THE FROBENIUS RECIPROCITY THEOREM AND ESSENTIALLY BOUNDED INDUCED REPRESENTATIONS

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Let G be a locally compact group with continuous unitary representations R_i acting on the Hilbert spaces $H(R_i)$, $i=1, 2$. Suppose that R_2 is irreducible. A closed subspace of $H(R_1)$, called the null space coming from R_1 at R_2 and denoted $F(R_1; R_2)$, is defined. Write $\mathscr{H}^c$ for the conjugate space of the Hilbert space $\mathscr{H}$. The following theorem is proved.

THEOREM 1. Let G be a compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively. Let U^L be the induced representation of L, and let M_H be the restriction of M to H. Then the following are equivalent:
  (a) The classical Frobenius Reciprocity Theorem.
  (b) $H(M) \otimes F(M_H; L)^c \cong F(U^L; M) \otimes H(L)^c$.

When G is not compact, both (a) and (b) may fail. A non-Hilbert Banach space induced representation (W^L) is defined. Let G be a locally compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively, where $H(L)$ is separable. Spaces $F_0(W^L; M) \otimes^* H(L)^c$ (shown to equal $F(U^L; M) \otimes H(L)^c$ when G is compact) and $QF(M_H; L)$ shown to equal $F(M_H; L)$ when L is compact) are defined. The following generalization of (b) is shown.

THEOREM 2. Let G, H, L be as above. Then

$$H(M) \otimes QF(M_H; L)^c \cong F_0(W^L; M) \otimes^* H(L)^c .$$

As these results indicate, the purpose of this paper is to establish yet another generalization of the Frobenius Reciprocity Theorem (FRT). The genesis of our approach is two-fold. The idea of employing the null space was suggested by Blattner's study [1] of the Glimm projection-valued measure. The notion of defining a non-Hilbert Banach space induced representation seemed natural in light of the work of Moore [8] and Kleppner [7].

We recall the following basic concepts. Suppose G is a locally compact group (we denote its neutral element by e). Endowing G with right invariant Haar measure, one uses the familiar notions of convolution and $*$ to define two Banach algebras, $L_1(G)$ (= the L_1-group algebra of G with norm denoted by $|| \cdot ||_1$) and $C^*(G)$

(= the C^*-group algebra of G with norm denoted by $||\cdot||_*$). Let R be a unitary (always assumed continuous) representation of G, and denote always by $H(R)$ the Hilbert space on which R acts. In terms of such a representation R of G, we define nondegenerate continuous *-representations (denoted by R again) of $L_1(G)$ and $C^*(G)$ in the usual way. The representation theories of $L_1(G)$ and $C^*(G)$ mirror that of G. The reader is referred to [1] for a precise statement of these facts.

1. **The null space.** Suppose $\mathscr{A}$ is a C^*-algebra. The Jacobson structure space of $\mathscr{A}$, written $\mathscr{A}^*$, is defined to be the space of kernels of irreducible nondegenerate *-representations of $\mathscr{A}$ endowed with the hull-kernel topology. If R is a *-representation acting on the Hilbert space $H(R)$, Glimm [5] has shown that there exists a unique projection-valued measure $E(R; \cdot)$ on the Borel field generated by the topology of $\mathscr{A}^*$ such that if C is a closed subset of $\mathscr{A}^*$, $E(R; C)$ is the projection on the manifold

$$F(R; C) = \{v \in H(R): R(a)v = 0 \text{ for all } a \in \bigcap C\} \ .$$

$E(R; \cdot)$ takes its values in the center of the von-Neumann algebra generated by $R(\mathscr{A})$.

Let G be a locally compact group with unitary representations R_1, R_2; suppose R_2 is irreducible. Regarding R_1 and R_2 as representations of $C^*(G)$ as well as of G, we consider the space $F(R_1; \text{Cl}\{\text{Ker } R_2\})$, where $\text{Cl}\{\text{Ker } R_2\}$ is the closure in the structure space topology of the point-set containing the kernel of the representation R_2, $\text{Ker } R_2$. Because $\text{Ker } R_2 \in \text{Cl}\{\text{Ker } R_2\}$, we have

$$F(R_1; \text{Cl}\{\text{Ker } R_2\}) = \{v \in H(R_1): R_1(f)v = 0 \text{ for all } f \in \text{Ker } R_2\} \ .$$

The projection $E(R_1; \text{Cl}\{\text{Ker } R_2\})$ will be called the *null projection coming from* R_1 *at* R_2, and will be denoted by $E(R_1; R_2)$. The space $F(R_1; \text{Cl}\{\text{Ker } R_2\})$ will be called the *null space coming from* R_1 *at* R_2 and will be denoted by $F(R_1; R_2)$.

LEMMA 1. *Let* G *be a compact group. Let* R_1 *be a unitary representation of* G *which contains a unitary irreducible representation* R_2 *of* G *exactly* n *times. Then* $F(R_1; R_2) \cong nH(R_2)$.

