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A CLASSIFICATION OF CENTERS

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The purpose of this paper is to classify centers according to isomorphisms. We define three types of isomorphism, and for two of these types give necessary and sufficient conditions for two centers to be isomorphic. We also give necessary and sufficient conditions for the third type of isomorphism to be equivalent to one of the other two.

These isomorphisms are discussed in a more general situation by Taro Ura [7]. This paper was motivated by discussions with Taro Ura and Otomar Hájek.

In our investigation we construct a section which generates a neighborhood of the center by using a theorem from the theory of fibre bundles. This section may be constructed directly, using the existence of a transversal through each noncritical point of the dynamical system. Much insight, which is otherwise lost, into the structure of a center is obtained from the fibre bundle approach.

The concept of a transversal is essential in our investigation. The basic material on transversal theory in planar dynamical systems is found in [3].

Throughout this paper R^+, R^1, and R^2 will denote the nonnegative reals, the reals, and the plane respectively.

Let (X, π) be a dynamical system on X, i.e., X is a topological space and π is a mapping of $X \times R^1$ onto X satisfying the following axioms (where $x\pi t = \pi(x, t)$ for $(x, t) \in X \times R^1$):

(1) Identity Axiom: $x\pi 0 = x$ for $x \in X$

(2) Homomorphism Axiom: $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, s \in R^1$

(3) Continuity Axiom: π is continuous on $X \times R^1$.

Then for $x \in X$, $x\pi R^1$ is called the trajectory through x and is denoted by $C(x)$. If $C(x) = \{x\}$, x is called a critical point. If there exists $t \in R^1$, $t \neq 0$, such that $x\pi t = x$, $C(x)$ is called periodic. If $C(x)$ is periodic and x is not a critical point, $C(x)$, is called a cycle.

1. Definition and properties of a center.

In the following (R^2, π) will denote a dynamical system on R^2 and P the set of noncritical periodic points of (R^2, π). Let $T: P \rightarrow R^1$ be the mapping which associates with each point $x \in P$ its fundamental period $T(x)$. For the proof of the following result see [3, VII, 4.15].

PROPOSITION 1.1. *T is a continuous mapping of P into R^1.*

DEFINITION 1.2. A critical point p of (R^2, π) is called a center if and only if there exists a neighborhood U of p such that $C(x)$ is a cycle for every $x \in U - \{p\}$.

PROPOSITION 1.3. *Let p be a center in (R^2, π). Then $\{p\}$ is both positively and negatively stable.*

Proof. Let U be a neighborhood of p as described in Definition 1.2. We will show that $D^+(p) = \{p\}$, where $D^+(p)$ denotes the positive prolongation of p (see [1, 1.4.1]). This will prove (by [1, 2.6.6]) that $\{p\}$ is positively stable. Let M be the component of $D^+(p)$ which contains p. By [1, 2.3.5], if $D^+(p)$ is compact, then it has exactly one component and if $D^+(p)$ is not compact, then none of its components is compact. We now have two cases:

$$M \cap (U - \{p\}) = \varnothing \quad \text{or} \quad M \cap (U - \{p\}) \neq \varnothing \ .$$

If $M \cap (U - \{p\}) = \varnothing$, then $M = \{p\}$ and $D^+(p) = M = \{p\}$. If

$$M \cap (U - \{p\}) \neq \varnothing \ , \quad \text{let} \quad y \in M \cap (U - \{p\}) \ .$$

Then there exist sequences $\{x_i\}_{i=1}^{\infty}$ in $U - \{p\}$ and $\{t_i\}_{i=1}^{\infty}$ in R^+, with $x_i \to p$ and $x_i \pi t_i \to y$. Since $x_i \in P$ for every i, we may assume $t_i \in [0, T(x_i))$. Since $C(y)$ is a cycle, $t_i \in [0, 2T(y))$ for all i sufficiently large by the continuity of T. Let $\{t_{i_n}\}_{n=1}^{\infty}$ be a convergent subsequence of $\{t_i\}_{i=1}^{\infty}$ with limit t_0. Then

$$y \leftarrow x_{i_n} \pi t_{i_n} \to p \pi t_0 = p \ .$$

This contradicts our assumption that $y \in M - \{p\}$. Thus $D_p^+ = \{p\}$ and $\{p\}$ is positively stable. Similarly $\{p\}$ is negatively stable.

DEFINITION 1.4. A cycle $C(x)$ of (R^2, π) decomposes R^2 into two components, one bounded and the other unbounded. int $C(x)$ and ext $C(x)$ will denote the bounded and unbounded components, respectively, of $R^2 - C(x)$.

PROPOSITION 1.5. *Let $C(x)$ be a cycle in (R^2, π). Then* $\operatorname{int} C(x)$ *and* $\operatorname{ext} C(x)$ *are invariant.*

Proof. The components of an invariant set are invariant.

In [3, VII, 4.8] it is proved that

PROPOSITION 1.6. *If $C(x)$ is a cycle in (R^2, π), then* $\operatorname{int} C(x)$ *contains a critical point.*

PROPOSITION 1.7. *Let p be a center in (R^2, π) and U be a neighborhood as described in Definition* 1.2. *Then there exists $x \in U$ such that*

(i) $\operatorname{int} C(x) \subset U$,
(ii) $p \in \operatorname{int} C(x)$, and
(iii) $p \in \operatorname{int} C(y)$ *for every* $y \in \operatorname{int} C(x) - \{p\}$.

