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A GENERALIZATION OF INJECTIVITY

JOHN A. BEACHY

In a category of modules the notions of ρ-injectivity (with respect to a torsion radical ρ) and quasi-injectivity can be generalized to a notion of injectivity with respect to two preradicals simultaneously. Using this general definition an analog of Baer's condition for injectivity is obtained, as well as other generalizations of results for injective and quasi-injective modules. An alternate approach (not requiring the existence of injective envelopes) is given for abelian categories, with the results stated in dual form for projectivity.

In the first section of the paper we give some preliminary definitions and results, including a definition of density with respect to a preradical which is weaker than the standard one, and the definitions of preradicals rad^M and Rad^M associated with a module M (the smallest preradical and smallest torsion preradical, respectively, for which M is torsion). In the second section we define and study the notion of (ρ, σ)-injectivity, for preradicals ρ and σ. A module Q is called (ρ, σ)-injective if every homomorphism $f: N_0 \rightarrow Q$, where N_0 is a ρ-dense submodule of N and $\ker(f)$ is σ-dense in N, can be extended to N. This definition is motivated by a theorem of L. Fuchs [3, Lemma 1] giving a characterization of quasi-injectivity. Many of the results are motivated by those of G. Azumaya in his paper on M-projective and M-injective modules [1]. We prove that a module is M-injective if and only if it is (ρ, σ)-injective, where ρ is the identity functor and σ is either rad^M or Rad^M. This approach depends heavily on the existence of injective envelopes in categories of modules. In the third section of the paper we drop this assumption and obtain slightly weaker results valid in any abelian category. These results are stated in their dual form, for projectivity, and we show that our definition specializes, for modules with a projective cover, to that of M-projectivity.

1. Preliminary definitions and results. We will use the terminology of J.-M. Maranda [6]. A subfunctor ρ of the identity functor on an abelian category $\underline{A}$ is called a preradical of $\underline{A}$. Thus a preradical ρ of $\underline{A}$ assigns to each object A of $\underline{A}$ a subobject $\rho(A)$ and to each morphism $f: A \rightarrow B$ in $\underline{A}$ its restriction $\rho(f): \rho(A) \rightarrow \rho(B)$. It is said to be idempotent if $\rho^2 = \rho$ and is called a torsion preradical if ρ is left exact. If $\rho(A/\rho(A)) = 0$ for all A in $\underline{A}$, then ρ is called a radical. An object A is ρ-torsion if $\rho(A) = A$ and ρ-torsionfree if $\rho(A) = 0$.

We are primarily interested in the category ${}_R\underline{M}$ of unital left R-modules over an associative ring R with identity. Any preradical ρ of ${}_R\underline{M}$ defines a closure operation on submodules in the following way: for a submodule N_0 of ${}_RN$ let the closure $C_\rho(N_0: N)$ of N_0 in N be the inverse image of $\rho(N/N_0)$ under the projection $p: N \to N/N_0$. The ρ-closure of N in its injective envelope $E(N)$ will be written simply as $C_\rho(N)$. A submodule N_0 of N is said to be ρ-closed in N if $C_\rho(N_0: N) = N_0$. Note that this is the case if and only if N/N_0 is ρ-torsionfree. This closure operation has the property that if $M, N \in {}_R\underline{M}$ with submodules M_0, N_0 respectively and $f \in \mathrm{Hom}_R(M, N)$, then $f(M_0) \subseteqq N_0$ implies $f(C_\rho(M_0: M)) \subseteqq C_\rho(N_0: N)$. This can be shown by considering the homomorphism from M/M_0 to N/N_0 induced by f. Associated with the closure operation is a notion of density.

DEFINITION 1.1. Let ρ be a preradical of ${}_R\underline{M}$. A submodule N_0 of a module ${}_RN$ is said to be ρ-dense in N if there exists an extension ${}_RM$ of N such that $N \subseteqq C_\rho(N_0: M)$.

LEMMA 1.2. *A submodule $N_0 \subseteqq {}_RN$ is ρ-dense in*

$$N \Longleftrightarrow N \subseteqq C_\rho(N_0\colon E(N)) .$$

Proof. If N_0 is ρ-dense in N then there exists an extension M of N with $N \subseteqq C_\rho(N_0: M)$. The inclusion $N \to E(N)$ extends to a homomorphism $f: M \to E(N)$, since $E(N)$ is injective, and then $f(N_0) \subseteqq N_0$ implies that $N = f(N) \subseteqq f(C_\rho(N_0: M)) \subseteqq C_\rho(N_0: E(N))$. The converse follows immediately from the definition.

PROPOSITION 1.3. *Let ρ be a preradical of ${}_R\underline{M}$, and N_1, N_2 be submodules of the module ${}_RN$, with $N_1 \subseteqq N_2 \subseteqq N$.*

(a) *If N_1 is ρ-dense in N, then N_2 is ρ-dense in N.*

(b) *If N_2 is ρ-dense in N, then N_2/N_1 is ρ-dense in N/N_1.*

Proof. (a) If N_1 is ρ-dense in N, then by Lemma 1.2 we have $N \subseteqq C_\rho(N_1: E(N))$. Since $N_1 \subseteqq N_2$ implies $C_\rho(N_1: E(N)) \subseteqq C_\rho(N_2: E(N))$, it follows that $N \subseteqq C_\rho(N_2: E(N))$, and N_2 is ρ-dense in N.