Proof. For a compact group, the fact that R_1 contains R_2 exactly n times means that R_1 is unitarily equivalent to a discrete direct sum of representations as follows: $R_1 \cong nR_2 \oplus (\bigoplus_\alpha R_\alpha)$, where each R_α is an irreducible unitary representation of G disjoint from R_2. For $v \in H(R_1)$, write

$$v = \bigoplus_{i=1}^{n} v_i \oplus (\bigoplus_\alpha v_\alpha), \; v_i \in H(R_2), \; v_\alpha \in H(R_\alpha) \; ,$$

whence

$$F(R_1; R_2) = \bigoplus_{i=1}^{n} \{v_i \in H(R_2): R_2(f)v_i = 0 \text{ for all } f \in \text{Ker } R_2\}$$

$\bigoplus \bigoplus_\alpha \{v_\alpha \in H(R_\alpha): R_\alpha(f)v = 0 \text{ for all } f \in \text{Ker } R_2\}$. Since the first n summands clearly sum to $nH(R_2)$, we need only show that for each α, the spaces $F(R_\alpha; R_2)$ consist of zero only. Because $E(R_\alpha; R_2)$ commutes with $R_\alpha(f)$ for all $f \in C^*(G)$, Schur's lemma implies that $E(R_\alpha; R_2)$ is either the zero operator or the identity operator I_α on $H(R_\alpha)$. If $E(R_\alpha; R_2) = I_\alpha$, then $H(R_\alpha) = F(R_\alpha; R_2)$, so that $\text{Ker } R_\alpha \supseteqq \text{Ker } R_2$. This implies (see Fell [4], p. 378) that $\text{Ker } R_\alpha \cong \text{Ker } R_2$, which in turn implies (see Glimm [6], p. 583) that $R_2 \cong R_\alpha$. This contradiction means $F(R_\alpha; R_2) = \{0\}$, as we wanted.

Let G be a locally compact group with unitary representation R_1 and irreducible unitary representation R_2. Since we will deal with the L_1-group algebra of G, let Ker_1 denote the kernel of a *-representation of $L_1(G)$. The closed subspace $F_1(R_1; R_2)$ of $H(R_1)$ defined by $F_1(R_1; R_2) = \{v \in H(R_1): R_1(f)v = 0 \text{ for all } f \in \text{Ker}_1 R_2\}$ will be called the L_1-*null space coming from* R_1 *at* R_2. The projection $E(R_1; R_2)$ on $F_1(R_1; R_2)$ commutes with $R_1(f)$ for all $f \in L_1(G)$. The results of Fell and Glimm cited in the proof of Lemma 1 can be shown to remain valid when $C^*(G)$ is replaced by $L_1(G)$ if R_2 is of finite degree. If G is a compact group, the fact that all of its irreducible representations are of finite degree will therefore mean that Lemma 1 *remains true when* $F(R_1; R_2)$ *is replaced by* $F_1(R_1; R_2)$.

Suppose now that G is a compact group with closed subgroup H. Let L be an irreducible unitary representation of H. Let $H(U^L)$ be the space of all Haar-measurable functions f on G with values in $H(L)$ such that (i) $f(x\gamma) = L(x)f(\gamma)$ for all $x \in H$, $\gamma \in G$, and (ii) $\int_G \|f(\gamma)\|^2 d\gamma < \infty$. As usual, let U^L, the induced representation of L, be the representation of G acting on $H(U^L)$, where $(U^L(\gamma)f)(\gamma') = f(\gamma'\gamma)$ for all $f \in H(U^L)$, γ, $\gamma' \in G$. Suppose M is an irreducible unitary representation of G; let M_H be the restriction of M to H. Suppose U^L contains M exactly n times. By Lemma 1 and the FRT, we have $F(U^L; M) \cong nH(M)$ and $F(M_H; L) \cong nH(L)$. Using a trivial dimension argument, we have

$$* \quad H(M) \otimes F(M_H; L) \cong F(U^L; M) \otimes H(L),$$ where "$\otimes$" denotes the algebraic tensor product. Since $H(L)$ and $H(M)$ are finite-dimensional, it is clear that * implies the FRT. Letting $\mathscr{H}^c$ be the conjugate space of a Hilbert space $\mathscr{H}$ (so that $\dim \mathscr{H} = \dim \mathscr{H}^c$), we get

THEOREM 1. *Let* G *be a compact group with closed subgroup* H. *Let* M, L *be irreducible unitary representations of* G, H, *re-*

spectively. Then the following statements are equivalent:
  (a) *The Frobenius Reciprocity Theorem.*
  (b) $H(M) \otimes F(M_H; L)^c \cong F(U^L; M) \otimes H(L)^c.$
  (c) $H(M) \otimes F_1(M_H; L)^c \cong F_1(U^L; M) \otimes H(L)^c.$

2. Essentially bounded induced representations. It is quite easy to see that statements (b) and (c) of Theorem 1 are false if we abandon the assumption that G is compact. Indeed, let G be the group of real numbers under addition and let $H = \{0\}$; let M be any character of G and let L be the irreducible identity representation of H. Then U^L is the right regular representation of G acting on $L_2(G)$ and $F(U^L; M)$ is the trivial subspace of $L_2(G)$ having dimension zero, whereas $F(M_H; L)$ has dimension one. However, if one considers the right regular representation of G acting on $L_\infty(G)$ instead of $L_2(G)$ one can define in the natural way a space "$F(U^L; M)$" having dimension one. We therefore define a new type of induced representation; we argue in Proposition 1 below that this definition is reasonable by re-examining the case when G is compact.