Proof. Let V be a disc neighborhood of p contained in U. Since p is positively stable there exists $x \in V - \{p\}$ such that $(C(x) =) C^+(x) \subset V$. Then $\operatorname{int} C(x) \subset V$ because V is simply connected. $\operatorname{int} C(x)$ contains a critical point by Proposition 1.6. This critical point must be p because p is the unique critical point in U. Similarly, $p \in \operatorname{int} C(y)$ for every $y \in \operatorname{int} C(x) - \{p\}$.

Thus we may reformulate Definition 1.2 as:

DEFINITION 1.2′. A critical point p of (R^2, π) is called a center if and only if there exists a cycle $C(x)$ such that $p \in \operatorname{int} C(x)$ and $\operatorname{int} C(x) - \{p\}$ consists of cycles. We choose a fixed $C(x_0)$ satisfying this condition and henceforth denote $\operatorname{int} C(x_0)$ by U.

PROPOSITION 1.8. *If $x \in U$, then $C(x)$ is both positively and negatively stable. Also $C(x_0)$ is stable relative to $\bar{U}$.*

Proof. See [3, VIII, 3.3].

PROPOSITION 1.9. *Let S be a transversal contained in U. Then $C(x) \cap S = \{x\}$ for every $x \in S$.*

Proof. Since S is a transversal and p is critical, $p \notin S$; thus $x \in S \subset U - \{p\}$ implies $C(x)$ is a cycle. A cycle intersects a transversal at a unique point, [3, VII, 4.4].

PROPOSITION 1.10. *Let p be a center and U be a neighborhood of p as described in Definition* 1.2′. *If C_1 and C_2 are distinct cycles in U, then $C_1 \subset \operatorname{int} C_2$ or $C_2 \subset \operatorname{int} C_1$.*

Proof. By Proposition 1.7 $p \in \operatorname{int} C_1$ and $p \in \operatorname{int} C_2$. Thus

$$\operatorname{int} C_1 \cap \operatorname{int} C_2 \neq \varnothing .$$

Thus $\operatorname{int} C_1 \subset \operatorname{int} C_2$ or $\operatorname{int} C_1 \cap \operatorname{ext} C_2 \neq \varnothing$. In the first case $\overline{\operatorname{int} C_1} \subset \overline{\operatorname{int} C_2}$. Therefore $C_1 \subset \operatorname{int} C_2$ or $C_1 \cap C_2 \neq \varnothing$. The latter is impossible because C_1 and C_2 are distinct trajectories. In the second case, $\partial(\operatorname{int} C_2) \cap \operatorname{int} C_1 \neq \varnothing$. Therefore $C_2 \cap \operatorname{int} C_1 \neq \varnothing$ and $C_2 \subset \operatorname{int} C_1$ since int C_1 is invariant.

COROLLARY 1.11. *If C_1 and C_2 are distinct cycles in U such that $C_1 \subset \operatorname{ext} C_2$, then $C_2 \subset \operatorname{int} C_1$.*

Proof. By Proposition 1.10, $C_2 \subset \operatorname{int} C_1$ or $C_1 \subset \operatorname{int} C_2$. C_1 cannot be contained in both $\operatorname{int} C_2$ and $\operatorname{ext} C_2$. Therefore $C_2 \subset \operatorname{int} C_1$.

2. Bundles and cross-sections.

DEFINITION 2.1. Let (R^2, π) be a dynamical system on R^2 and let $x, y \in R^2$. We define a relation $\sim$ on R^2 by letting $x \sim y$ if and only if $x \in C(y)$.

Evidently $\sim$ is an equivalence relation. The topology on $R^2/\sim$ will be the quotient topology.

PROPOSITION 2.2. *Let e be the natural mapping of R^2 onto $R^2/\sim$. Then e is an open mapping.*

Proof. e is open if and only if $e^{-1}eG$ is open for every open set $G \subset R^2$. Now, $e^{-1}eG = G\pi R^1 = \bigcup_{t \in R^1} G\pi t$, and $G\pi t$ is open for every $t \in R^1$ since $\pi_t \colon R^2 \approx R^2$. Hence $G\pi R^1$ is open and e is an open mapping.

PROPOSITION 2.3. *If V is an invariant subset of R^2, then $e(V)$ is homeomorphic to $V/(\sim \cap V \times V)$.*

Proof. Since e is an open mapping, the result follows from § I, 3.5 of [2].

We shall now write $e(V)$ as $V/\sim$ where it is understood that $\sim$ is restricted to $V \times V$.

PROPOSITION 2.4. *$e \mid U$ is a closed mapping of U onto $U/\sim$.*

Proof. $e \mid U$ is closed if and only if $e^{-1}eF = F\pi R^1$ is closed in U for every set F which is closed in U. Let $x \in \overline{F\pi R^1} \cap U$. Then there exist sequences $\{x_i\}_{i=1}^{\infty}$ in F and $\{t_i\}_{i=1}^{\infty}$ in R^1 such that $x_i\pi t_i \rightarrow x$. Thus $C(x_i) \rightarrow C(x)$. Let $y \in U - \operatorname{int} C(x)$. Then $C(x) \subset \operatorname{int} C(y)$ by Corollary 1.11 and $\overline{\operatorname{int} C(y)}$ is a compact neighborhood of $C(x)$. Thus $x_i \in \overline{\operatorname{int} C(y)}$ for i sufficiently large. Let $\{x_{i_n}\}_{n=1}^{\infty}$ be a convergent subsequence of $\{x_i\}_{i=1}^{\infty}$ with limit z. Then $z \in F \cap C(x)$ since F is closed and $C(x_i) \rightarrow C(x)$. Thus $x \in C(z) \subset F\pi R^1$ and $F\pi R^1$ is closed.

