PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

F'-SPACES AND THEIR PRODUCT WITH P-SPACES

W. W. COMFORT, NEIL HINDMAN, AND S. NEGREPONTIS

The F'-spaces studied here, introduced by Leonard Gillman and Melvin Henriksen, are by definition completely regular Hausdorff spaces in which disjoint cozero-sets have disjoint closures. The principal result of this paper gives a sufficient condition that a product space be an F'-space and shows that the condition is, in a strong sense, best possible. A fortuitous corollary in the same vein responds to a question posed by Gillman: When is a product space basically disconnected (in the sense that each of its cozero-sets has open closure)?

A concept essential to the success of our investigation was suggested to us jointly by Anthony W. Hager and S. Mrowka in response to our search for a (simultaneous) generalization of the concepts "Lindelöf" and "separable." Using the Hager-Mrowka terminology, which differs from that of Frolík in [3], we say that a space is weakly Lindelöf if each of its open covers admits a countable subfamily with dense union. § 1 investigates F'-spaces which are (locally) weakly Lindelöf; § 2 applies standard techniques to achieve a product theorem less successful than that of § 3; § 4 contains examples, chiefly elementary variants of examples from [5] or Kohls' [8], and some questions.

1. F'-spaces and their subspaces.

Following [5], we say that a (completely regular Hausdorff) space is an F-space provided that disjoint cozero-sets are completely separated (in the sense that some continuous real-valued function on the space assumes the value 0 on one of the sets and the value 1 on the other). It is clear that any F-space is an F'-space and (by Urysohn's Lemma) that the converse is valid for normal spaces. Since each element of the ring $C^*(X)$ of bounded real-valued continuous functions on X extends continuously to the Stone-Čech compactification βX of X, it follows that X is an F-space if and only if βX is an F-space. These and less elementary properties of F-spaces are discussed at length in [5] and [6], to which the reader is referred also for definitions of unfamiliar concepts.

F-spaces are characterized in 14.25 of [6] as those spaces in which each cozero-set is C^*-embedded. We begin with the analogous characterization of F'-spaces. All hypothesized spaces in this paper are understood to be completely regular Hausdorff spaces.

THEOREM 1.1. *X is an F'-space if and only if each cozero-set in X is C^*-embedded in its own closure.*

Proof. To show that $\operatorname{coz} f$ (with $f \in C(X)$ and $f \geqq 0$, say) is C^*-embedded in $\mathrm{cl}_X \operatorname{coz} f$ it suffices, according to Theorem 6.4 of [6], to show that disjoint zero-sets A and B in $\operatorname{coz} f$ have disjoint closures in $\mathrm{cl}_X \operatorname{coz} f$. There exists $g \in C^*(\operatorname{coz} f)$ with $g > 0$ on A, $g < 0$ on B. It is easily checked that the function h, defined on X by the rule

$$h = \begin{Bmatrix} fg & \text{on} & \operatorname{coz} f \\ 0 & \text{on} & Zf \end{Bmatrix}$$

lies in $C^*(X)$, and that the (disjoint) cozero-sets $\operatorname{pos} h$, $\operatorname{neg} h$, contain A and B respectively. Since $\mathrm{cl}_X \operatorname{pos} h \cap \mathrm{cl}_X \operatorname{neg} h = \emptyset$, we see that A and B have disjoint closures in X, hence surely in $\mathrm{cl}_X \operatorname{coz} f$.

The converse is trivial: If U and V are disjoint cozero-sets in X, then the characteristic function of U, considered as function on $U \cup V$, lies in $C^*(U \cup V)$, and its extension to a function in $C^*(\mathrm{cl}_X(U \cup V))$ would have the values 0 and 1 simultaneously at any point in $\mathrm{cl}_X U \cap \mathrm{cl}_X V$.

The "weakly Lindelöf" concept described above allows us to show that certain subsets of F'-spaces are themselves F', and that certain F'-spaces (for example, the separable ones) are in fact F-spaces. We begin by recording some simple facts about weakly Lindelöf spaces.

Recall that a subset S of X is said to be regularly closed if $S = \mathrm{cl}_X \operatorname{int}_X S$.

LEMMA 1.2. (a) *A regularly closed subset of a weakly Lindelöf space is weakly Lindelöf;*

(b) *A countable union of weakly Lindelöf subspaces of a (fixed) space is weakly Lindelöf;*

(c) *Each cozero-set in a weakly Lindelöf space is weakly Lindelöf.*

Proof. (a) and (b) follow easily from the definition, and (c) is obvious since for $f \in C^*(X)$ the set $\operatorname{coz} f$ is the union of the regularly closed sets $\mathrm{cl}_X\{x \in X : |f(x)| > 1/n\}$.

Lemma 1.2(c) shows that any point with a weakly Lindelöf neighborhood admits a fundamental system of weakly Lindelöf neighborhoods. For later use we formalize the concept with a definition.