DEFINITION 1. Let G be a locally compact group with closed subgroup H. Let L be a unitary representation of H and let $L_\infty(G; H(L))$ be the Banach space of all Haar-measurable essentially bounded functions from G to $H(L)$ (for f in $L_\infty(G; H(L))$, let $\| f \|_\infty = $ ess sup $\| f(\gamma) \|$). Let $B(W^L)$ be the Banach space consisting of all g in $L_\infty(G; H(L))$ such that $g(x\gamma) = L(x)g(\gamma)$ for all $x \in H$, $\gamma \in G$. Define a representation W^L of G on $B(W^L)$ by the rule $(W^L(\gamma)g)(\gamma') = g(\gamma'\gamma)$ for all $g \in B(W^L)$, γ, $\gamma' \in G$; W^L is called the L_∞-*induced representation of L acting on* $B(W^L)$.

It is clear that $B(W^L)$ is invariant under W^L, and that W^L maps G into the isometries on $B(W^L)$. It should be noted that W^L is not in general a strongly continuous representation of G. We can define W^L as a representation of $L_1(G)$ by the rule that for $f \in L_1(G)$, $W^L(f)$ is the operator on $B(W^L)$ sending $g \in B(W^L)$ to $W^L(f)g \in B(W^L)$, where

$$(W^L(f)g)(a) = \int_G f(\gamma)(W^L(\gamma^{-1})g)(a)d\gamma$$

$$= \int_G f(\gamma)g(a\gamma^{-1})d\gamma \ ,$$

for all $a \in G$. Because $\| W^L(f)g \|_\infty \leq \| f \|_1 \| g \|_\infty$, the representation $f \to W^L(f)$ is a strongly continuous representation of $L_1(G)$. Note that since in general $\| f \|_* \leq \| f \|_1$, it does not seem to be the case that one can similarly define a strongly continuous representation of $C^*(G)$. It is for this reason that we will have to concern ourselves

with $L_1(G)$ at the expense of $C^*(G)$.

As $L_\infty(G; H(L))$ can be regarded as the dual of $L_1(G; H(L))$ (defined in the natural way) when $H(L)$ is separable, W^L can be shown to be a *weak-* continuous* representation of G when in fact $H(L)$ is separable.

3. The case G is compact. Let G be any locally compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively. Let us define a space $F_1(W^L; M)$ in the natural way:

$$F_1(W^L; M) = \{g \in B(W^L): W^L(f)g = 0 \text{ for all } f \in \text{Ker } M\} .$$

When G is *compact*, we have the following result.

PROPOSITION 1. *Let G, H, M and L be as in Lemma 1. Then $F_1(U^L; M) = F_1(W^L; M)$.*

Proof. Since $H(L)$ is finite-dimensional, we can write (relative to some basis of $H(L)$) every $g \in H(U^L)$ as a "column-vector valued" function $g = (g_k)$, where $g_k \in L_2(G)$, $k = 1, \cdots, r = \dim H(L)$. It plainly suffices to show that $F_1(U^L; M) \subseteq F_1(W^L; M)$, so we will be done if we establish that if $g = (g_k) \in F_1(U^L; M)$ then each g_k is in fact continuous (and hence in $L_\infty(G)$).

We have $F_1(U^L; M) = \{g \in H(U^L): U^L(f)g = 0 \text{ for all } f \in \text{Ker}_1 M\}$. Let $m = \dim H(M)$. There exist m^2 continuous functions h_{ij} on G of the form $h_{ij}(\gamma) = (M(\gamma^{-1})u_i, v_j)$ $(u_i, v_j \in H(M), i, j = 1, \cdots, m)$ such that $\text{Ker}_1 M = \{f \in L_1(G): \int_G f(\gamma)h_{ij}(\gamma)d\gamma = 0, i, j = 1, \cdots, m\}$. Clearly, if $g = (g_k) \in H(U^L)$ and if each g_k is a linear combination of the continuous functions h'_{ij} (where $h'_{ij}(\gamma) = h_{ij}(\gamma^{-1})$), then $U^L(f) = 0$ for all $f \in \text{Ker}_1 M$. Conversely, we will show that if $g = (g_k) \in F_1(U^L; M)$, then each g_k is indeed a linear combination of the functions h'_{ij}.

