$\frac{1}{2}(27m^2+15m)+1$	$(-1)^{m+1}$
2	$\frac{1}{2}(27m^2+21m)+2$	$(-1)^{m+1}$.

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [3, p. 283] viz.,

$$\prod_{n=0}^{\infty}[(1-x^{2kn+k-l})(1-x^{2kn+k+l})(1-x^{2kn+2k})] = \sum_{-\infty}^{+\infty}(-1)^m x^{km^2+lm}\,. \tag{10}$$

In establishing this lemma k and l are given values which are in conformity with the quadratic expressions in m given in Lemma 3. As an illustration we have

$$\begin{aligned}\sum_{v=2}(\mp x^v) &= \sum_{-\infty}^{+\infty}(-1)^{m+1}x^{\frac{1}{2}(27m^2+21m)+2} \\ &= -x^2\prod_{n=0}^{\infty}[(1-x^{27n+3})(1-x^{27n+24})(1-x^{27n+27})]\,.\end{aligned} \tag{11}$$

LEMMA 4. *Writing $v \equiv i$ simply for $v \equiv i$ (mod 3)*

$$\sum_{v\equiv 0}(\mp x^v) = \prod_{n=0}^{\infty}[(1-x^{27n+12})(1-x^{27n+15})(1-x^{27n+27})]$$

$$\sum_{v\equiv 1}(\mp x^v) = -x\prod_{n=0}^{\infty}[(1-x^{27n+6})(1-x^{27n+21})(1-x^{27n+27})]\,.$$

$$\sum_{v\equiv 2}(\mp x^v) = -x^2\prod_{n=0}^{\infty}[(1-x^{27n+3})(1-x^{27n+24})(1-x^{27n+27})]\,.$$

Lemma 5, given below is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v\equiv i}(\mp x^v)$ as given in the above lemma. The following fact also is to be taken into consideration.

$$(12)\quad \prod_{n=0}^{\infty}(1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27})]/f(x)$$
$$= \prod_{n=0}^{\infty}[(1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27})]/[(1 - x)(1 - x^2)(1 - x^3)\cdots]$$
$$= \sum_{n=0}^{\infty} {}^{27}_{r}p(n)x^n .$$

LEMMA 5.

$$U_0 \equiv \sum_{n=0}^{\infty} {}^{27}_{12}p(n)x^n \pmod 3$$
$$U_1 \equiv -\sum_{n=0}^{\infty} {}^{27}_{6}p(n-1)x^n \pmod 3$$
$$U_2 \equiv -\sum_{n=0}^{\infty} {}^{27}_{3}p(n-2)x^n \pmod 3 .$$

We require another set of congruences which are obtained from the classical result, due to Catalan [1, p. 290].

$$(13)\quad p(n-1) + 2p(n-2) - 5p(n-5) - 7p(n-7) + \cdots = \sigma(n) ,$$

and another result due to Glaisher [1, p. 312]

$$(14)\quad p(n-1) + 2^2p(n-2) - 5^2p(n-5) - 7^2p(n-7) + \cdots$$
$$= -\frac{1}{12}[5\sigma_3(n) - (18n-1)\sigma(n)] .$$

These results can be rewritten according to our notation as

$$(15)\quad \sum_v [\mp vp(n-v)] = -\sigma(n) ,$$

$$(16)\quad \sum_v [\mp v^2p(n-v)] = \frac{1}{12}[5\sigma_3(n) - (18n-1)\sigma(n)] .$$

Now from (5) we have

$$u_r = \sum_v (\mp v^r x^v)/f(x)$$
$$(17)\quad = \sum_v (\mp v^r x^v) \,.\, \sum_{n=0}^{\infty} p(n)x^n$$
$$= \sum_{n=1}^{\infty}\{\sum_v [\mp v^r p(n-v)]\}x^n ,\ r > 0 .$$

It is now easy to establish the validity of the following lemma from the above three relations (15), (16) and (17).