DEFINITION 1.3. The space X is locally weakly Lindelöf at its point x if x admits a weakly Lindelöf neighborhood in X. A space locally weakly Lindelöf at each of its points is said to be locally weakly Lindelöf.

THEOREM 1.4. *Let A and B be weakly Lindelöf subsets of the* *space X, each missing the closure (in X) of the other. Then there exist disjoint cozero-sets U and V for X for which*

$$A \subset \mathrm{cl}_X(A \cap U)\ , \qquad B \subset \mathrm{cl}_X(B \cap V)\ .$$

Proof. For each $x \in A$ there exists $f_x \in C^*(X)$ with $f_x(x) = 0$, $f_x \equiv 1$ on $\mathrm{cl}_X B$. Similarly, for each $y \in B$ there exists $g_y \in C^*(X)$ with $g_y(y) = 0$, $g_y \equiv 1$ on $\mathrm{cl}_X A$. Taking $0 \leqq f_x \leqq 1$ and $0 \leqq g_y \leqq 1$ for each x and y, we define

$$U_x = f_x^{-1}[0, 1/2)\ , \qquad V_y = g_y^{-1}[0, 1/2)\ ,$$
$$W_x = f_x^{-1}[0, 1/2]\ , \qquad Z_y = g_y^{-1}[0, 1/2]\ .$$

Then, with $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ sequences chosen in A and B respectively so that $A \cap (\bigcup_n U_{x_n})$ is dense in A and $B \cap (\bigcup_n V_{y_n})$ is dense in B, we set

$$U_n^{\sim} = U_{x_n} \backslash \bigcup_{k \leqq n} Z_{y_k}\ , \qquad V_n^{\sim} = V_{y_n} \backslash \bigcup_{k \leqq n} W_{x_k}$$

and, finally, $U = \bigcup_n U_n^{\sim}$, $V = \bigcup_n V_u^{\sim}$.

The theorem just given has several elementary corollaries.

COROLLARY 1.5. *Two weakly Lindelöf subsets of an F'-space, each missing the closure of the other, have disjoint closures (which are weakly Lindelöf).*

COROLLARY 1.6. *Any weakly Lindelöf subspace of an F'-space is itself an F'-space.*

Proof. If A and B are disjoint cozero-sets in the weakly Lindelöf subset Y of the F'-space X, we have from 1.2(c) that A and B are themselves weakly Lindelöf, and that

$$A \cap \mathrm{cl}_X B = A \cap \mathrm{cl}_Y B = \varnothing \quad \text{and} \quad B \cap \mathrm{cl}_X A = B \cap \mathrm{cl}_Y A = \varnothing\ .$$

From 1.5 it follows that

$$\varnothing = \mathrm{cl}_X A \cap \mathrm{cl}_X B \supset \mathrm{cl}_Y A \cap \mathrm{cl}_Y B\ .$$

COROLLARY 1.7. *Each weakly Lindelöf subspace of an F'-space is C^*-embedded in its own closure.*

Proof. Disjoint zero-sets of the weakly Lindelöf subspace Y of the F'-space X are contained in disjoint cozero subsets of Y, which by 1.2(c) and 1.5 have disjoint closures in X.

Corollaries 1.6 and 1.7 furnish us with a sufficient condition that an F'-space be an F-space.

THEOREM 1.8. *Each F'-space with a dense Lindelöf subspace is an F-space.*

Proof. If Y is a dense Lindelöf subspace of the F'-space X, then Y is F' by 1.6, hence (being normal) is an F-space. But by 1.7 Y is C^*-embedded in X, hence in βX, so that $\beta Y = \beta X$. Now Y is an F-space, hence βY, hence βX, hence X.

COROLLARY 1.9. *A separable F'-space is an F-space.*

The following simple result improves 3B.4 of [6]. Its proof, very similar to that of 1.4, is omitted.

THEOREM 1.10. *Any two Lindelöf subsets of a (fixed) space, neither meeting the closure of the other, are contained in disjoint cozero-sets.*

An example given in [5] shows that there exists a (nonnormal) F'-space which is not an F-space. For each such space X the space βX, since it is normal, cannot be an F'-space; for (as we have observed earlier) X is an F-space if and only if βX is an F-space. Thus not every space in which an F$'$-space is dense and C^*-embedded need be an F'-space. The next result shows that passage to C^*-embedded subspaces is better behaved.

THEOREM 1.11. *If Y is a C^*-embedded subset of the F'-space X, then Y is an F'-space.*

Proof. Disjoint cozero-sets in Y are contained in disjoint cozero-sets in X, whose closures (in X, even) are disjoint.

We shall show in Theorem 4.2 that the F' property is inherited not only by C^*-embedded subsets, but by open subsets as well.

2. On the product of a (locally) weakly Lindelöf space and a P-space. A P-point in the space X is a point x with the property that each continuous real-valued function on X is constant throughout some neighborhood of x. If each point of X is a P-point, then X is said to be a P-space. The P-spaces are precisely those spaces in which each G_δ subset is open.