If $g = (g_k) \in F_1(U^L; M)$, then $U^L(f)g = 0$ whenever in particular $f \in \text{Ker}_2 M(= L_2(G) \cap \text{Ker}_1 M)$. Hence, for $f \in \text{Ker}_2 M$, $k = 1, \cdots, r$, we have $\int_G f(\gamma)g_k(a\gamma^{-1})d\gamma = (f \circ g_k)(a) = 0$ for almost all $a \in G$, and since the convolutions $f \circ g_k$ are continuous we have $(f \circ g_k)(a) = 0$ for all $a \in G$. Letting $a = e$, we infer that if $U^L(f)g = 0$ for $f \in \text{Ker}_2 M$, then $\int_G f(\gamma)g_k(\gamma^{-1})d\gamma = 0$, $k = 1, \cdots, r$.

Define, for each h_{ij}, $g_k \in L_2(G)$, linear functionals F_{ij}, F_k on $L_2(G)$ as follows:

$$F_{ij}(f) = \int_G f(\gamma)h_{ij}(\gamma)d\gamma , \qquad i, j = 1, \cdots, m$$

$$F_k(f) = \int_G f(\gamma)g_k(\gamma^{-1})d\gamma , \qquad k = 1, \cdots, r .$$

Hence $\operatorname{Ker} F_k \supseteq \bigcap_{i,j=1}^{m} \operatorname{Ker} F_{ij}$, $k = 1, \cdots, r$. We conclude that there exist constants c_{kij} such that for $k = 1, \cdots, r$, we have

$$F_k = \sum_{i,j=1}^{m} c_{kij} F_{ij} ,$$

whence $g_k = \sum_{i,j=1}^{m} c_{kij} h'_{ij}$, as we wanted to show.

For purposes of generalization, we extract the following corollary from the above proof.

COROLLARY. *Let G, H, M and L be as in Lemma 1. For any u, v in $H(M)$ let $g_{u,v}$ be the function on G defined by $g_{u,v}(\gamma) = (M(\gamma)u, v)$ for all $\gamma \in G$. Let S_M be the linear span of all such functions $g_{u,v}$. Let S_M^r be the collection of all functions g mapping G into $H(L)$ (with $r = \dim H(L)$) which can be written (relative to some fixed basis of $H(L)$) as column-vector valued functions with entries $g_1, \cdots, g_r$ in S_M. Then*

$$F_1(U^L; M) = F_1(W^L; M) = \{g \in S_M^r : g(x\gamma)$$
$$= L(x)g(\gamma) \text{ for all } x \in H, \ \gamma \in G\} .$$

4. The space $F_0(W^L; M) \otimes^* H(L)^c$. Let G be a locally compact group and let $H = \{e\}$. If M is an irreducible unitary representation of G and if L is the trivial irreducible representation of H, one calculates $F_1(W^L; M)$ to be the space

$$F_1(W^L; M) = \Big\{g \in L_\infty(G) : \int_G f(\gamma)g(\gamma^{-1})d\gamma = 0 \text{ for all } f \in L_1(G) \text{ such that}$$

$$\int_G f(\gamma)(M(\gamma^{-1})u, v)d\gamma = 0 \text{ for all } u, v \in H(M)\Big\} .$$

Elementary considerations thus show that $F_1(W^L; M)$ is just the weak-* closure in $L_\infty(G)$ (= the dual of $L_1(G)$) of the space S_M = linear span $\{g_{u,v} \in L_\infty(G) : g_{u,v}(\gamma) = (M(\gamma)u, v), u, v \in H(M)\}$. Because the weak-* closure of such a space is a very difficult thing to determine precisely, we feel constrained to consider the simpler space $S_M(= F_1(W^L; M)$ when M is of finite degree).

Let us now suppose that H is any closed subgroup of G with irreducible unitary representation L of *finite degree* r. We *define* a space $F_0(W^L; M)$ as follows:

$$F_0(W^L; M) = \{g \in S_M^r : g(x\gamma) = L(x)g(\gamma) \text{ for all } x \in H, \ \gamma \in G\} ,$$

where S_M^r is as in the corollary of Proposition 1. Of course, when G is compact, $F_1(U^L; M) = F_1(W^L; M) = F_0(W^L; M)$.

Suppose $\mathscr{H}_0$ is a pre-Hilbert space with inner product $(,)$.

Define a new pre-Hilbert space $\mathscr{H}_0^c$ as follows: (1) the underlying space of $\mathscr{H}_0^c$ is $\mathscr{H}_0$; (2) if $v \in \mathscr{H}_0$, the corresponding member of $\mathscr{H}_0^c$ is denoted by $\bar{v}$; (3) scalar multiplication is given by $a\bar{v} = \overline{\bar{a}v}$; (4) the inner product is given by $(\bar{u}, \bar{v}) = (v, u)$. When $\mathscr{H}_0$ is a Hilbert space, $\mathscr{H}_0^c$ is just the conjugate space of $\mathscr{H}_0$. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be pre-Hilbert spaces containing the elements u, v, respectively. Define an operator $u \otimes \bar{v}$ of rank 1 from $\mathscr{H}_2$ into $\mathscr{H}_1$ by $(u \otimes \bar{v})w = (w, v)u$ for all $w \in \mathscr{H}_2$. The space of linear combinations of such operators is just the space of linear operators of finite rank mapping $\mathscr{H}_2$ into $\mathscr{H}_1$, and is isomorphic to the algebraic tensor product of $\mathscr{H}_1$ and $\mathscr{H}_2^c$; we denote this space by $\mathscr{H}_1 \otimes \mathscr{H}_2^c$.

