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ON THE ZEROS OF THE SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(n)}(z) + p(z)y(z) = 0$.

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In this paper sufficient conditions for disconjugacy and for nonoscillation of the equation $y^{(n)}(z) + p(z)y(z) = 0$ are given. For $n = 2m$ a theorem ensuring that no solution of this equation has two zeros of multiplicity m is obtained. Here the invariance of the equation under linear transformations of z is used.

In [6] Nehari considered the equation

$$(1) \qquad y^{(n)}(z) + p_{n-1}(z)y^{(n-1)}(z) + \cdots + p_0(z)y(z) = 0 ,$$

where the analytic functions $p_i(z)$, $i = 0, \cdots, n-1$ are regular in a given domain D, and obtained a disconjugacy theorem for bounded convex domains and a nonoscillation theorem for the unit disk. Equation (1) is called *disconjugate* in a domain D, if no nontrivial solution of (1) has more than $(n-1)$ zeros in D. (The zeros are counted by their multiplicity). The equation is called nonoscillatory in D, if no nontrivial solution has an infinite number of zeros in D.

In this paper we obtained related results for a special case of (1); i.e., for the equation

$$(2) \qquad y^{(n)}(z) + p(z)y(z) = 0 ,$$

where the analytic function $p(z)$ is regular in the unit disk.

Section 1 deals with the invariance of equation (2), where $p(z)$ is analytic in a general domain, under the linear transformation

$$(3) \qquad \zeta = \frac{az + b}{cz + d} , \qquad ad - bc \neq 0 ,$$

(Theorem 1). The invariance of

$$(4) \qquad y''(z) + p(z)y(z) = 0$$

played an important role in Nehari's results on this second order equation [3; 5].

In § 2 we obtain sufficient conditions for disconjugacy and nonoscillation of equation (2) in the unit disk (Theorem 2 and Theorem 4 respectively). From Theorem 2 and the invariance of (2) under the linear transformations (3) we get a sufficient condition for the disconjugacy of (2) in non-Euclidean disks (Theorem 3).

In § 3 we deal with equations of even order $n = 2m$, and obtain a condition on $p(z)$, which ensures that no solution of (2) has two zeros of multiplicity m. For the proof of this Theorem 5 we apply Theorem 1 and the method used in [5].

1. Invariance under linear transformations.

THEOREM 1. *The equation*

$$\frac{d^n y}{dz^n} + p(z)y(z) = 0 \tag{2}$$

is transformed by the linear mapping

$$\zeta = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{3'}$$

into an equation of the same form

$$\frac{d^n w_1}{d\zeta^n} + P_1(\zeta)w_1(\zeta) = 0. \tag{2'}$$

Here

$$w_1(\zeta) = (a - c\zeta)^{n-1}w(\zeta) \tag{5}$$

and

$$P_1(\zeta) = \left(\frac{dz}{d\zeta}\right)^n P(\zeta) = (a - c\zeta)^{-2n}P(\zeta), \tag{6}$$

where

$$w(\zeta) = y(z) = y\left(\frac{d\zeta - b}{-c\zeta + a}\right) \tag{7}$$

and

$$P(\zeta) = p(z) = p\left(\frac{d\zeta - b}{-c\zeta + a}\right). \tag{8}$$

Proof. It is easily verified that

$$(a - c\zeta)^{n+1}\frac{d^n w_1}{d\zeta^n} = \frac{d^n y}{dz^n}.$$

Applying this and (5)—(8) to equation (2) we obtain

$$\frac{d^n y}{dz^n} + p(z)y(z) = \frac{d^n y}{dz^n} + p(z)w_1(\zeta)(a - c\zeta)^{1-n}$$
$$= \left[\frac{d^n w_1}{d\zeta^n} + P_1(\zeta)w_1(\zeta)\right](a - c\zeta)^{n+1} ,$$

which proves the statement of our theorem.

The assumption $ad - bc = 1$ in (3′) was made just for convenience. In the general case (3), formula (6) has to be replaced by

$$P_1(\zeta) = \left(\frac{dz}{d\zeta}\right)^n P(\zeta) = \frac{(a - c\zeta)^{-2n}}{(ad - bc)^{-n}} P(\zeta) .$$

The converse of Theorem 1 is also true: the only transformations $\zeta = \psi(z)$, which leave the form of equation (2), for $n \geqq 3$, invariant are the linear transformations (3). This follows from a theorem of Wilczynski [11, p. 26]. For $n = 2$ equation (4) is invariant for any univalent transformation $\zeta = \psi(z)$; however if $\psi(z)$ is not linear, the connection between $p(z)$ and $P_1(\zeta)$ is more complicated than (6).