The following diagram, a sub-graph of one found in [5] and in [8], is convenient for reference.

$$\begin{tikzcd}[column sep=small]
 & P \arrow[dr] & & & \\
\text{discrete} \arrow[ur] \arrow[dr] & & \text{basically disconnected} \arrow[r] & F \arrow[r] & F' . \\
 & \text{extremally disconnected} \arrow[ur] & & &
\end{tikzcd}$$

In the interest of making this paper self-contained, we now include from [2] a proof of the fact that if a product space $X \times Y$ is an F'-space, then both X and Y are F'-spaces and either X or Y is a P-space. Indeed, the first conclusion is obvious. For the second, let x_0 and y_0 be points in X and Y respectively belonging to the boundary of the sets $\operatorname{coz} f$ and $\operatorname{coz} g$ respectively (with $f \in C(X)$ and $g \in C(Y)$ and $f \geqq 0$ and $g \geqq 0$). Then the function h, defined on $X \times Y$ by the rule $h(x, y) = f(x) - g(y)$, assumes both positive and negative values on each neighborhood in $X \times Y$ of (x_0, y_0). Thus $\operatorname{pos} h$ and $\operatorname{neg} h$ are disjoint cozero-sets in $X \times Y$ each of whose closure contains (x_0, y_0).

We are going to derive, in 2.4, a simple condition sufficient that a product space be an F'-space.

THEOREM 2.1. *Let X be a P-space, let Y be weakly Lindelöf, and let $f \in C^*(X \times Y)$. Then the real-valued function F, defined on X by the rule*

$$F(x) = \sup\{f(x, y) : y \in Y\} ,$$

lies in $C^*(X)$.

Proof. To check the continuity of F at $x_0 \in X$, let $\varepsilon > 0$ and first find $y_0 \in Y$ such that $f(x_0, y_0) > F(x_0) - \varepsilon$. There is a neighborhood $U \times V$ of (x_0, y_0) throughout which $f > F(x_0) - \varepsilon$, and for $x \in V$ we have $F(x) \geqq f(x, y_0) > F(x_0) - \varepsilon$.

To find a neighborhood U' of x_0 throughout which $F \leqq F(x_0) + \varepsilon$, first select for each $y \in Y$ a neighborhood $U_y \times V_y$ of (x_0, y) throughout which $f < F(x_0) + \varepsilon/2$. Because Y is weakly Lindelöf there is a sequence $\{y_k\}_{k=1}^{\infty}$ in Y with $\bigcup_k V_{y_k}$ dense in Y. With $U' = \bigcap_k U_{y_k}$ we check easily that U' is a neighborhood of x_0 for which $F(x) \leqq F(x_0) + \varepsilon$ whenever $x \in U'$.

COROLLARY 2.2. *Let X be a P-space and Y a weakly Lindelöf space, and let π denote the projection from $X \times Y$ onto X. Then for each cozero-set A in $X \times Y$, the set πA is open-and-closed in X.*

Proof. If $A = \operatorname{coz} f$ with $f \in C^*(X \times Y)$ and $f \geqq 0$, then πA is the cozero-set of the function F defined as in 2.1, hence is closed (since X is a P-space).

The following lemma asserts, in effect, that for suitably restricted spaces X and Y, the closure in $X \times Y$ of each cozero-set may be computed by taking closures of vertical slices. When $A \subset X \times Y$ we denote $\operatorname{cl}_{X \times Y} A$ by the symbol $\bar{A}$, and $A \cap (\{x\} \times Y)$ by A_x.

LEMMA 2.3. *Let X be a P-space and let Y be locally weakly Lindelöf at each of its non-P-points. Then* $\bar{A} = \bigcup_{x \in X} \overline{A_x}$ *for each cozero-set A in $X \times Y$.*

Proof. The inclusion $\supset$ is obvious, so we choose $(x, y) \in \bar{A}$. We must show that $\{x\} \times V$ meets A_x for each neighborhood V in Y of y. If y is a P-point of Y then (x, y) is a P-point of $X \times Y$, so that indeed

$$(x, y) \in (\{x\} \times V) \cap A_x .$$

If y is not a P-point of Y and V_0 is a weakly Lindelöf neighborhood of y in Y with $V_0 \subset V$, then $(X \times V_0) \cap A$ is a cozero-set in $X \times V_0$ and 2.2 applies to yield: $\pi[(X \times V_0) \cap A]$ is open-and-closed in X. Since $(x, y) \in \mathrm{cl}_{X \times V_0}[(X \times V_0) \cap A]$, we have

$$\begin{aligned} x = \pi(x, y) \in \pi\, \mathrm{cl}_{X \times V_0}[(X \times V_0) \cap A] &\subset \mathrm{cl}_X \pi[(X \times V_0) \cap A] \\ = \pi[(X \times V_0) \cap A] , \end{aligned}$$

so that $(\{x\} \times V) \cap A_x \supset (\{x\} \times V_0) \cap A_x \neq \varnothing$ as desired.

The elementary argument just given yields the following result, which we shall improve upon in 3.2.