(b) If N_2 is ρ-dense in N, then since $N \subseteqq C_\rho(N_2: E(N))$ it follows that $N/N_1 \subseteqq C_\rho(N_2/N_1: E(N)/N_1)$.

The usual definition of ρ-density (see [5] and [6]) states that a submodule N_0 is ρ-dense in N if $\rho(N/N_0) = N/N_0$. Note that this occurs if and only if $N \subseteqq C_\rho(N_0: M)$ for every extension M of N. Our definition has some of the usual properties, as shown in the above proposition, and in addition guarantees that N is always ρ-dense in $C_\rho(N)$. The next proposition shows that the two definitions are equivalent when ρ is a torsion preradical.

PROPOSITION 1.4. *The following conditions are equivalent for a preradical ρ of ${}_R\underline{M}$.*

(a) *ρ is a torsion preradical.*

(b) *For all $N \in {}_R\underline{M}$, a submodule N_0 is ρ-dense in*

$$N \Longleftrightarrow \rho(N/N_0) = N/N_0 .$$

Proof. Recall that ρ is a torsion preradical if and only if for all modules N and submodules N_0, $N_0 \subseteqq \rho(N)$ implies $\rho(N_0) = N_0$.

(a) $\Rightarrow$ (b). If N_0 is ρ-dense in N, then there exists an extension M of N with $N \subseteqq C_\rho(N_0: M)$. This implies that N/N_0 is isomorphic to a submodule of $\rho(M/N_0)$, and since ρ is assumed to be a torsion preradical, we must have $\rho(N/N_0) = N/N_0$. On the other hand, it is always true that $\rho(N/N_0) = N/N_0$ implies that N_0 is ρ-dense in N.

(b) $\Rightarrow$ (a). If N_0 is a submodule of ${}_RN$ such that $N_0 \subseteqq \rho(N)$, then the zero submodule is ρ-dense in N_0, since $C_\rho(0: N) = \rho(N)$. By assumption we must have $\rho(N_0) = N_0$, and this shows that ρ is a torsion preradical.

If ρ and σ are preradicals such that $\rho(M) \subseteqq \sigma(M)$, for all $M \in {}_R\underline{M}$, we write $\rho \leqq \sigma$. The smallest preradical of ${}_R\underline{M}$, the zero functor, is denoted by 0 and the largest preradical, the identity functor, is denoted by ∞. If $M \in {}_R\underline{M}$ and ρ is a preradical with $\rho(M) = M$, then since ρ is a preradical we must have $f(M) \subseteqq \rho(N)$, for any $N \in {}_R\underline{M}$ and $f \in \mathrm{Hom}_R(M, N)$. Letting $\mathrm{rad}^M(N) = \Sigma f(M)$, where f runs through all elements of $\mathrm{Hom}_R(M, N)$, we have $\mathrm{rad}^M \leqq \rho$. It can be verified that rad^M is an idempotent preradical, and is the smallest preradical for which M is torsion. Furthermore, $\mathrm{rad}^M(N) = N$ if and only if M generates N in the categorical sense. The module M is called cofaithful if it generates every injective module in ${}_R\underline{M}$. This occurs if and only if ${}_RR$ can be embedded in a finite direct sum of copies of M (see [2]). There is also a smallest torsion preradical for which M is torsion. This can be shown by considering for each module ${}_RN$ the intersection of all submodules which are the torsion submodule of N for some torsion preradical for which M is torsion. The formal definition and some properties follow. Using the notation of Fuchs [3] we denote by $\bar{\Omega}(M)$ the set of all left ideals of R which contain a finite intersection of left ideals of the form $\mathrm{Ann}(m) = \{r \in R\colon rm = 0\}$, for some $m \in M$.

DEFINITION 1.5. Let ${}_RM \in {}_R\underline{M}$. The smallest preradical and smallest torsion preradical of ${}_R\underline{M}$ for which M is torsion will be denoted by rad^M and Rad^M, respectively.

PROPOSITION 1.6. *Let* ${}_RM \in {}_R\underline{M}$.

(a) *For all* $N \in {}_R\underline{M}$, $\mathrm{Rad}^M(N) = N \cap \mathrm{rad}^M(E(N))$.

(b) *The left ideal* A *of* R *is* Rad^M-*dense in* $R \Leftrightarrow A \in \bar{\Omega}(M)$.

(c) *If* N *is a submodule or factor module of* M *then* $\mathrm{Rad}^N \leqq \mathrm{Rad}^M$.