We thus see that $F_0(W^L; M) \otimes H(L)^c$ can be regarded as the space of all matrix-valued functions N on G such that the matrix entries of N are functions in S_M and such that $N(x\gamma) = L(x)N(\gamma)$ for all $x \in H$, $\gamma \in G$. If $e_1, \cdots, e_r$ is an orthonormal basis of $H(L)$ and if $N \in F_0(W^L; M) \otimes H(L)^c$, then the matrix entries of N, $(N(\cdot)e_i, e_j)$, $i, j = 1, \cdots, r$, have the form

$$(N(\cdot)e_i, e_j) = \sum_k (M(\cdot)v_{kij}, u_{kij}), \quad v_{kij}, u_{kij} \in H(M) \ .$$

Since the collection of all these v_{kij} is finite, we can write

$$(N(\cdot)e_i, e_j = \sum_{k=1}^n (M(\cdot)v_k, u'_{kij}) \ ,$$

where the v_k are linearly independent and independent of the particular indices i, j. These considerations lead naturally to a definition.

DEFINITION 2. Let G be a locally compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively, where $H(L)$ is separable. Define the space $F_0(W^L; M) \otimes^* H(L)^c$ to be the space of all functions N on G such that:

(a) For all $\gamma \in G$, $N(\gamma)$ is a trace-class operator on the Hilbert space $H(L)$.

(b) $N(x\gamma) = L(x)N(\gamma)$ for all $x \in H$, $\gamma \in G$.

(c) There exists an orthonormal basis (ONB) $\{e_i\}$ of $H(L)$ and a finite collection of linearly independent elements $v_1, \cdots, v_n$ in $H(M)$ such that

$$(N(\cdot)e_i, e_j) = \sum_{k=1}^n (M(\cdot)v_k, u_{kij}) \ , \qquad i, j = 1, 2, \cdots \ .$$

It is clear that $F_0(W^L; M) \otimes^* H(L)^c = F_1(W^L; M) \otimes H(L)^c$ when L is of finite degree.

We will need a few more definitions and results. If $\mathscr{H}$ is a Hilbert space, let $\mathscr{L}(\mathscr{H})$ be the space of all the bounded linear operators mapping $\mathscr{H}$ into itself and let $\mathscr{L}_1(\mathscr{H})$ be the subspace of $\mathscr{L}(\mathscr{H})$ consisting of all the trace-class operators in $\mathscr{L}(\mathscr{H})$. If Tr denotes the usual trace functional and if $T \in \mathscr{L}_1(\mathscr{H})$, the functional $A \to \mathrm{Tr}\,(TA)$ on $\mathscr{L}(\mathscr{H})$ is continuous in the ultraweak topology of $\mathscr{L}(\mathscr{H})$ (see [2]).

LEMMA 2. *If G is a group with irreducible unitary representation R and if $T \in \mathscr{L}_1(H(R))$ satisfies $\mathrm{Tr}\,(TR(\gamma)) = 0$ for all $\gamma \in G$, then $T = 0$.*

Proof. By the von Neumann density theorem, the space $\mathfrak{A} = $ linear span $\{R(\gamma)\colon \gamma \in G\}$ is ultraweakly dense in $\mathscr{L}(H(R))$. Since $\mathrm{Tr}(TA) = 0$ for all $A \in \mathfrak{A}$, the ultraweak continuity of $\mathrm{Tr}\,(T\cdot)$ implies $\mathrm{Tr}\,(TA) = 0$ for all $A \in \mathscr{L}(H(R))$. Therefore $\mathrm{Tr}\,(TT^*) = 0$, so that $T = 0$.

We finally recall the following special form of the *principle of n-fold transitivity*: Let G be a group with irreducible representation R and let $\mathfrak{A} = $ linear span $\{R(\gamma)\colon \gamma \in G\}$. Let $v_1, \cdots, v_n$ be a collection of linearly independent elements of $H(R)$ and let v be any other element in $H(R)$. Then there exist operators $A_k \in \mathfrak{A}$ such that $A_k v_i = \delta_{ik} v$, $i, k = 1, \cdots, n$, where δ_{ik} is the Kronecker delta.