THEOREM 2.4. *Let Y be an F'-space which is locally weakly Lindelöf at each of its non-P-points. Then $X \times Y$ is an F'-space for each P-space X.*

Proof. If A and B are disjoint cozero-sets in $X \times Y$, then from 2.3 we have

$$\bar{A} \cap \bar{B} = (\textstyle\bigcup_{x \in X} \overline{A_x}) \cap (\bigcup_{x \in X} \overline{B_x}) = \bigcup_{x \in X} (\overline{A_x} \cap \overline{B_x}) = \bigcup_{x \in X} \varnothing = \varnothing .$$

The theorem just given furnishes a proof for 2.5(b) below, announced earlier in [2]. (In a letter of December 27, 1966, Professor Curtis has asserted his agreement with the authors' beliefs that (a) the argument given in [2] contains a gap and (b) this error does not in any way affect the other interesting results of [2].)

COROLLARY 2.5. *Let X be a P-space and let Y be an F'-space such that either*

(a) *Y is locally Lindelöf; or*

(b) *Y is locally separable.*

Then $X \times Y$ is an F'-space.

Note added September 16, 1968. The reader may have observed already a fact noticed only lately by the authors: Each F'-space in which each open subset is weakly Lindelöf is extremally disconnected (in the sense that disjoint open subsets have disjoint closures). [For the proof, let U and V be disjoint open sets in such a space Y, suppose that $p \in \mathrm{cl}\, U \cap \mathrm{cl}\, V$, and for each point y in U find a cozero-set U_y of Y with $y \in U_y \subset U$. The cover $\{U_y : y \in U\}$ admits a countable subfamily $\mathscr{U}$ whose union is dense in U. If $\mathscr{V}$ is constructed similarly for V, then $\cup \mathscr{U}$ and $\cup \mathscr{V}$ are disjoint cozero-sets in X whose closures contain p.] It follows that each separable F'-space, and hence each locally separable F'-space, is extremally disconnected, and hence basically disconnected. Thus the conclusion to Corollary 2.5(b) is unnecessarily weak. In view of 3.4 we have in fact: If X is a P-space and Y is a locally separable F'-space, then $X \times Y$ is basically disconnected.

3. When the product of spaces is F'.

It is clear that for each collection $\{\mathscr{W}_\alpha\}_{\alpha \in A}$ of open covers of a locally weakly Lindelöf space Y and for each y in Y one can find a neighborhood U of y and for each α a countable subfamily $\mathscr{V}_\alpha$ of $\mathscr{W}_\alpha$ such that $U \subset \mathrm{cl}_Y(\cup \mathscr{V}_\alpha)$. (Indeed, the neighborhood U may be chosen independent of the collection $\{\mathscr{W}_\alpha\}_{\alpha \in A}$.)

When, in contrast to this strong condition, such a neighborhood U is hypothesized to exist for each countable collection of covers of Y, we shall say that Y is countably locally weakly Lindelöf (abbreviation: CLWL). The formal definition reads as follows:

DEFINITION 3.1. The space Y is CLWL if for each countable collection $\{\mathscr{W}_n\}$ of open covers of Y and for each y in Y there exist a neighborhood U of y and (for each n) a countable subfamily $\mathscr{V}_n$ of $\mathscr{W}_n$ with $U \subset \mathrm{cl}_Y(\cup \mathscr{V}_n)$.

A crucial property of CLWL spaces is disclosed by the following lemma, upon which the results of this section depend.

For f in $C(X \times Y)$, we denote by f_x that (continuous) function on Y defined by the rule $f_x(y) = f(x, y)$.

LEMMA 3.2. *Let $f \in C^*(X \times Y)$, where X is a P-space and Y is* CLWL. *If $(x_0, y_0) \in X \times Y$, then there is a neighborhood $U \times V$ of (x_0, y_0) such that $f_x \equiv f_{x_0}$ on V whenever $x \in U$.*

Proof. For each y in Y and each positive integer n there is a neighborhood $U_n(y) \times V_n(y)$ of (x_0, y) for which

$$|f(x', y') - f(x_0, y)| < 1/n \quad \text{whenever} \quad (x', y') \in U_n(y) \times V_n(y) .$$

Since for each n the family $\{V_n(y) : y \in Y)$ is an open cover of Y, there exist a neighborhood V of y_0 and (for each n) a countable subset Y_n of Y for which $V \subset \mathrm{cl}_Y(\cup\{V_n(y) : y \in Y_n\})$.

We define the neighborhood U of x_0 by the rule

$$U = \bigcap_n (\cap \{U_n(y) : y \in Y_n\}) .$$

To check that neighborhood $U \times V$ of (x_0, y_0) is as desired, suppose that there is a point (x', y') in $U \times V$ with $f(x', y') \neq f(x_0, y')$. Choosing an integer n and a neighborhood $U' \times V'$ of (x', y') such that $|f(x, y) - f(x_0, y')| > 1/n$ whenever $(x, y) \in U' \times V'$, we see that since $y' \in V \subset \mathrm{cl}_Y(\cup \{V_{3n}(y) : y \in Y_{3n}\})$ and $V' \cap V_{3n}(y')$ is a neighborhood of y' there exist points $\bar{y}$ in Y_{3n} and $\bar{\bar{y}}$ in $[V' \cap V_{3n}(y')] \cap V_{3n}(\bar{y})$.