The following material on bundles is to be found in [6].

DEFINITION 2.5. A bundle β is a collection as follows:

(1) A space B called the *bundle space*,

(2) a space X called the *base space*,

(3) a map $p: B \to X$ of B onto X called the *projection*,

(4) a space Y called the *fibre*,

(5) an effective topological transformation group G of Y (i.e., $g \circ y = y$ for all $y \in G$ implies g is the identity) called the *group of the bundle*,

(6) a family $\{V_j\}$ of open sets covering X indexed by a set J, the V_j's are called the *coordinate neighborhoods*, and

(7) for each j in J, a homeomorphism

$$\varphi_j: V_j \times Y \to p^{-1}(V_j)$$

called the coordinate function.

The coordinate functions are required to satisfy the following conditions:

(8) $p\varphi_j(x, y) = x$ for $x \in V_j$, $y \in Y$

(9) if the map $\varphi_{j,x}: Y \to p^{-1}(x)$ is defined by setting $\varphi_{j,x}(y) = \varphi_j(x, y)$ then for each pair i, j in J, and each $x \in V_i \cap V_j$, the homeomorphism $\varphi_{j,x}^{-1}\varphi_{i,x}: Y \to Y$ coincides with the operation of an element of G and

(10) for each pair i, j in J, the map

$$g_{ji}: V_j \cap V_i \to G$$

defined by $g_{ji}(x) = \varphi_{j,x}^{-1}\varphi_{i,x}$ is continuous.

Let $U - \{p\}$ be the bundle space, $U - \{p\}/\sim$ be the base space, the canonical mapping e of $U - \{p\}$ onto $U - \{p\}/\sim$ be the projection. Then S^1 (the one-sphere) is the fibre; as the group take S^1 (with complex multiplication). $U - \{p\}$ can be covered by a countable family $\{U_j\}_{j=1}^{\infty}$ of open invariant sets which are generated by arc transversals $\{T_j\}_{j=1}^{\infty}$ minus their end points: if a_j and b_j are the end-points of T_j, then $U_j = (T_j - (\{a_j\} \cup \{b_j\}))\pi R^1$. If we set $V_j = U_j/\sim$, then $\{V_j\}_{j=1}^{\infty}$ is an open covering of $U - \{p\}/\sim$. For any $(C(x), \xi) \in V_j \times S^2$ define

$$\varphi_j((C(x), \xi)) = (C(x) \cap T_j)\pi\lambda_\xi T(x)$$

where $\xi = \exp[i\lambda_\xi 2\pi]$ and $\lambda_\xi \in [0, 1)$. It is easily verified that the above satisfies (1) through (8). We will verify that it also satisfies (9) and (10) and is hence a bundle. Let $\delta = \delta(x) \in S^1$ be such that

$$(C(x) \cap T_i)\pi(\lambda_\delta T(x)) = C(x) \cap T_j .$$

It can be shown that $\delta(\cdot)$ is continuous. Then

$$\begin{aligned}\varphi_{j,x}^{-1}\varphi_{i,x}(\xi) &= \varphi_{j,x}^{-1}((C(x) \cap T_i)\pi\lambda_\xi T(x)) \\ &= \varphi_{j,x}^{-1}((C(x) \cap T_j)\pi(\lambda_\xi - \lambda_\delta)T(x)) \\ &= \xi\delta^{-1} .\end{aligned}$$

Thus $\varphi_{j,x}^{-1}\varphi_{i,x}$ coincides with multiplication by δ^{-1} and is continuous since δ is a continuous function.

PROPOSITION 2.6. *$U - \{p\}/\sim$ is homeomorphic with* $(0, 1)$.

Proof. First $U - \{p\}/\sim$ is connected and locally connected since $U - \{p\}$ is such. Second, $U - \{p\}/\sim$ is a regular T_1 space since $e \mid U$ is a closed mapping. Since the topology of $U - \{p\}$ has a countable base and e is an open mapping, the topology of $U - \{p\}/\sim$ has a countable base. By Urysohn's metrization theorem $U - \{p\}/\sim$ is metrizable. It is known that if a metric space X is separable, connected, and locally connected, and such that on removing any point y of X the remaining set $X - \{y\}$ consists of exactly two components, then it is the homeomorphic image of $(0, 1)$, [8]. Take any $C(x) \in U - \{p\}/\sim$. Then $(U - \{p\}) - C(x)$ consists of two components C_1 and C_2. (Indeed, $C(x)$ is a Jordan curve in $U \approx R^2$.) For $i = 1, 2$, $e(C_i)$ is both open and closed since C_i is both open and closed and $e \mid U$ is both open and closed. $(U - \{p\})/\sim - C(x) = ((U - \{p\}) - C(x))/\sim = e((U - \{p\}) - C(x) = e(C_1 \cup C_2) = e(C_1) \cup e(C_2)$. Thus $(U - \{p\})/\sim - C(x)$ has exactly two components. Hence $U - \{p\}/\sim$ is homeomorphic with $(0, 1)$.

DEFINITION 2.7. A space Y will be called solid with respect to a space X, if for every closed subset A of X and mapping $f: A \to Y$, there exists a mapping $f': X \to Y$ such that $f' \mid A = f$.