2. Disconjugacy and nonoscillation.

THEOREM 2. *Let the analytic function $p(z)$ be regular in $|z| < 1$. If*

(9) $$|p(z)| \leqq \frac{n!}{(1 - |z|)(1 + |z|)^{n-1}}, \quad |z| < 1 ,$$

then equation

(2) $$y^{(n)}(z) + p(z)y(z) = 0$$

is disconjugate in $|z| < 1$.

We remark that for $n = 2$, (9) becomes

$$|p(z)| \leqq \frac{2}{1 - |z|^2} , \qquad |z| < 1 ,$$

which is a condition of Pokornyi [8; 5] for disconjugacy of equation (4) in the unit disk.

In the case of equation (2) and $|z| < 1$, the general theorem [6, p. 328] gives that

$$\overline{\lim_{r\to 1}} \frac{(2r)^{n-1}}{(n - 1)!} \int_{|\zeta|=r} |p(\zeta)|\,|d\zeta| < 2$$

implies the disconjugacy of (2) in $|z| < 1$.

Using [4, p. 127, Ex. 8] this corollary to Nehari's theorem follows from Theorem 2.

As the function $f_n(r) = n!/(1-r)(1+r)^{n-1}$ is monotonic decreasing in $0 \leqq r \leqq (n-2)/n$, it follows by the maximum principle that, for $n > 2$, (9) is equivalent to

$$|p(z)| \leqq \frac{n!}{(1-|z|)(1+|z|)^{n-1}}, \quad \frac{n-2}{n} \leqq |z| < 1 .$$

Proof. For proving this theorem we use "divided differences" [6; 7, Chapter 1]. We denote by $[z, z_1, \cdots, z_k]$ the k – th divided difference of $y(z)$, i.e., we set

$$[z, z_1] = \frac{y(z) - y(z_1)}{z - z_1},$$

$$[z, z_1, \cdots, z_k] = \frac{[z, z_1, \cdots, z_{k-1}] - [z_1, z_2, \cdots, z_k]}{z - z_k}, \quad k = 2, \cdots, n .$$

If C is a closed contour in the unit disk, such that $z, z_1, \cdots, z_n$ are in the interior of C, then it follows from the definition that

$$[z, z_1, \cdots, z_n] = \frac{1}{2\pi i}\int_C \frac{y(\zeta)}{(\zeta - z)(\zeta - z_1)\cdots(\zeta - z_n)}d\zeta .$$

The right hand side is defined also when some of the z_i's coincide and may thus serve as a definition of the left hand side also in that case (where the divided differences would have to be defined with the help of derivatives). Clearly then $[z, z_1, \cdots, z_n]$ is continuous in all its arguments. Moreover, if $y(z_1) = \cdots = y(z_n) = 0$, we obtain

$$[z, z_1, \cdots, z_n] = \frac{y(z)}{\prod_{i=1}^{n}(z - z_i)} . \tag{10}$$

To prove the theorem, assume now, by negation, that (2) has a nontrivial solution $y(z)$ which vanishes at the n-points $z_1, \cdots, z_n$ of the open unit disk E. These n points cannot all coincide, as $y(z^*) = y'(z^*) = \cdots = y^{(n-1)}(z^*) = 0$ implies $y \equiv 0$. Therefore there are at least two distinct points. Let H be the convex hull of the points $z_1, \cdots, z_n$. H is therefore either a segment or a convex polygon.

Let z be any point in H; we use now Hermite's formula for the divided difference of $y(z)$ [7, p. 9]

$$[z, z_1, \cdots, z_n] = \int\cdots\int y^{(n)}(t_0 z + t_1 z_1 + \cdots + t_n z_n)dt_1 \cdots dt_n , \tag{11}$$

where the integral is extended over the n dimensional simplex of volume $1/n!$ given by

$$t_i \geqq 0 \qquad i = 0, \cdots, n\,; \qquad \sum_{i=0}^{n} t_i = 1\,. \tag{12}$$

We remark that formula (11) is proved in [7, p. 9] only made the assumption that all the z_i's are distinct. As however both sides are continuous in $z_1, \cdots, z_n$, this formula is valid also in the case where some of the z_i's coincide. The point $\zeta = t_0 z + \cdots + t_n z_n$, where the t_i satisfy (12), belongs to the convex hull of the $n+1$ points $z, z_1, \cdots z_n$, and as $z \in H$, it follows that $\zeta \in H$.