Since $(x', \bar{\bar{y}}) \in U' \times V'$, we have

$$|f(x', \bar{\bar{y}}) - f(x_0, y')| > 1/n .$$

But since $(x', \bar{\bar{y}}) \in U \times V_{3n}(\bar{y}) \subset U_{3n}(\bar{y}) \times V_{3n}(\bar{y})$, and $(x_0, \bar{\bar{y}}) \in U_{3n}(\bar{y}) \times V_{3n}(\bar{y})$, and $(x_0, \bar{\bar{y}}) \in U_{3n}(y') \times V_{3n}(y')$, we have

$$\begin{aligned} |f(x', \bar{\bar{y}}) - f(x_0, y')| &\leqq |f(x', \bar{\bar{y}}) - f(x_0, \bar{y})| \\ &\quad + |f(x_0, \bar{y}) - f(x_0, \bar{\bar{y}})| + |f(x_0, \bar{\bar{y}}) - f(x_0, y')| \\ &< 1/3n + 1/3n + 1/3n = 1/n . \end{aligned}$$

We have seen in § 2 that if the product space $X \times Y$ is an F'-space then both X and Y are F'-spaces and either X or Y is a P-space. It is clear that every discrete space is a P-space, and that the product of any F'-space with a discrete space is an F'-space; the example given by Gillman in [4], however, shows that the product of a P-space with an F'-space may fail to be an F'-space. Thus it appears natural to ask the question: Which F'-spaces have the property that their product with each P-space is an F'-space? We now answer this question.

THEOREM 3.3. *In order that $X \times Y$ be an F'-space for each P-space X, it is necessary and sufficient that Y be an F'-space which is* CLWL.

Proof. Sufficiency. Let $f \in C^*(X \times Y)$, and let $(x_0, y_0) \in X \times Y$. We may suppose without loss of generality that there is a neighborhood V' of y_0 in Y for which

$$V' \cap \operatorname{pos} f_{x_0} = \varnothing .$$

But then, choosing $U \times V$ as in Lemma 3.2, we see that

$$U \times (V \cap V') \cap \operatorname{pos} f = \varnothing ,$$

so that $(x_0, y_0) \notin \operatorname{cl} \operatorname{pos} f$.

Necessity. (A preliminary version of the construction below—in the context of weakly Lindelöf spaces, not of CLWL spaces—was communicated to us by Anthony W. Hager in connection with a project not closely related to that of the present paper. We appreciate professor Hager's helpful letter, which itself profited from his collaboration with S. Mrowka.)

We have already seen that Y must be an F'-space. If Y is not CLWL then there are a sequence $\{\mathscr{W}_n\}$ of open covers of Y and a point y_0 in Y with the property that for each neighborhood U of y_0 there is an integer $n(U)$ for which the relation

$$U \subset \mathrm{cl}_Y(\cup \mathscr{V})$$

fails for each countable subfamily $\mathscr{V}$ of $\mathscr{W}_{n(U)}$.

Let $\mathscr{U}$ denote the collection of neighborhoods of y_0. With each $U \in \mathscr{U}$ we associate the family $\Sigma(U)$ of countable intersections of sets of the form $Y \backslash W$ with $W \in \mathscr{W}_{n(U)}$, and we write

$$\tau(U) = \{(A, U) : A \in \Sigma(U)\} .$$

From the definition of $n(U)$ it follows that $(\mathrm{int}_Y A) \cap U \neq \varnothing$ whenever $A \in \Sigma(U)$. The space X is the set $\{\infty\} \cup \bigcup_{U \in \mathscr{U}} \tau(U)$, topologized as follows: Each of the points (A, U), for $A \in \Sigma(U)$, constitutes an open set, so that X is discrete at each of its points except for ∞; and a set containing the point ∞ is a neighborhood of ∞ if and only if it contains, for each $U \in \mathscr{U}$, some point $(A, U) \in \tau(U)$ and each point of the form (B, U) with $B \subset A$ and $(B, U) \in \tau(U)$. Since $\bigcap_{k=1}^{\infty} A_k \in \Sigma(U)$ whenever each $A_k \in \Sigma(U)$, it follows that each countable intersection of neighborhoods of ∞ is a neighborhood of ∞, so that X is a P-space. Like every Hausdorff space with a basis of open-and-closed sets, X is completely regular. It remains to show that $X \times Y$ is not an F'-space.