PROPOSITITION 2.8. *S^1 is solid with respect to $U - \{p\}/\sim$.*

Proof. It suffices, by Proposition 2.6, to show that S^1 is solid with respect to $(0, 1)$. We will only indicate the proof. Let I denote $(0, 1)$ and A be a closed subset of I. The components of $I - A$ are open intervals and there are at most countably many of them. If $A = I$ there is nothing to show. Let $f: A \to S^1$ be continuous, $A \neq I$. Let V be a component of $I - A$. Since $A \neq I$, V must have an end-point a contained in $(0, 1)$. If a is the only end-point of V in $(0, 1)$ define $f^1: V \to S^1$ by $f^1(x) = f(a)$ for all $x \in V$. If V has another end-point b contained in $(0, 1)$, we have two cases: $f(a) \neq f(b)$ or $f(a) = f(b)$. If $f(a) = f(b)$ define $f^1: V \to S^1$ by $f^1(x) = f(a)$ for all $x \in V$. If $f(a) \neq f(b)$, then the points $f(a)$ and $f(b)$ are the end-points of two subarcs of S^1. Let S_1 be the one of shorter arc length, and if the two arcs are of equal length S_1 is chosen to be either arc. Then there exists a homeomorphism f^1 of $\bar{V}$ onto S_1 such that $f^1(a) = f(a)$ and $f^1(b) = f(b)$. We repeat this construction for every component of $I - A$ and let g denote the union of all such mappings. The continuity of g follows from the fact that in any compact subinterval of I there can be only a finite number of components of $I - A$ whose end-points have f images which are diametrically opposite.

The following theorem from [6, 12.2] gives the existence of cross-sections to bundles $p: B \to X$, i.e., a continuous mapping $f: X \to B$ such that $pf(x) = x$ for every $x \in X$.

THEOREM. *Let X be a normal space with the property that every covering of X by open sets is reducible to a countable covering. Let β be a bundle over X with fibre Y which is solid. Let f be a cross-section of β defined on a closed subset A of X. Then f can be extended to a cross-section over all of X.* (Taking $A = \varnothing$, it follows that β has a cross-section.)

It should be noted that in the proof of this theorem it is not necessary that Y be solid, but only that Y be solid with respect to X, i.e., that any continuous mapping $f: A \to Y$, A closed in X, be continuously extendable to a mapping $f': X \to Y$. Hence

PROPOSITION 2.9. *There exists a continuous map $f: U - \{p\}/\sim \to U - \{p\}$ such that $ef(C(x)) = C(x)$ for every $C(x) \in U - \{p\}/\sim$.*

COROLLARY 2.10. *Let f be as in Proposition 2.9 and $S = f(U - \{p\}/\sim)$; then S is homeomorphic with* $(0, 1)$.

Proof. This is a consequence of the fact that if $\alpha: X \to Y$ has a cross-section $\beta: Y \to X$, then Y is homeomorphic with $\beta(Y)$.

COROLLARY 2.11. $C(x) \cap S = \{x\}$ *for each* $x \in S$ *and* $S\pi R^1 = U - \{p\}$.

PROPOSITION 2.12. *Let $h: (0,1) \to S$ be a homeomorphism. Then either* $\lim_{t\to 1} h(t) = p$ *or* $\lim_{t\to 0} h(t) = p$.

Proof. Let $x \in S$ and $a \in (0, 1)$ be such that $h(a) = x$. Then either $h((0, a)) \subset \operatorname{int} C(x)$ or $h((a, 1)) \subset \operatorname{int} C(x)$ since $S\pi R^1 = U - \{p\}$ and $S \cap C(x) = \{x\}$. Since this is true for every $x \in S$ we must have

$$\bar{S} - (S \cup C(x_0) \cup \{p\}) = \varnothing \ .$$

Thus if $h((0, a)) \subset \operatorname{int} C(x)$, then $\lim_{t\to 0} h(t) = p$ since $\overline{\operatorname{int} C(x)}$ is compact. Similarly if $h((a, 1)) \subset \operatorname{int} C(x)$, then $\lim_{t\to 1} h(t) = p$.

COROLLARY 2.13. *$S \cup \{p\}$ is homeomorphic with* $[0, 1)$.

REMARK 2.14. Let $x \in S$ and S_1 be the subarc of $S \cup \{p\}$ with end-points x and p. In what follows we will assume $x = x_0$ and $S = S_1$.

3. Type-N-isomorphisms.

The classification of dynamical systems in terms of the following types of isomorphisms is due to Ura [7].

Let (X_1, π_1) and (X_2, π_2) be two dynamical systems. An isomorphism of (X_1, π_1) onto (X_2, π_2) is a pair of mappings (h, φ) which satisfies one of the sets of conditions which follow. An isomorphism which satisfies the condition of Type N will be called a type-N-isomorphism. If there exists a type-N-isomorphism of (X_1, π_1) onto (X_2, π_2), then we say that (X_1, π_1) and (X_2, π_2) are type-N-isomorphic.

Type 1. (Topological isomorphisms.)

(1) h is a homeomorphism of X_1 onto X_2.

(2) φ is a homeomorphic group-isomorphism of the real additive group R^1 onto itself, i.e., $\varphi(t) = ct$ for some nonzero constant c.

(3) (Homomorphism condition) $h(x\pi_1 t) = h(x)\pi_2\varphi(t)$ for all $x \in X_1$ and $t \in R^1$.

Type 2.

(1) h is a homeomorphism of X_1 onto X_2.