From (10), (11) and (2) it follows that

$$\frac{y(z)}{\prod_{i=1}^{n}(z - z_i)} = -\int_{t_1 \cdots t_n} \cdots \int p(\zeta) y(\zeta) dt_1 \cdots dt_n\,, \tag{13}$$

where $\zeta = t_0 z + t_1 z_1 + \cdots + t_n z_n \in H$. Let ζ_0 be a point, or one of the points, in which $|p(z)y(z)|$ attains its maximum in H. (This maximum is positive, otherwise $p(z)y(z) \equiv 0$, and as $y(z) \not\equiv 0$, it follows that $p(z) \equiv 0$. Equation (2) becomes $y^{(n)}(z) = 0$, which is clearly disconjugate). As

$$|p(\zeta_0)y(\zeta_0)| \geqq |p(z)y(z)|\,, \qquad z \in H\,, \tag{14}$$

it follows by (13) that for every $z \in H$,

$$|y(z)| \leqq \prod_{i=1}^{n} |z - z_i| \frac{|y(\zeta_0)|\,|p(\zeta_0)|}{n!}\,.$$

Choosing now $z = \zeta_0$ and using $y(\zeta_0) \neq 0$ we obtain

$$|p(\zeta_0)| \prod_{i=1}^{n} |\zeta_0 - z_i| \geqq n!\,. \tag{15}$$

We prove that for ζ_0 satisfying (14),

$$\prod_{i=1}^{n} |\zeta_0 - z_i| < (1 - |\zeta_0|)(1 + |\zeta_0|)^{n-1}\,; \tag{16}$$

(cf [10, Th. 2)].

Let us assume first that the convex hull H of $z_1, \cdots, z_n$ is a polygon. Then, by the maximum principle, ζ_0 is on the boundary of H. Therefore ζ_0 is on a segment, the endpoints of which are two of the n given points $z_1, \cdots, z_n$. We denote these points by z_1, z_2 Clearly,

$$|\zeta_0 - z_i| < 1 + |\zeta_0|\,. \qquad i = 3, \cdots, n\,. \tag{17}$$

Denoting by z_1^*, z_2^* the endpoints $|z_1^*| = |z_2^*| = 1$ of the chord determined by z_1 and z_2, we obtain

$$|\zeta_0 - z_1||\zeta_0 - z_2| < |\zeta_0 - z_1^*||\zeta_0 - z_2^*| . \tag{18}$$

As the product of the segments of a chord through ζ_0 depends only on ζ_0, we have

$$|\zeta_0 - z_1^*||\zeta_0 - z_2^*| = (1 - |\zeta_0|)(1 + |\zeta_0|) .$$

This and (18) give

$$|\zeta_0 - z_1||\zeta_0 - z_2| < (1 - |\zeta_0|)(1 + |\zeta_0|) . \tag{19}$$

(17) and (19) imply (16).

If H is a segment and ζ_0 one of the points of the segment in which $|p(z)y(z)|$ becomes maximum, then we denote by z_1, z_2 the endpoints of H and by z_1^*, z_2^* the endpoints of the corresponding chord. (17) and (19) hold and therefore (16) is again valid.

(15), which followed from the assumption that (2) is not disconjugate in $|z| < 1$, and (16) imply

$$|p(\zeta_0)| \geqq \frac{n!}{\prod_{i=1}^{n} |\zeta_0 - z_i|} > \frac{n!}{(1 - |\zeta_0|)(1 + |\zeta_0|)^{n-1}} , \tag{20}$$

which contradicts assumption (9). This contradiction concludes the proof of the theorem.