Since for $U \in \mathscr{U}$ there is no countable subfamily $\mathscr{V}$ of $\mathscr{W}_{n(U)}$ for which $U \subset \mathrm{cl}_Y(\cup \mathscr{V})$, the set $(\mathrm{int}_Y A) \cap U$ is uncountable whenever $U \in \mathscr{U}$ and $A \in \Sigma(U)$. Thus whenever $(A, U) \in \tau(U)$ we choose distinct points $p_{(A,U)}$ and $q_{(A,U)}$ in $(\mathrm{int}_Y A) \cap U$ and disjoint neighborhoods $F_{(A,U)}$ and $G_{(A,U)}$ of $p_{(A,U)}$ and $q_{(A,U)}$ respectively, with $F_{(A,U)} \cup G_{(A,U)} \subset (\mathrm{int}_Y A) \cap U$. Because Y is completely regular there exist continuous functions $f_{(A,U)}$ and $g_{(A,U)}$ mapping Y into $[0, 1]$ such that

$$f_{(A,U)}(p_{(A,U)}) = 1 , \qquad f_{(A,U)} \equiv 0 \text{ off } F_{(A,U)} ,$$
$$g_{(A,U)}(q_{(A,U)}) = 1 , \qquad g_{(A,U)} \equiv 0 \text{ off } G_{(A,U)} .$$

Now for each positive integer k we define functions f_k and g_k on $X \times Y$ by the rules $f_k(x, y) = g_k(x, y) = 0$ if $x = \infty$ or if $x = (A, U)$ with $k \neq n(U)$; $f_k((A, U), y) = f_{(A,U)}(y)$ if $k = n(U)$; $g_k((A, U), y) = g_{(A,U)}(y)$ if $k = n(U)$. Each function f_k is continuous at each point $((A, U), y) = (x, y) \in X \times Y$ (with $x \neq \infty$), since f_k agrees either with the function 0 or with the continuous function $f_{(A,U)} \circ \pi_Y$ on the open subset $\{(A, U)\} \times Y$ of $X \times Y$. Similarly, each function g_k is continuous at each point $(x, y) \in X \times Y$ with $x \neq \infty$. To check the continuity (of f_k, say) at the point $(\infty, y) \in X \times Y$, find $W \in \mathscr{W}_k$ for which $y \in W$ and write

$$V = \{\infty\} \cup \bigcup_{k \neq n(U)} \tau(U) \cup \bigcup_{k=n(U)} \{(B, U) : B \subset Y \backslash W\} .$$

Then $V \times W$ is a neighborhood of (∞, y) on which f_k is identically 0: For if $(A, U) \in \tau(U)$ with $k \neq n(U)$ we have $f_k((A, U), y) = 0$, and if $A \in \Sigma(U)$ with $A \subset Y \backslash W$ and $k = n(U)$, then (since $y \in W \subset Y \backslash \mathrm{int}_Y A \subset Y \backslash F_{(A, U)}$) we have

$$f_k((A, U), y) = f_{(A, U)}(y) = 0 .$$

We notice next that if k and m are positive integers then $\mathrm{coz}\, f_k \cap \mathrm{coz}\, g_m = \emptyset$: Indeed, if $f_k((A, U), y) \neq 0$ and $g_m((A, U), y) \neq 0$, then $k = n(U)$ and $m = n(U)$, so that $y \in F_{(A, U)} \cap G_{(A, U)}$, a contradiction. Thus, defining

$$f = \sum_{k=1}^{\infty} f_k/2^k \quad \text{and} \quad g = \sum_{k=1}^{\infty} g_k/2^k$$

we have $f \in C^*(X \times Y)$ and $g \in C^*(X \times Y)$ and $\mathrm{coz}\, f \cap \mathrm{coz}\, g = \emptyset$. Nevertheless for each neighborhood $V \times U_0$ of (∞, y_0) we have $(A_0, U_0) \in V$ for some $A_0 \in \Sigma(U_0)$, so that

$$\begin{aligned} f((A_0, U_0), p_{(A_0, U_0)}) &\geqq f_{n(U_0)}((A_0, U_0), p_{(A_0, U_0)})/2^{n(U_0)} \\ &= f_{(A_0, U_0)}(p_{(A_0, U_0)})/2^{n(U_0)} \\ &= 1/2^{n(U_0)} > 0 \end{aligned}$$

and $(V \times U_0) \cap \mathrm{coz}\, f \neq \emptyset$. Likewise $(V \times U_0) \cap \mathrm{coz}\, g \neq \emptyset$, and it follows that $(\infty, y_0) \in \mathrm{cl}\, \mathrm{coz}\, f \cap \mathrm{cl}\, \mathrm{coz}\, g$. Thus $X \times Y$ is not an F'-space.

The proof of Theorem 3.3 being now complete, we turn to the corollary which we believe responds adequately to Gillman's request in [4] for a theorem characterizing those pairs of spaces (X, Y) for which $X \times Y$ is basically disconnected.

COROLLARY 3.4. *In order that $X \times Y$ be basically disconnected for each P-space X, it is necessary and sufficient that Y be a basically disconnected space which is* CLWL.