(2) φ is a continuous mapping of $X_1 \times R^1$ onto R^1 such that for every fixed $x \in X_1$, $\varphi(x, \cdot)$ is a homeomorphic group-isomorphism of the real additive group R^1 onto itself such that $\varphi(x, 0) = 0$, i.e., there exists a continuous mapping $\varphi_1 \colon X_1 \to R^1$ such that $\varphi(x, t) = \varphi_1(x)t$ for all $x \in X_1$ and $t \in R^1$.

(3) (Homomorphism Condition)
$h(x\pi_1 t) = h(x)\pi_2\varphi(x, t)$ for all $x \in X_1$ and $t \in R^1$.

Type 2′. (Phase-map with reparameterization [4].)

(1) h is a homeomorphism of X_1 onto X_2.

(2) φ is a continuous mapping of $X_1 \times R^1$ onto R^1 such that for every fixed $x \in X_1$, $\varphi(x, \cdot)$ is a homeomorphism of R^1 onto R^1 such that $\varphi(x, 0) = 0$.

(3) (Homomorphism Condition)
$h(x\pi_1 t) = h(x)\pi_2\varphi(x, t)$ for all $x \in X_1$ and $t \in R^1$.

REMARK. Type $1 \subset$ Type $2 \subset$ Type 2′.

Under certain restrictions we will show that isomorphisms of types 2 and 2′ are equivalent for centers, and for $i = 1, 2$ give necessary and sufficient conditions for two centers to be type-N-isomorphic. The proof of the following assertion is in [7].

PROPOSITION 3.1. *"type-N-isomorphic" is an equivalence relation on the family of all dynamical systems.*

4. Classification of centers. We will now classify centers in terms of type-N-isomorphisms. Let (R^2, π_0) be the dynamical system defined by

$$\dot{x} = y \qquad \dot{y} = -x .$$

The phase portrait consists of a single critical point—the origin—and cycles of fundamental period 2π which are concentric circles about the origin. Let $x \in R^2$ and $t \in R^1$; then $x\pi_0 t = xe^{it}$. Let

$$U_0 = \{x \in R^2: |x| \leqq 1\} \quad \text{and} \quad (R^2, \pi),\ U,\ T(\cdot)\ ,$$

be as before.

PROPOSITION 4.1. *$(U_0 - \{0\}, \pi_0)$ and $(U - \{p\}, \pi)$ are type-2-isomorphic.*

Proof. Let S be an arc such that $S\pi R^1 = U$ and let $f: [0, 1] \to S$ be a homeomorphism such that $f(0) = p$. If $x \in U_0 - \{0\}$, there exists a unique $t_x \in [0, 2\pi)$ such that $x\pi_0 t_x = |x|$. Define $h: U_0 \to U$ as follows:

$$h(x) = \begin{cases} f(|x|)\pi - \dfrac{t_x}{2\pi} T(f(|x|)) & \text{if } x \in U_0 - \{0\} \\ p & \text{if } x = 0 \end{cases}$$

h is easily verified to be continuous. Let $x, y \in U_0 - \{0\}$ be such that $h(x) = h(y)$. Then

$$f(|x|)\pi - \frac{t_x}{2\pi} T(f(|x|)) = f(|y|)\pi - \frac{t_y}{2\pi} T(f(|y|)) .$$

Thus $f(|x|)$ and $f(|y|)$ are on the same trajectory and both are elements of S. Hence $f(|x|) = f(|y|)$ and $|x| = |y|$ since f is a homeomorphism. Next, $t_x, t_y \in [0, 2\pi)$ implies $t_x = t_y$. Thus $x = |x|e^{it_x} = |y|e^{it_y} = y$; this shows that h is one-to-one.

If $y \in U - \{p\}$ there exists a $\tau_y \in [0, T(y))$ such that $y\pi\tau_y \in S$. Then $h^{-1}(y) = f^{-1}(y\pi\tau_y)\exp[-2\pi i\tau_y/T(y)]$ and h is onto. Since each continuous, one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism, h is a homeomorphism of U_0 onto U.

Now let $x \in U_0 - \{0\}$ and $t \in R^1$. Then $x\pi_0 t_x = |x| = (x\pi_0 t)\pi_0 t_{x\pi_0 t}$ implies $t_x = t + t_{x\pi_0 t} + 2n\pi$ for some integer n.

$$h(x\pi_0 t) = f(|x\pi_0 t|)\pi - \frac{t_{x\pi_0 t}}{2\pi} T(f(|x\pi_0 t|))$$

$$= f(|x|)\pi - \frac{(t_x - t - 2n\pi)}{2\pi} T(f(|x|))$$

$$= f(|x|)\pi - \frac{t_x - t}{2\pi} T(f(|x|))$$

$$= h(x)\pi \frac{t}{2\pi} T(f(|x|)) .$$

Since $h(x)$ and $f(|x|)$ are on the same trajectory, we have $T(h(x)) = T(f(|x|))$. Thus

$$h(x\pi_0 t) = h(x)\pi \frac{t}{2\pi} T(h(x)) .$$

Set $\varphi(x, t) = (t/2\pi)T(h(x))$ for all $x \in U_0 - \{0\}$ and for all $t \in R^1$. Evidently $(h \mid U_0 - \{0\}, \varphi)$ satisfies the conditions of type 2.