For the proof of the next theorem it is convenient to state some simple consequences of Theorem 2. The transformation $\zeta = z/\rho$ maps $|z| < \rho$ on $|\zeta| < 1$, and equation (2) is transformed into (2′) with $P_1(\zeta) = \rho^n p(z)$. As (2) is disconjugate in $|z| < \rho$ if (2′) is disconjugate in $|\zeta| < 1$, we obtain a sufficient condition for disconjugacy of (2) in $|z| < \rho$, namely

$$|p(z)| \leqq \frac{n!}{(\rho - |z|)(\rho + |z|)^{n-1}} , \quad |z| < \rho .$$

Using the minimum of the function $n!/(\rho - r)(\rho + r)^{n-1}$ for $0 \leqq r < \rho$, we obtain another, weaker, sufficient condition for disconjugacy of (2) in $|z| < \rho$,

$$|p(z)| \leqq \frac{n!}{(n-1)^{n-1}} \left(\frac{n}{2\rho}\right)^n , \quad |z| < \rho . \tag{21}$$

We remark that for $\rho = 1, n = 2$ the value of the constant in (21) is 2. The exact constant in this case is $\pi^2/4$ [3, Th. 2].

THEOREM 3. *Let the analytic function $p(z)$ be regular in $|z| < 1$*

and assume that there exists ρ, $0<\rho<1$, *such that*

$$|p(z)|(1-|z|^2)^n \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n(1-\rho^2)^n, \quad |z|<1\,. \tag{22}$$

Then equation (2) *is disconjugate in every non-Euclidean disk of radius* $1/2\log[(1+\rho)/(1-\rho)]$.

Proof. Let ρ satisfy (22) and let G be a given disk in $|z|<1$ with non-Euclidean radius $1/2\log[(1+\rho)/(1-\rho)]$. By mapping the unit disk on itself, G can be mapped onto a disk G_1 given by $|\zeta|<\rho$. Equation (2) is transformed into (2′). As for linear mappings $\zeta=\zeta(z)$ of the unit disk on itself

$$\left|\frac{d\zeta}{dz}\right| = \frac{1-|\zeta|^2}{1-|z|^2}\,,$$

we obtain

$$|P_1(\zeta)| = |p(z)|\left|\frac{dz}{d\zeta}\right|^n = |p(z)|\frac{(1-|z|^2)^n}{(1-|\zeta|^2)^n}\,. \tag{23}$$

From (23) together with (22) it follows that

$$|P_1(\zeta)| \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n\frac{(1-\rho^2)^n}{(1-|\zeta|^2)^n}\,,$$

which for $|\zeta|<\rho$ gives

$$|P_1(\zeta)| \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n.$$

By (21), this is a sufficient condition for disconjugacy of (2′) in G_1, $|\zeta|<\rho$, and therefore (2) is disconjugate in G. Theorem 3 is thus proved.

This theorem can be stated as follows: if

$$|p(z)|(1-|z|^2)^n \leqq C < \infty\,, \qquad |z|<1\,, \tag{24}$$

then equation (2) is disconjugate in every non-Euclidean disk of radius $1/2\log[(1+\rho_0)/(1-\rho_0)]$, where $\rho_0=g^{-1}(C)$ and

$$g(\rho) = \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n(1-\rho^2)^n\,.$$

$g(\rho)$ is a monotonic decreasing function. Therefore the smallest C satisfying (24) gives the biggest non-Euclidean radius.

For $n=2$ non-Euclidean disks of disconjugacy were considered in [2] and [9].

THEOREM 4. *Assume that the analytic function $p(z)$ is regular in $|z| < 1$. Let $n \geqq 3$ and let C be a positive constant. If*

$$|p(z)| \leqq \frac{C}{1 - |z|}, \qquad |z| < 1, \tag{25}$$

then equation (2) *is nonoscillatory in* $|z| < 1$.

In the case $n = 2$, equation (4) is nonoscillatory in $|z| < 1$, if there exists x_1, $0 < x_1 < 1$, such that

$$|p(z)| \leqq \frac{2}{1 - |z|^2}, \quad x_1 < |z| < 1. \tag{26}$$

Proof. Assume that equation (2) has a solution with an infinite number of zeros in the unit disk. We can then find a sequence of zeros $z_1, z_2, \cdots$ tending to z^* on the boundary, $|z^*| = 1$. For any ρ, $0 < \rho < 1$, let $G(\rho)$ be the intersection of the disk $|z - z^*| < \rho$ with the unit disk. Any $G(\rho)$ contains an infinite number of zeros. Denote n of these zeros by $z_1, \cdots, z_n$. As in the proof of Theorem 2, we denote the convex hull of these n points by H and choose $\zeta_0 \in H$ such that (14) holds. We choose z_1 and z_2 as in that proof; (15) and (19) are again valid.