Proof. Sufficiency. Let $(x_0, y_0) \in \mathrm{cl}\, \mathrm{coz}\, f$, where $f \in C^*(X \times Y)$, and let V' be a neighborhood of y_0 in Y for which $V' \subset \mathrm{cl}\, \mathrm{coz}\, f_{x_0}$. Choosing $U \times V$ as in Lemma 3.2, we see that $U \times (V \cap V')$ is a neighborhood in $X \times Y$ of (x_0, y_0) for which

$$U \times (V \cap V') \subset \mathrm{cl}\, \mathrm{coz}\, f .$$

Necessity. That Y must be basically disconnected is clear. That Y must be CLWL follows from 3.3 and the fact that each basically disconnected space is an F'-space.

4. **Some examples and questions.** If the point x of the topological space X admits a neighborhood (X itself, say) which is an F-space, then each neighborhood U of x in X contains a neighborhood V which is an F-space: Indeed, if $f \in C(X)$ with $x \in \operatorname{coz} f \subset U$ and we set $V = \operatorname{coz} f$, then each pair (A, B) of disjoint cozero-sets of V is a pair of disjoint cozero-sets in X, which accordingly may be completely separated in X, hence in V.

The paragraph above shows that any point with a neighborhood which is an F-space admits a fundamental system of F-space neighborhoods. The statement with "F" replaced throughout by "F'" follows from the implication (b) $\Rightarrow$ (d) of Theorem 4.2 below. The following definitions are natural.

DEFINITION 4.1. The space X is locally F (resp. locally F') at the point $x \in X$ if x admits a neighborhood in X which is an F-space (resp. an F'-space).

Clearly each F-space is locally F, and each locally F space is locally F'. Gillman and Henriksen produce in 8.14 of [5] an F'-space which is not an F-space, and their space is easily checked to be locally F. In the same spirit we shall present in 4.3 an F'-space which is not locally F. We want first to make precise the assertion that the F' property, unlike the F property, is a local property.

THEOREM 4.2. *For each space X, the following properties are equivalent*:

(a) *X is an F'-space*;
(b) *X is locally F'*;
(c) *each cozero-set in X is an F'-space*;
(d) *each open subset of X is an F'-space.*

Proof. That (a) $\Rightarrow$ (b) is clear. To see that (b) $\Rightarrow$ (c), let U be a cozero-set in X and let A and B be disjoint (relative) cozero subsets of U. Then A and B are disjoint cozero subsets of X. Suppose $p \in \mathrm{cl}_U A \cap \mathrm{cl}_U B$. Then, if V is the hypothesized F'-space neighborhood of p, we have $p \in \mathrm{cl}_V(A \cap V) \cap \mathrm{cl}_V(B \cap V)$. This contradicts the fact that V is an F'-space.

If (c) holds and A and B are disjoint (relative) cozero-sets of an open subset U of X, then for any point p in $\mathrm{cl}_U A \cap \mathrm{cl}_U B$ there exists a cozero-set V in X for which $p \in V \subset U$. It follows that

$$p \in \mathrm{cl}_V(A \cap V) \cap \mathrm{cl}_V(B \cap V) ,$$

contradicting the fact that V is an F'-space. This contradiction shows that (d) holds.

The implication (d) $\Rightarrow$ (a) is trivial.

EXAMPLE 4.3. *An F'-space not locally F.* Let X be any F'-space which is not an F-space, let D be the discrete space with $|D| = \aleph_1$, and let $Y = (X \times D) \cup \{\infty\}$, where ∞ is any point not in $X \times D$ and Y is topologized as follows: A subset of $X \times D$ is open in Y if it is open in the usual product topology on $X \times D$, and ∞ has an open neighborhood basis consisting of all sets of the form $\{\infty\} \cup (X \times E)$ with $|D \backslash E| \leqq \aleph_0$. Then ∞ admits no neighborhood which is an F-space, since each neighborhood of ∞ contains (for some $d \in D$) the set $X \times \{d\}$, which is homeomorphic to X itself, as an open-and-closed subset. Yet Y is an F'-space since ∞ is a P-point of Y and each other point of Y belongs to an F'-space, $X \times D$, which is dense in Y.

We have observed already that a Lindelöf F'-space, being normal, is an F-space. We show next that the Lindelöf condition cannot be replaced by the locally Lindelöf property.

EXAMPLE 4.4. *A locally Lindelöf F'-space which is not F.* The space $X = L' \times L \backslash \{(\omega_2, \omega_1)\} \cup \bigcup_{\alpha < \omega_1} D_\alpha$ defined in 8.14 of [5] does not fill the bill here because the space L' of ordinals $\leqq \omega_2$ (with each $\gamma < \omega_2$ isolated and with neighborhoods of ω_2 as in the order topology) is not Lindelöf. When the space is modified by the replacement of L' by $\beta L'$, the resulting space (X' say) fails to be an F-space just as in [5]. Yet L' is a P-space, so that $\beta L'$ is a compact F-space, and therefore (by Theorem 3.3 above, or by Theorem 6.1 of [9]) $\beta L' \times L$ is a Lindelöf F'-space. Thus X' is a locally Lindelöf space which is locally F', hence is a locally Lindelöf F'-space.