PROPOSITION 4.2. *The following three conditions are equivalent:*
(i) (U_0, π_0) *and* (U, π) *are type-2-isomorphic.*
(ii) (U_0, π_0) *and* (U, π) *are type-2′-isomorphic.*
(iii) $\lim_{y \to p} T(y)$ *exists, is finite, and nonzero.*

Proof. We shall show that (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii). Assume $\lim_{y \to p} T(y)$ exists and equals λ, $0 < \lambda \in R^1$. Let h and φ be as in the proof of Proposition 4.1 and define $\bar{\varphi}: U_0 \times R^1 \to R^1$ as follows:

$$\bar{\varphi}(x, t) = \begin{cases} \varphi(x, t) & \text{if } x \in U_0 - \{0\} \text{ and } t \in R^1 \\ \dfrac{t\lambda}{2\pi} & \text{if } x = 0 \text{ and } t \in R^1 . \end{cases}$$

Evidently $\bar{\varphi}$ is a continuous extension of φ to $U_0 \times R^1$ and $(h, \bar{\varphi})$ satisfies the conditions of type 2.

Now assume (h, φ) is a type-2′-isomorphism of (U_0, π_0) onto (U, π). $h(0) = h(0\pi t) = h(0)\pi\varphi_1(0, t)$ for every $t \in R^1$. Thus $h(0)$ is critical and must equal p. Since h is a homeomorphism, $h(x) = p$ if and only if $x = 0$. Let $x \in U_0 - \{0\}$. Then $h(x) = h(x\pi_0 2\pi) = h(x)\pi\varphi(x, 2\pi)$ and $h(x) \neq h(x\pi_0 t)$ for $0 < t < 2\pi$ imply that $|\varphi(x, 2\pi)|$ is the fundamental period of $h(x)$, i.e., $|\varphi(x, 2\pi)| = T(h(x))$ for all $x \in U_0 - \{0\}$. By the continuity of $\varphi(\cdot, 2\pi)$, we have that $\lim_{x \to 0} T(h(x))$ exists and is finite. $\varphi(x, \cdot)$ a homeomorphism such that $\varphi(x, 0) = 0$ implies $\lim_{x \to 0} T(h(x)) \neq 0$. Since h is a homeomorphism, $\lim_{y \to p} T(y)$ exists, is finite, and nonzero. This completes the proof.

Let (R^2, π_1) and (R^2, π_2) be two dynamical systems with centers p_1 and p_2 respectively. For $i = 1, 2$, let U_i be a neighborhood of p_i as described in Remark 2.14, S_i be the arc which generates U_i, and T_i be the mapping which associates with $x \in U_i - \{p_i\}$ its fundamental period $T_i(x)$.

THEOREM 4.3. *$(U_1 - \{p_1\}, \pi_1)$ and $(U_2 - \{p_2\}, \pi_2)$ are type-2-isomorphic.*

Proof. This is an immediate consequence of Propositions 3.1 and 4.1.

PROPOSITION 4.4. *If $f: S_1 \to S_2$ is a homeomorphism, then there exists a type-2-isomorphism (h, φ) of $(U_1 - \{p_1\}, \pi_1)$ onto $(U_2 - \{p_2\}, \pi_2)$ such that $h \mid S_1 = f$ and $\varphi(x, t) = tT_2(h(x))/T_1(x)$ for all $x \in U_1 - \{p_1\}$ and for all $t \in R^1$.*

Proof. Analogous to that of Proposition 4.1.

DEFINITION 4.5. Let (X_1, π_1) and (X_2, π_2) be dynamical systems. A homeomorphism h of X_1 onto X_2 is said to be trajectory preserving if and only if $h(C_1(x)) = C_2(h(x))$ for every $x \in X_1$.

PROPOSITION 4.6. *(U_1, π_1) and (U_2, π_2) are type-2-isomorphic if and only if there exists a trajectory preserving homeomorphism $h: U_1 \to U_2$ such that* $\lim_{y \to p_1} T_2(h(y))/T_1(y)$ *exists, is finite, and nonzero.*

Proof. Let h be a trajectory preserving homeomorphism of U_1 onto U_2 such that $\lim_{y \to p_1} T_2(h(y))/T_1(y)$ exists, is finite, and nonzero. Then $h(x\pi_1 R^1) = h(x)\pi_2 R^1$, and, for all $x \in S_1$, $h(x)\pi_2 R^1 \cap h(S_1) = \{h(x)\}$ since $x\pi_1 R^1 \cap S_1 = \{x\}$. $h \mid S_1$ is a homeomorphism of S_1 onto $h(S_1)$. By Proposition 4.4 there exists a homeomorphism g of U_1 onto U_2 such that $(g \mid U_1 - \{p_1\}, \varphi)$ is a type-2-isomorphism of $(U_1 - \{p_1\}, \pi_1)$ onto $(U_2 - \{p_2\}, \pi_2)$. Moreover $g \mid S_1 = h \mid S_1$ and $\varphi(x, t) = tT_2(g(x))/T_1(x)$. Then $\varphi(x, t) = tT_2(h(x))/T_1(x)$ for all $x \in S - \{p\}$ since $g \mid S_1 = h \mid S_1$ and $\lim_{x \to p_1} tT_2(h(x))/T_1(x) = \lambda t$ for some nonzero λ by our assumption on h. Define $\bar{\varphi}: U_1 \times R^1 \to R^1$ as follows:

$$\bar{\varphi}(x, t) = \begin{cases} \varphi(x, t) & \text{if } x \in U_1 - \{p_1\} \text{ and } t \in R^1 \\ \lambda t & \text{if } x = p_1 \text{ and } t \in R^1 . \end{cases}$$

$\bar{\varphi}$ is evidently a continuous extension of φ and $(g, \bar{\varphi})$ a type-2-isomorphism of (U_1, π_1) onto (U_2, π_2).