If $n \geqq 3$, then clearly

$$|\zeta_0 - z_i| < 2\rho \qquad i = 3, \cdots, n.$$

Using this and (19) we obtain

$$\prod_{i=1}^{n} |\zeta_0 - z_i| < (1 - |\zeta_0|)(1 + |\zeta_0|)(2\rho)^{n-2} < (1 - |\zeta_0|)2^{n-1}\rho^{n-2}. \tag{27}$$

From (15) and (27) it follows that

$$|p(\zeta_0)| > \frac{n!}{(1 - |\zeta_0|)2^{n-1}\rho^{n-2}}. \tag{28}$$

For any given C, we can find ρ such that

$$\frac{n!}{2^{n-1}\rho^{n-2}} > C. \tag{29}$$

From (28) and (29) we obtain a contradiction to our assumption (25), which completes the proof of the first part of the theorem ($n \geqq 3$).

For $n = 2$, we choose ρ such that $\rho = 1 - x_1$. (15) and (19) imply

$$|p(\zeta_0)| > \frac{2}{1 - |\zeta_0|^2}. \tag{30}$$

As $x_1 < |\zeta_0| < 1$, (30) contradicts (26), which completes the proof of the second part of Theorem 4 ($n = 2$).

By [9, Th. 1] the condition

$$|p(z)| \leqq \frac{1}{(1-|z|^2)^2}, \ |z| > x_0 , \qquad 0 < x_0 < 1$$

is sufficient for nonoscillation of (4) in $|z| < 1$; hence the second part ($n = 2$) of Theorem 4 follows from this theorem.

Nehari has given a nonoscillation theorem for the general equation (1) in any bounded convex domain. In the case of the unit disk and the special equation (2) his sufficient condition becomes

$$\lim_{r\to 1}\int_0^{2\pi} |p(re^{i\theta})|\, d\theta < \infty . \tag{31}$$

This sufficient condition (31) implies our condition (25). (See [4, p. 127, Ex. 8]).

3. Equations of even order $n = 2m$; nonexistence of solutions with two zeoros of multiplicity m.

THEOREM 5. *Let the analytic function $p(z)$ be regular in $|z|<1$. The equation*

$$y^{(2m)}(z) + (-1)^{m+1}p(z)y(z) = 0 \tag{32}$$

has no solution having two zeros of multiplicity m in $|z| < 1$ if

$$|p(z)| \leqq P(|z|) , \tag{33}$$

where $P(x)$ is a function with the following properties:

(a) *$P(x)$ is positive and continuous for $-1 < x < 1$;*

(b) *$P(-x) = P(x)$;*

(c) *$(1-x^2)^{2m}P(x)$ is nonincreasing if x varies from 0 to 1;*

(d) *the differential equation*

$$u^{(2m)}(x) + (-1)^{m+1}P(x)u(x) = 0 \tag{34}$$

has no solution with two zeros of multiplicity m in $-1 < x < 1$.

Proof. (cf. [5]). Suppose the theorem is false and there exists a solution of (32) with zeros of multiplicity m at α and $\beta(|\alpha| < 1, |\beta| < 1, \alpha \neq \beta)$. The circle passing through α and β and orthogonal to $|z| = 1$ is divided by $|z| = 1$ into two arcs. We denote the arc inside $|z| < 1$ by C. Without loss of generality, we may assume that C is in the upper half plane and symmetric with respect to the imaginary axis. The linear transformation

$$z = \frac{\zeta + i\rho}{1 - i\rho\zeta}\,, \qquad 0 \leqq \rho < 1\,, \tag{35}$$

maps $|z| < 1$ on $|\zeta| < 1$ and C on the linear segment $-1 < \zeta < 1$. With the aid of Theorem 1 and (23), equation (32) is transformed into the equation

$$w^{(2m)}(\zeta) + (-1)^{m+1}q(\zeta)w(\zeta) = 0\,, \tag{36}$$

with

$$|q(\zeta)| = |p(z)|\left|\frac{dz}{d\zeta}\right|^{2m} = |p(z)|\frac{(1-|z|^2)^{2m}}{(1-|\zeta|^2)^{2m}}\,. \tag{37}$$