The condition that a space be locally weakly Lindelöf at each of its non-P-points is more easily worked with then the condition that it be CLWL. A converse to Theorem 2.4 would, therefore, be a welcome replacement for the "necessity" part of Theorem 3.3. The following example shows that the converse to Theorem 2.4 is invalid.

EXAMPLE 4.5. *A CLWL F'-space with a non-P-point at which it is not locally weakly Lindelöf.* Let Y be the space $D \times D \cup \{\infty\}$ with D the discrete space for which $|D| = \aleph_1$ and (after the fashion of 8.5 of [5]) adjoin to Y a copy of the integers N so that ∞ becomes a point in $\beta N \backslash N$. The resulting space $Y' = Y \cup N$ is topologized so that each point $y \neq \infty$ contitutes by itself an open set, while a set containing ∞ is a neighborhood of ∞ if it contains both a set drawn from the ultrafilter on N corresponding to ∞ and a set of the form $D \times E$ with $|D \backslash E| \leqq \aleph_0$. Then ∞ is not a P-point of Y', since the function whose value at the integer $n \in N \subset Y'$ is $1/n$ and whose value at each other point of Y' is 0 is constant on no neighborhood of ∞; and Y' is not locally weakly Lindelöf at ∞ since each neighborhood of ∞ contains as an open-and-closed subset a homeomorph of the uncountable discrete space D. The only nonisolated point of Y', ∞, can belong to a set of the form $(\mathrm{cl}\operatorname{coz} f)\backslash \operatorname{coz} f$ only when $\infty \in \mathrm{cl}(\operatorname{coz} f \cap N)$, so that Y' is an F'-space. If, finally, $\mathscr{W}_n$ is a sequence of open covers of Y' and a neighborhood U of ∞ in Y is chosen so that for each n we have $U \subset W_n$ for some $W_n \in \mathscr{W}_n$ (as is possible, since Y is a P-space), then evidently $U \cup N$ is a neighborhood of ∞ in Y' contained in $\mathrm{cl}_{Y'}(\cup \mathscr{V}_n)$ for a suitable countable subfamily $\mathscr{V}_n$ of $\mathscr{W}_n$. Thus Y' is CLWL.

Theorem 1.8 does not provide an answer to the following problem, which we have been unable to solve.

QUESTION 4.6. Is each weakly Lindelöf F'-space an F-space?

On the basis of Theorem 3.3 and Corollary 3.4 and the fact that the class of F-spaces is nestled properly between the classes of F'- and of basically disconnected spaces, one wonders whether the obvious F-space analogue of 3.3 and 3.4 is true. We have not been able to settle this question, though one of us hopes to pursue it in a later communication. We close with a formal statement of this question, and of a related problem.

QUESTION 4.7. In order that $X \times Y$ be an F-space for each P-space X, is it sufficient that Y be an F-space which is CLWL?

QUESTION 4.8. Do there exist a P-space X and an F-space Y such that $X \times Y$ is an F'-space but not an F-space?

This paper has benefited, both mathematically and grammatically, from an unusually thoughtful and instructive referee's report. It is a pleasure to thank the referee for his helpful criticism.

REFERENCES

1. W. W. Comfort and Stelios Negrepontis, *Extending continuous functions on $X \times Y$ to subsets of $\beta X \times \beta Y$*, Fund. Math. **59** (1966), 1-12.
2. Philip C. Curtis, Jr., *A note concerning certain product spaces*, Archiv der Math. **11** (1960), 50-52.
3. Zdenek Frolík, *Generalizations of compact and Lindelöf spaces*, Czech. Math. J. (84) **9** (1959), 172-217 (Russian; English Summary).

4. Leonard Gillman, *A P-space and an extremally disconnected space whose product is not an F-space*, Archiv der Math. **11** (1960), 53-55.
5. Leonard Gillman and Melvin Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), 366-391.
6. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, D. Van Nostrand Company, Inc., Princeton, N. J. 1960.
7. Neil Hindman, *On the product of F′-spaces and of basically disconnected spaces*, Notices Amer. Math. Soc. **14** (1967), 825; (abstract).
8. Carl W. Kohls, *Hereditary properties of some special spaces*, Archiv der Math. **12** (1961), 129-133.
9. S. Negrepontis, *On the product of F-spaces*, Trans. Amer. Math. Soc. **135** (1969), 443-457.

Received December 12, 1967, and in revised form March 20, 1968. The first two authors gratefully acknowledge support received from the National Science Foundation under grant NSF-GP-8357 (and, earlier, under grant NSF-GP-7056, administered by the University of Massachusetts when these authors were associated with that institution). The third author gratefully acknowledges support received from the Canadian National Research Council under grant A-4035 and from the Summer (1967) Research Institute of the Canadian Mathematical Congress at Kingston, Ontario.

WESLEYAN UNIVERSITY
MIDDLETOWN, CONNECTICUT
MCGILL UNIVERSITY
MONTREAL, QUEBEC