PROPOSITION 2. *Let G be a locally compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively, where $H(L)$ is separable. Suppose N is a function mapping G into $\mathscr{L}_1(H(L))$ such that $N(x\gamma) = L(x)N(\gamma)$ for all $x \in H$, $\gamma \in G$. Then the following two statements are equivalent:*

(i) *There exists $A \in H(M) \otimes H(M)^c$ such that $\mathrm{Tr}\,(N(\gamma)) = \mathrm{Tr}\,(M(\gamma)A)$ for all $\gamma \in G$.*

(ii) *$N \in F_0(W^L; M) \otimes^* H(L)^c$.*

Proof. (a) (i) $\Rightarrow$ (ii). Writing A in the form $A = \sum_{i=1}^n v_i \otimes \bar{u}_i$, where $v_1, \cdots, v_n$ are linearly independent elements in $H(M)$, we have, for all $\gamma \in G$,

$$\mathrm{Tr}\,(N(\gamma)) = \sum_{i=1}^n (M(\gamma)v_i, u_i)\,.$$

Let v be any element of $H(M)$. By the principle of n-fold transitivity there exist operators $A_{v,k}$, $k = 1, \cdots, n$, of the form $\sum_p a_p M(\gamma_p)$ such that $A_{v,k} v_i = \delta_{ik} v$. For $k = 1, \cdots, n$, $\gamma \in G$, we define operators

$N(v, k; \gamma) = \sum_p a_p N(\gamma \gamma_p)$ satisfying conditions (a), (b) of Definition 2. We have

$$\text{Tr } (N(v, k; \gamma)) = \sum_{i=1}^{n} (M(x\gamma)A_{v,k}v_i, u_i) = (M(\gamma)v, u_k) \ .$$

Hence, for all $x \in H$, we see that

$$\text{Tr } \left(\left(\sum_{i=1}^{n} N(v_i, i; \gamma) \right) L(x) \right) = \sum_{i=1}^{n} (M(x\gamma)v_i, u_i) = \text{Tr } (N(\gamma)L(x)) \ .$$

Lemma 2 implies $N(\cdot) = \sum_{i=1}^{n} N(v_i, i; \cdot)$.

Let $\{e_i\}$ be an ONB of $H(L)$. We will have shown that N satisfies (c) of Definition 2, if, for each $k = 1, \cdots, n$, it is the case that $(N(v_k, k; \cdot)e_i, e_j) = (M(\cdot)v_k, u_{kij})$, where $u_{kij} \in H(M)$. To show this, observe that for all $v \in H(M)$, $x \in H$, $\gamma \in G$,

$$\overline{\text{Tr } (N(v, k; \gamma)^* L(x))} = \text{Tr } (N(v, k; x^{-1}\gamma)) = (M(\gamma)v, M(x)u_k) \ .$$

Hence, for $x_p \in H$ and scalars b_p, we have

$$\text{Tr } \left(N(v \cdot k; e)^* \sum_p b_p L(x_p) \right) = \left(\sum_p b_p M(x_p)u_k, v \right) \ .$$

By the von Neumann density theorem, the space of operators of the form $\sum_p b_p L(x_p)$ is ultraweakly dense in $\mathscr{L}(H(L))$. Choose a sequence of operators A_{nij} (of the form $\sum_p b_{np} L(x_{np})$) such that $A_{nij} \to E_{ij}$ ultraweakly (where E_{ij} is defined by $E_{ij}e_m = \delta_{im}e_j$). Let $w_{nij} = \sum_p b_{np} M(x_{np})u_k$. Since $\text{Tr } (N(v, k; e)^* A_{nij}) = (w_{nij}, v)$ for all $v \in H(M)$, the ultraweak continuity of the trace functional implies that

$$| (w_{nij} - w_{mij}, v) | \to 0$$

as $n, m \to \infty$; i.e., $\{w_{nij}\}$ is a weak Cauchy sequence. Since $H(M)$ is reflexive and thus weakly complete, there exists $u_{kij} \in H(M)$ such that

$$(u_{kij}, M(\gamma)v) = \lim_{n \to \infty} (w_{nij}, M(\gamma)v) = \lim_{n \to \infty} \text{Tr } (N(v, k; \gamma)^* A_{nij})$$
$$= \text{Tr } (N(v, k; \gamma)^* E_{ij}) = N(v, k; \gamma)^* e_j, e_i) \ .$$

This is,

$$(N(v_k, k; \gamma)e_i, e_j) = (M(\gamma)v_k, u_{kij}) \ .$$

  (b) (ii) $\Rightarrow$ (i). If $N \in F_0(W^L; M) \bigotimes^* H(L)^c$, there exist linearly independent elements $v_1, \cdots, v_n$ in $H(M)$, an ONB $\{e_i\}$ of $H(L)$, and elements $u_{kij} \in H(M)$ such that

$$(N(\gamma)e_i, e_j) = \sum_{k=1}^{n} (M(\gamma)v_k, u_{kij}) \ , \qquad i, j = 1, 2, \cdots \ .$$