It follows from (35) that $|\zeta| \leqq |z|$ if $-1 < \zeta < 1$. Hence, by assumption (c) it follows that

$$(1-|z|^2)^{2m}P(|z|) \leqq (1-|\zeta|^2)^{2m}P(|\zeta|)\,, \quad -1 < \zeta < 1\,.$$

Combining this with (33) and (37) we obtain

$$|q(\zeta)| \leqq P(|\zeta|)\,, \qquad -1 < \zeta < 1\,. \tag{38}$$

Thus, our assumption that (32) has a solution with two zeros at α and β of multiplicity m implies that (36) has a solution $w(\zeta)$ possessing two zeros of multiplicity m at a and b, $-1 < a < b < 1$. Let $w(\zeta)$ be this solution. Multiplying equation (36) by $\bar{w}(\zeta)$ and integrating from a to b along the real axis, we obtain

$$\int_a^b w^{(2m)}(x)\bar{w}(x)dx + (-1)^{m+1}\int_a^b q(x)\,|w(x)|^2\,dx = 0\,.$$

Integrating by parts m times and noting that all the integrated parts vanish, we get

$$\int_a^b w^{(m)}(x)\bar{w}^{(m)}(x)dx = \int_a^b q(x)\,|w(x)|^2\,dx\,.$$

By (38) and assumption (b) it follows that

$$\int_a^b |w^{(m)}(x)|^2\,dx \leqq \int_a^b P(x)\,|w(x)|^2\,dx\,. \tag{39}$$

If we write $w(x) = \sigma(x) + i\tau(x)$, both σ and τ have zeros of multiplicity m at a and b and we have $|w^{(m)}|^2 = [\sigma^{(m)}]^2 + [\tau^{(m)}]^2$. (39) becomes

$$\int\{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\}dx \leqq \int_a^b P(x)[\sigma^2(x) + \tau^2(x)]dx\,. \tag{40}$$

Let now λ be the lowest eigenvalue of the real differential system given by

$$u^{(2m)}(x) + (-1)^{m+1}\lambda P(x)u(x) = 0 \tag{41}$$

with $a \leqq x \leqq b$, $-1 < a < b < 1$, and the boundary conditions

$$\begin{aligned} u(a) &= u'(a) = \cdots = u^{(m-1)}(a) = 0 \\ u(b) &= u'(b) = \cdots = u^{(m-1)}(b) = 0 \,. \end{aligned}$$

As σ and τ are admissible comparison functions for this problem, it follows by Rayleigh's inequality that

$$\begin{aligned} \lambda \int_a^b P(x)\sigma^2(x)dx &\leqq \int_a^b [\sigma^{(m)}(x)]^2 dx \\ \lambda \int_a^b P(x)\tau^2(x)dx &\leqq \int_a^b [\tau^{(m)}(x)]^2 dx \,. \end{aligned} \tag{42}$$

Combining (42) with (40) we obtain

$$\int_a^b \{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\}dx \leqq \frac{1}{\lambda}\int_a^b \{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\}dx \,. \tag{43}$$

Hence, $\lambda \leqq 1$. If $\lambda = 1$, then equation (41) becomes (34), and the first eigenfunction of the corresponding system contradicts assumption (d). If $\lambda < 1$, we take $a < c < b$ and consider equation (41) for $a \leqq x \leqq c$, with the boundary conditions

$$\begin{aligned} u(a) &= u'(a) = \cdots = u^{(m-1)}(a) = 0 \\ u(c) &= u'(c) = \cdots = u^{(m-1)}(c) = 0 \,. \end{aligned}$$

Let $\lambda_p(c)$ be the first eigenvalue of this system. By the minimum characterization,

$$\lambda_p(c) = \operatorname{Min} \frac{\int_a^c v^{(m)^2}(x)dx}{\int_a^c Pv^2(x)dx} \,, \tag{44}$$

where the minimum is taken over the class of all functions $v(x)$ in C^m (or D^m) satisfying

$$\begin{aligned} v(a) &= v'(a) = \cdots = v^{(m-1)}(a) = 0 \\ v(c) &= v'(c) = \cdots = v^{(m-1)}(c) = 0 \,. \end{aligned}$$