Now assume that (h, φ) is a type-2-isomorphism of (U_1, π_1) onto (U_2, π_2). Then $\varphi_x(\cdot)$ a homeomorphic group isomorphism of R^1 onto itself such that $\varphi_x(0) = 0$; thus there exists a continuous function $f: U_1 \to R^1$ such that $\varphi_x(t) = f(x)t$ for all $x \in U_1$ and for all $t \in R^1$. Indeed, $f(x) = \varphi(x, 1)$. If $x \in U_1 - \{p_1\}$, then

$$h(x) = h(x\pi_1 T_1(x)) = h(x)\pi_2\varphi(x, T_1(x))$$

and $h(x) \neq h(x\pi_1 t)$ for $0 < t < T_1(x)$. Thus $|\varphi(x, T_1(x))|$ is the fundamental period of $h(x)$. Thus $|\varphi_x(T_1(x))| = |f(x)T_1(x)| = |f(x)|\, T_1(x) = T_2(h(x))$. Therefore $|f(x)| = T_2(h(x))/T_1(x)$ and $\lim_{x\to p_1} T_2(h(x))/T_1(x) = |f(p_1)| \neq 0$ since f is continuous and φ_{p_1} is a homeomorphic group isomorphism of R^1 onto itself. This completes the proof.

COROLLARY 4.7. *If both* $\lim_{x\to p_1} T_1(x)$ *and* $\lim_{y\to p_2} T_2(y)$ *exist, are finite, and nonzero, then* (U_1, π_1) *and* (U_2, π_2) *are type-2-isomorphic.*

Proof. Since S_1 and S_2 are both homeomorphic to $[0, 1]$ (by Remark 2.14), S_1 and S_2 are homeomorphic. By Proposition 4.4 there exists a trajectory preserving homeomorphism of $U_1 - \{p_1\}$ onto $U_2 - \{p_2\}$. This can be extended to a trajectory preserving homeomorphism h of U_1 onto U_2 by mapping p_1 onto p_2. Then $\lim_{x\to p_1} T_2(h(x))/T_1(x)$ exists, is finite, and nonzero since both $\lim_{x\to p_1} T_1(x)$ and $\lim_{y\to p_2} T_2(y)$ are such. The result follows from Proposition 4.6.

By assumption U_1 and U_2 are neighborhoods of p_1 and p_2 respectively such that there exist $x_1, x_2 \in R^2$ with $\operatorname{int} C_1(x_1) = U_1$ and $\operatorname{int} C_2(x_2) = U_2$. Moreover x_i can be chosen so that $\bar{S}_i = S_i \cup \{x_i\} \cup \{p_i\}$, $i = 1, 2$. (See Remark 2.14.)

COROLLARY 4.8. *If* $\lim_{x\to p_1} T_1(x) = \lim_{y\to p_2} T_2(y)$ (*with values* 0 *and* ∞ *as allowed*), $T_1(x_1) = T_2(x_2)$ *and both* $T_1 \mid S_1$ *and* $T_2 \mid S_2$ *are one-to-one,* (U_1, π_1) *and* (U_2, π_2) *are type-2-isomorphic.*

Proof. Since $T_i(S_i)$ is connected, $T_i(S_i)$ is an interval for $i=1, 2$. Moreover $T_1(S_1) = T_2(S_2)$ since

$$T_1(x_1) = T_2(x_2) \text{ and } \lim_{x\to p_1} T_1(x) = \lim_{y\to p_2} T_2(y)\ .$$

If V is a compact subset of S_i, then $T_i \mid V$ is a homeomorphism because a continuous, one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism. Since this is true for every compact subset V of S_i, T_i is a homeomorphism, $i = 1, 2$. Define g: $S_1 \to S_2$ as follows:

$$g(x) = \begin{cases} T_2^{-1}T_1(x) & \text{if } x \in S_1 - \{p_1\} \\ p_2 & \text{if } x = p_1\ . \end{cases}$$

Evidently g is a homeomorphism of S_1 onto S_2. By Proposition 4.4 g can be extended to a trajectory preserving homeomorphism h: $U_1 \to U_2$. Then

$$\begin{aligned}\lim_{y\to p_1}\frac{T_2(h(y))}{T_1(y)} &= \lim_{y\to p_1}\frac{T_2(h(C_1(y)\cap S_1))}{T_1(C_1(y)\cap S_1)}\\ &= \lim_{\substack{y\to p_1\\ y\in s_1}}\frac{T_2(h(y))}{T_1(y)} = \lim_{\substack{y\to p_1\\ y\in s_1}}\frac{T_2(g(y))}{T_1(y)}\\ &= \lim_{\substack{y\to p_1\\ y\in s_1}}\frac{T_2(T_2^{-1}T_1(y))}{T_1(y)} = 1\ .\end{aligned}$$

The result now follows by Proposition 4.6.