LEMMA 6.

$$u_1 = -\sum_{n=1}^{\infty} \sigma(n)x^n$$

$$u_2 = \frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n-1)\sigma(n)]x^n .$$

The next lemma can be easily obtained by the substitution of the above values of u_1 and u_2 in (7).

LEMMA 7.

$$U_0 - 1 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n-1)\sigma(n)]x^n ,$$

$$U_1 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n+11)\sigma(n)]x^n ,$$

$$U_2 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n-13)\sigma(n)]x^n .$$

The congruences given in Lemma 8 are elementary and can be readily proved.

LEMMA 8.

$$\sigma(3n-1) \equiv 0 \pmod 3 .$$

$$\sigma(3^{\lambda}n) \equiv \sigma(n) \pmod 3 , \quad \lambda \geq 0 .$$

4. Proof of the theorems. By comparing the coefficients of like powers of x in the expressions (modulo 3) for U_i given in Lemmas 5 and 7 we obtain the following congruences for $n > 0$.

$$ {}^{27}_{12}p(n) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n-1)\sigma(n)] \pmod 3 \tag{18}$$

$$ -{}^{27}_{6}p(n-1) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n+11)\sigma(n)] \pmod 3 \tag{19}$$

$$ -{}^{27}_{3}p(n-2) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n-13)\sigma(n)] \pmod 3 . \tag{20}$$

Remembering the well-known congruence, [4 ; 2, p. 167],

$$\sigma_k(n) \equiv 0 \pmod M \text{ for almost all } n \tag{21}$$

for arbitrarily fixed M and odd k, it is a straightforward matter to deduce Theorem 1 from the above congruences.

To establish Theorem 2 we obtain by a process of addition or subtraction of (18), (19) and (20) in pairs the following.

$$(22)\qquad -{}_{12}^{27}p(n) - {}_{6}^{27}p(n-1) \equiv {}_{12}^{27}p(n) + {}_{3}^{27}p(n-2) \equiv {}_{6}^{27}p(n-1) - {}_{3}^{27}p(n-2) \equiv \sigma(n) \pmod 3 .$$

Now writing $3n+2$ for n in (22) and making use of the first relation of Lemma 8 we obtain the theorem immediately.

To derive a generalization from (22) we write $3^\lambda n$ for n in it and make use of the last congruence of Lemma 8 to obtain,

$$(23)\qquad -{}_{12}^{27}p(3^\lambda n) - {}_{6}^{27}p(3^\lambda n-1) \equiv {}_{12}^{27}p(3^\lambda n) + {}_{3}^{27}p(3^\lambda n-2) \equiv {}_{6}^{27}p(3^\lambda n-1) - {}_{3}^{27}p(3^\lambda n-2) \equiv \sigma(n) \pmod 3 .$$

We need write $3n-1$ for n in (23) and use the first congruence of Lemma 8 to arrive at the more general Theorem 3.

THEOREM 3. *With respect to the modulus* 3

$$-{}_{12}^{27}p(3^{\lambda+1}n - 3^\lambda) \equiv {}_{6}^{27}p(3^{\lambda+1}n - 3^\lambda - 1) \equiv {}_{3}^{27}p(3^{\lambda+1}n - 3^\lambda - 2) .$$

Finally, it might be of interest to note that the three restricted partition functions ${}_{r}^{27}p(n)$, $r = 3, 6$ and 12, are connected by the identical relation,

$$(24)\qquad {}_{12}^{27}p(n) = {}_{6}^{27}p(n-1) + {}_{3}^{27}p(n-2) ,\quad n > 0 .$$

This is seen to be true by a joint consideration of (6), Lemma 4, and (12). The first relation gives

$$(25)\qquad \sum_{i=0}^{2}\sum_{v\equiv i}(\mp x^v)/f(x) = 1 .$$

We substitute the values of $\sum_{v\equiv i}(\mp x^v)$ in the product form as given in Lemma 4, and then make use of (12) in order to express the left hand side of (25) as a power series in x whose coefficients are simple linear functions of the restricted partition functions. Now (24) is obtained directly by equating to zero the coefficient of x^n, $n > 0$.

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