Exactly as in (a), use n-fold transitivity to define, for all $v \in H(M)$, $k = 1, \cdots, n$, functions $N(v, k; \cdot) \in F_0(W^L; M) \bigotimes^* H(L)^e$ satisfying

$$(N(v, k; \cdot)e_i, e_j) = (M(\cdot)v, u_{kij}), \qquad i, j = 1, 2, \cdots.$$

Let E_m be the projection on the space spanned by $e_1, \cdots, e_m$; since $H(L)$ is separable, these E_m converge ultraweakly to the identity operator on $H(L)$. We see that

$$\mathrm{Tr}\,(N(v, k; e)E_m) = \sum_{i=1}^m (v, u_{kii}) = (v, u_{km}),$$

where the $u_{km} = \sum_{i=1}^m u_{kii}$ must converge weakly to an element u_k in $H(M)$ and $\mathrm{Tr}\,(N(v, k; e)) = (v, u_k)$. For $v = M(\gamma)v_k$ (so that $N(v, k; e) = N(v_k, k; \gamma)$) we see that

$$\mathrm{Tr}\,(N(v_k, k; \gamma)) = (M(\gamma)v_k, u_k) = \mathrm{Tr}\,(M(\gamma)(v_k \otimes \bar{u}_k)).$$

Letting $A = \sum_{k=1}^n v_k \otimes \bar{u}_k \in H(M) \bigotimes H(M)^e$, we have, for all $\gamma \in G$, $\mathrm{Tr}\,(N(\gamma)) = \mathrm{Tr}\,(\sum_{k=1}^n N(v_k, k; \gamma)) = \mathrm{Tr}\,(M(\gamma)A)$.

5. **A quasi-null space.** Let G be a locally compact group with unitary representations R_1 and R_2, where R_2 is irreducible. We will define a subspace $QF(R_1; R_2)$ of $H(R_1)$, called the *quasi-null space coming from* R_1 *at* R_2, as follows: $QF(R_1; R_2) = \{u \in H(R_1):$ for all $\gamma \in G$ and all $v \in H(R_1)$, there exists an operator $T(u, v) \in \mathscr{L}_1(H(R_2))$ such that $(R_1(\gamma)u, v) = \mathrm{Tr}\,(T(u, v)^*R_2(\gamma))\}$. If $u \in QF(R_1; R_2)$, we have for all $v \in H(R_1)$ and all $f \in L(G)$ that

$$(R_1(f)u, v) = \int_G f(\gamma)(R_1(\gamma^{-1})u, v)d\gamma$$
$$= \mathrm{Tr}\,(T(u, v)^*R_2(f)),$$

so that $(R_1(f)u, v) = \mathrm{Tr}\,(T(u, v)^*R_2(f))$ for all $f \in C^*(G)$. Thus $f \in \mathrm{Ker}\,R_2$ implies $R_1(f)u = 0$. We see that $QF(R_1; R_2) \subseteqq F(R_1; R_2)$.

PROPOSITION 3. *Let G be a locally compact group. Let R_1 be a unitary representation of G and let R_2 be an irreducible unitary representation of G such that $R_2(f)$ is a compact operator for all $f \in C^*(G)$. Then $QF(R_1; R_2) = F(R_1; R_2)$.*

Proof. Let $u \in F(R_1; R_2)$. We have $(R_1(f)u, v) = 0$ for all $v \in H(R_1)$ and for all $f \in \mathrm{Ker}\,R_2$. For any $v \in H(R_2)$, the continuous functional $F_{u,v}$ on $C^*(G)$ defined by $F_{u,v}(f) = (R_1(f)u, v)$ therefore induces a continuous functional $F'_{u,v}$ on the quotient space $C^*(G)/\mathrm{Ker}\,R_2$. As R_2 is compact, the mapping $f \to R_2(f)$ maps $C^*(G)$ onto $\mathscr{L}_c(H(R_2))$

(= the space of compact operators on $H(R_2)$), and thus induces an isomorphism

$$R_2^\sharp: C^*(G)/\mathrm{Ker}\, R_2 \cong \mathscr{L}_c(H(R_2)) \;.$$

Since $\mathscr{L}_c(H(R_2))$ is a C^*-algebra this isomorphism is an isometry ([3, pp. 18, 82]). For $f \in C^*(G)$, let $f^\sharp$ denote its equivalence class in $C^*(G)/\mathrm{Ker}\, R_2$. Using the well-known fact that the dual of $\mathscr{L}_c(H(R_2))$ is $\mathscr{L}_1(H(R_2))$ ([3, p. 82]), we conclude that there exists a trace-class operator $T(u, v)^*$ such that $F'_{u,v}(f^\sharp) = \mathrm{Tr}\,(T(u, v)^* R_2^\sharp(f^\sharp))$ for all $f^\sharp \in C^*(G)/\mathrm{Ker}\, R_2$; i.e., $(R_1(f)u, v) = \mathrm{Tr}\,(T(u, v)^* R_2(f))$ for all $f \in C^*(G)$. A simple convergence argument shows that $(R_1(\gamma)u, v) = \mathrm{Tr}\,(T(u, v)^* R_2(\gamma))$ for all $\gamma \in G$; i.e., $u \in QF(R_1; R_2)$.