EXAMPLE 4.9. If $\lim_{x\to p_1} T_1(x) = \lim_{x\to p_2} T_2(x) = 0$, it is not necessarily true that (U_1, π_1) and (U_2, π_2) are type-2-isomorphic. Let (U_0, π_0) be as before and define π_1 and π_2 as follows (f and g shall be chosen later):

$$x\pi_1 t = x\pi_0\frac{t}{f(x)} \qquad \text{for all } x\in U_0 \text{ and for all } t\in R^1$$

$$x\pi_2 t = x\pi_0\frac{t}{g(x)} \qquad \text{for all } x\in U_0 \text{ and for all } t\in R^1\ .$$

If there exists a type-2-isomorphism (h, φ) of (U_0, π_1) onto (U_0, π_2) then by Proposition 4.6 $\lim_{y\to 0} T_2(h(x))/T_1(x)$ exists and is nonzero. Note that $T_1(x) = f(x)$ and $T_2(x) = g(x)$. Restricting our attention to S_1 and S_2, the problem may be reduced to the following:

Given continuous functions $f, g\colon [0, 1]\to[0, 1]$ such that $f(0) = g(0) = 0$ and $f(x) > 0 < g(x)$ for $x\in(0, 1]$. Does there exist a homeomorphism $h_1\colon[0, 1]\to[0, 1]$ such that $\lim_{x\to 0} f(h_1(x))/g(x)$ exists and is nonzero? It is not hard to see that there exist functions f and g satisfying our assumptions and such that $\lim_{x\to 0} f(h_1(x))/g(x)$ does not exist for any homeomorphism $h_1\colon[0, 1]\to[0, 1]$. Hence for these choices of f and g, (U_0, π_1) and (U_0, π_2) are not type-2-isomorphic.

Similarly, if $\lim_{x\to p_1} T_1(x) = \lim_{y\to p_2} T_2(y) = +\infty$, it is not necessarily true that (U_1, π_1) and (U_2, π_2) are type-2-isomorphic.

PROPOSITION 4.10. *(U_1, π_1) and (U_2, π_2) are type-1-isomorphic if and only if there exists a trajectory preserving homeomorphism h of U_1 onto U_2 and a constant λ such that $T_2(h(x)) = \lambda T_1(x)$ for all $x\in U_1 - \{p\}$.*

Proof. Assume (U_1, π_1) and (U_2, π_2) are type-1-isomorphic. Then there exist a homeomorphism $h\colon U_1\to U_2$ and a nonzero constant λ such that $h(x\pi_1 t) = h(x)\pi_2\lambda t$ for all $x\in U_1$ and for all $t\in R^1$. Evidently h is trajectory preserving. Let $x\in U_1 - \{p_1\}$. Then

$$h(x) = h(x\pi_1 T_1(x)) = h(x)\pi_2\lambda T_1(x)$$

and $h(x) \neq h(x)\pi_2\lambda t$ for $t \in (0, T_1(x))$. Thus

$$T_2(h(x)) = |\lambda T_1(x)| = |\lambda| \, T_1(x) .$$

Now let h be a trajectory preserving homeomorphism of U_1 onto U_2 and λ be a nonzero constant such that $T_2(h(x)) = \lambda T_1(x)$ for every $x \in U_1 - \{p\}$. Then $h \mid S_1$ is a homeomorphism of S_1 onto $h(S_1)$. By Proposition 4.4 there exists a homeomorphism $g: U_1 - \{p_1\} \rightarrow U_2 - \{p_2\}$ such that $g \mid S_1 = h \mid S_1$ and (g, φ) is a type-2-isomorphism of

$$(U_1 - \{p_1\}, \pi_1) \quad \text{onto} \quad (U_2 - \{p_2\}, \pi_2)$$

where $\varphi(x, t) = T_2(g(x))t/T_1(x)$ for all $x \in U - \{p_1\}$ and $t \in R^1$. Then $\varphi(x, t) = T_2(h(x))t/T_1(x)$ for all $x \in S_1 - \{p_1\}$. Thus $\varphi(x, t) = \lambda t$ for all $x \in S_1 - \{p_1\}$. Define $\bar{\varphi}: U_1 \times R^1 \rightarrow R^1$ as follows:

$$\bar{\varphi}(x, t) = \lambda t .$$

Then it is easy to show $g(x\pi_1 t) = g(x)\pi_2\lambda t$ for all $x \in U_1 - \{p_1\}$ and for all $t \in R^1$. g can be extended to a homeomorphism $\bar{g}$ of U_1 onto U_2 by mapping p_1 onto p_2. Then $(\bar{g}, \bar{\varphi})$ is a type-1-isomorphism of (U_1, π_1) onto (U_2, π_2).

COROLLARY 4.11. *(U_0, π_0) and (U_1, π_1) are type-1-isomorphic if and only if $T_1(\cdot)$ is constant on $U_1 - \{p_1\}$.*

BIBLIOGRAPHY

1. N. P. Bhatia and G. P. Szegö, *Dynamical systems: stability theory and applications*, Springer-Verlag, New York, 1967.
2. N. Bourbaki, *Elements of mathematics, general topology, Part I*, Addison-Wesley, Reading, Massachusetts, 1966.
3. Otomar Hájek, *Dynamical systems in the plane*, Academic Press, London, 1968.
4. ———, *Categôrical concepts in dynamical systems theory*, Proceedings of Topological Dynamics Symposium, Fort Collins, Colorado, W. A. Benjamin, 1968.
5. John L. Kelley, *General topology*, D. Van Nostrand Co., Inc., Princeton, 1965.
6. Norman Steenrod, *The topology of Fibre bundles*, Princeton University Press, Princeton, 1951.
7. Taro Ura, *Local isomorphisms and local parallelizability in dynamical systems theory*, Math. Systems Theory **2** (1968).
8. A. J. Wood, *The topological characterization of an open linear interval*, Proc. London Math. Soc. **41** (1936), 191-198.

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