Hence, $\lambda_p(c)$ is increasing as c goes from b to a. From (44) it follows that

$$\lambda_p(c) \geqq \lambda_k(c) \,, \tag{45}$$

where k is a constant satisfying

$$k > P(x) > 0 \quad \text{in} \quad [a, b] \,.$$

Denoting $c - a = l$ and $lt = x - a$, the system

$$\begin{aligned} & u^{(2m)}(x) + (-1)^{m+1}\lambda k u(x) = 0 \\ & u(a) = \cdots = u^{(m-1)}(a) = 0 \\ & u(c) = \cdots = u^{(m-1)}(c) = 0 \end{aligned}$$

is transformed into the system

$$\begin{aligned} & u^{(2m)}(t) + (-1)^{m+1}\Lambda k u(t) = 0 \\ & u(0) = \cdots = u^{(m-1)}(0) = 0 \\ & u(1) = \cdots = u^{(m-1)}(1) = 0 . \end{aligned}$$

Denoting the first eigenvalue of this system by Λ_k, it follows that

$$\Lambda_k = \lambda_k(c) l^{2m} . \tag{46}$$

From (45) and (46) it follows that as c goes to a $(l \to 0)$, $\lambda_p(c)$ tends to ∞. Hence, there exists a value c_1, $a < c_1 < b$, such that $\lambda(c_1) = 1$, and we again obtain a contradiction to our assumption (d). This completes the proof of Theorem 5.

For $m = 1$ Theorem 5 reduces to [5, Th. 1].

We bring now some examples. For $m = 2$, i.e. for the differential equation of the fourth order,

$$y^{(4)}(z) - p(z)y(z) = 0 ,$$

the following functions may serve as examples in Theorem 5 :

$$P_1(x) = (0.753\,\pi)^4 = 31.28 \cdots , \tag{47}$$

$$P_2(x) = \frac{9}{(1 - x^2)^4} \tag{48}$$

and

$$P_3(x) = \frac{24}{(1 - x^2)^2} . \tag{49}$$

$P_1(x)$, $P_2(x)$ and $P_3(x)$ clearly satisfy assumptions (a), (b), (c) of the theorem. In order to show that $P_1(x)$ satisfies assumption (d), we consider the equation

$$u^{(4)}(x) - k^4 u(x) = 0 ,$$

which has $u(x) = C_1 \cos kx + C_2 \sin kx + C_3 \cos hkx + C_4 \sin hkx$ as general solution. The requirement $u(\pm 1) = u'(\pm 1) = 0$ implies $\tan hk = \pm \tan k$, the smallest solution of which is $k = 2.3650 = 0.753\,\pi$. The equation

$$u^{(4)}(x) - (0.753\,\pi)^4 u(x) = 0$$

has therefore a solution with double zeros at ± 1; in other words, the first eigenvalue λ_1 of the system

$$(50) \qquad \begin{aligned} &u^{(4)}(x) - \lambda(0.753\,\pi)^4 u(x) = 0 \\ &u(\pm 1) = u'(\pm 1) = 0 \end{aligned}$$

equals 1. As for any a, b, $-1 < a < b < 1$, the eigenvalues of the system

$$(51) \qquad \begin{aligned} &u^{(4)}(x) - \lambda(0.753\,\pi)^4 u(x) = 0 \\ &u(a) = u'(a) = u(b) = u'(b) = 0 \end{aligned}$$

are greater than the eigenvalues of (50), the system (51) cannot have an eigenvalue equal to 1. $P_1(x)$ thus satisfies assumption (d).

The following inequalities due to Beesack [1, p. 494]

$$\int_{-1}^{1} v''^2 dx > \int_{-1}^{1} \frac{9v^2}{(1-x^2)^4}\, dx\ , \quad v \in D''\ , \quad v(\pm 1) = v'(\pm 1) = 0$$

unless $v = A(1-x^2)^{3/2}$, and

$$\int_{-1}^{1} v''^2 dx > \int_{-1}^{1} \frac{24v^2}{(1-x^2)^2}\, dx\ , \quad v \in D''\ , \quad v(\pm 1) = v'(\pm 1) = 0$$

unless $v = A(1-x^2)^2$, imply that $P_2(x)$ and $P_3(x)$ satisfy assumption (d).

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