REMARK. When R_2 above is of finite degree, one shows by a similar technique that $QF(R_1; R_2) = F_1(R_1; R_2)$.

6. A Frobenius reciprocity theorem. Let G be a locally compact group with closed subgroup H. Let M, L be irreducible unitary representations of G, H, respectively. For $A \in H(M) \otimes H(M)^c$, write $A = \sum_{i=1}^n v_i \otimes \bar{u}_i$, where $v_1, \cdots, v_n$ are linearly independent in $H(M)$. Let us observe the equivalence of the following two statements:
   (i) For each $\gamma \in G$, there exists $T_A(\gamma) \in \mathscr{L}_1(H(L))$ such that $\mathrm{Tr}\,(M(x\gamma)A) = \mathrm{Tr}\,(T_A(\gamma)L(x))$ for all $x \in H$.
   (ii) $A \in H(M) \otimes QF(M_H; L)^c$.
   To show (i) $\Rightarrow$ (ii), note that for all $x \in H$, $\gamma \in G$,

$$\mathrm{Tr}\,(M(x^{-1}\gamma)A) = \sum_{i=1}^n (M(x^{-1}\gamma)v_i, u_i) = \mathrm{Tr}\,(T_A(\gamma)L(x^{-1})) \;.$$

As in the proof of Proposition 2, use n-fold transitivity to find, for all $v \in H(M)$, $k = 1, \cdots, n$, operators of the form $\sum_p a_p M(\gamma_p)$, for which

$$(M(x)u_k, v) = \mathrm{Tr}\,(T(u_k, v)^* L(x)) \;,$$

where

$$T(u_k, v) = \sum_p a_p T_A(\gamma_p) \in \mathscr{L}_1(H(L)) \;.$$

Conversely, for $A = \sum_{i=1}^n v_i \otimes \bar{u}_i$ with $u_1, \cdots, u_n \in QF(M_H; L)$, if we let $T_A(\cdot) = \sum_{i=1}^n T(u_i, M(\cdot)v_i)$ statement i follows.
   It is easy to see that $T_A(\cdot)$ in i is unique. Indeed, if $T'_A(\cdot)$ also satisfies i, the fact that $\mathrm{Tr}\,((T_A(\gamma) - T'_A(\gamma))L(x)) = 0$ for all $x \in H$, $\gamma \in G$, implies, by Lemma 2, that $T'_A = T_A$. It follows that the map $A \to T_A$ is linear.

THEOREM 2. *Let G be a locally compact group with closed subgroup H. Let M be an irreducible unitary representation of G and L an irreducible unitary representation of H, where $H(L)$ is separable. Then the linear map $A \to T_A$ maps $H(M) \otimes QF(M_H; L)^c$ isomorphically onto $F_0(W^L; M) \otimes^* H(L)^c$.*

Proof. The map $A \to T_A(\cdot)$ is injective, since $T_A(\cdot) = 0$ implies $\text{Tr}\,(M(\gamma)A) = 0$ for all $\gamma \in G$; by Lemma 2, $A = 0$. Let $A \in H(M) \otimes QF(M_H; L)^c$. For all $x, y \in H$, $\gamma \in G$, we have $\text{Tr}\,(T_A(x\gamma)L(y)) = \text{Tr}\,(M(y)M(x\gamma)A) = \text{Tr}\,(L(x)T_A(\gamma)L(y))$. By Lemma 2, $T_A(x\gamma) = L(x)T_A(\gamma)$. By Proposition 2, the fact that $\text{Tr}\,(T_A(\gamma)) = \text{Tr}\,(M(\gamma)A)$ for all $\gamma \in G$ implies that $T_A \in F_0(W^L; M) \otimes^* H(L)^c$.

To see that $A \to T_A$ is surjective, let $N \in F_0(W^L; M) \otimes^* H(L)^c$. By Proposition 2, there exists $A \in H(M) \otimes H(M)^c$ such that for all $x \in H$, $\gamma \in G$, $\text{Tr}\,(N(\gamma)L(x)) = \text{Tr}\,(N(x\gamma)) = \text{Tr}\,(M(x\gamma)A)$, so that by the equivalence of statements (i) and (ii) preceding this theorem, we have $A \in H(M) \otimes QF(M_H; L)^c$. By the uniqueness of T_A we must have $N = T_A$, as we needed to show.

We have already noted that when L and M are of finite degree, we have $F_0(W^L; M) = F_1(W^L; M)$ and $QF(M_H; L) = F(M_H; L) = F_1(M_H; L)$. Indeed, if L is compact, $QF(M_H; L) = F(M_H; L)$. Further, if G is compact, $F_1(W^L; M) = F_1(U^L; M) = F(U^L; M)$. In light of Theorem 1, we have a complete generalization as well as another proof of the classical Frobenius Reciprocity Theorem.

It is to be hoped that this approach somehow lays bare the skeleton of the structure of this reciprocity situation, and, moreover, that the quasinull space yields some information as the manner in which an irreducible representation is contained in some other representation.

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