(d) $\mathrm{Rad}^M = \infty \Leftrightarrow M$ *is cofaithful.*

Proof. (a) Since Rad^M is a torsion preradical and $\mathrm{rad}^M \leqq \mathrm{Rad}^M$, it follows that $\mathrm{Rad}^M(N) = N \cap \mathrm{Rad}^M(E(N)) \supseteqq N \cap \mathrm{rad}^M(E(N))$, for all $N \in {}_R\underline{M}$. On the other hand, setting $\rho(N) = N \cap \mathrm{rad}^M(E(N))$ for all $N \in {}_R\underline{M}$ defines a torsion preradical of ${}_R\underline{M}$. That ρ is a preradical follows from the fact that if $f \in \mathrm{Hom}_R(N, Q)$ then f extends to $g\colon E(N) \to E(Q)$, with $g(\mathrm{rad}^M(E(N))) \subseteqq \mathrm{rad}^M(E(Q))$, and consequently $f(N \cap \mathrm{rad}^M(E(N))) \subseteqq Q \cap \mathrm{rad}^M(E(Q))$. Furthermore, ρ is a torsion preradical, since if N_0 is a submodule of N, then $E(N_0)$ is a direct summand of $E(N)$, and so $\mathrm{rad}^M(E(N_0)) = E(N_0) \cap \mathrm{rad}^M(E(N))$. Hence $\rho(N_0) = N_0 \cap \mathrm{rad}^M(E(N_0)) = N_0 \cap \mathrm{rad}^M(E(N)) = N_0 \cap \rho(N)$. Thus ρ is a torsion preradical with $\rho(M) = M$ and $\rho \leqq \mathrm{Rad}^M$, and so we must have $\rho = \mathrm{Rad}^M$.

(b) We will show that for any injective module ${}_RQ$, $x \in \mathrm{rad}^M(Q)$ if and only if $\mathrm{Ann}(x) \in \bar{\Omega}(M)$. The result then follows from part (*a*). If ${}_RQ$ is injective and $x \in \mathrm{rad}^M(Q)$, then by the remarks preceding the proposition, $x = \sum_{i=1}^n f_i(m_i)$, for $m_i \in M$ and $f_i \in \mathrm{Hom}_R(M, Q)$. It is clear that $\bigcap_{i=1}^n \mathrm{Ann}(m_i) \subseteqq \mathrm{Ann}(x)$, and $\mathrm{Ann}(x) \in \bar{\Omega}(M)$. On the other hand, if $x \in Q$ and $\mathrm{Ann}(x) \supseteqq A = \bigcap_{i=1}^n \mathrm{Ann}(m_i)$, for elements $m_i \in M$, then the homomorphism $g\colon R/A \to Rx$ with $g(1) = x$ is well-defined. Since R/A can be embedded in the direct sum M^n of n copies of M, the homomorphism g can be extended by the injectivity of Q to $f\colon M^n \to Q$, with $f(m_1, m_2, \cdots, m_n) = x$. For the components f_i of f we have $x = \sum_{i=1}^n f_i(m_i)$, and so $x \in \mathrm{rad}^M(Q)$.

(c) If N is a submodule or factor module of M, then since $\mathrm{Rad}^M(M) = M$ and Rad^M is a torsion preradical we must have $\mathrm{Rad}^M(N) = N$. It follows from the definition of Rad^N that $\mathrm{Rad}^N \leqq \mathrm{Rad}^M$.

(d) If $\mathrm{Rad}^M = \infty$, then for any injective module Q, $\mathrm{rad}^M(Q) = \mathrm{Rad}^M(Q) = Q$ and M generates Q. Conversely, if M is cofaithful, then for any module ${}_RN$, $\mathrm{Rad}^M(N) = N \cap \mathrm{rad}^M(E(N)) = N$.

The definitions in 1.5 can be dualized, and we give here only the constructions of preradicals rad_M and Rad_M associated with a module ${}_RM$. It can be shown that the preradical rad_M defined below is the largest preradical for which M is torsionfree, and is in fact a radical. We have $\mathrm{Rad}_M \leqq \mathrm{rad}_M$, although Azumaya has shown [1, Prop. 7] that the preradicals coincide for any projective module.

DEFINITION 1.7. Let ${}_RM \in {}_R\underline{M}$. Let rad_M and Rad_M be the preradicals defined by $\mathrm{rad}_M(N) = \bigcap_{f \in \mathrm{Hom}_R(N,M)} \ker(f)$ and $\mathrm{Rad}_M(N) = \mathrm{Ann}(M) \cdot N$, for all modules $N \in {}_R\underline{M}$.

The following lemma and its dual will be used in both of the following sections. Recall that a monomorphism $f\colon A \to B$ in an abelian category $\underline{A}$ is essential if for all $C \in \underline{A}$ and $g\colon B \to C$ in $\underline{A}$, gf is a monomorphism implies that g is a monomorphism. Equivalently, the monomorphism $f\colon A \to B$ is essential if and only if $g \neq 0$ implies $h \neq 0$ in every pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd} .$$

LEMMA 1.8. *If*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & C \arrow[d, "f"] \arrow[dl, "g"] \\ A \arrow[r, "i"'] & B \end{tikzcd}$$

is a pullback diagram in an abelian category, with $i\colon A \to B$ an essential monomorphism, then $gp = h$ implies $ig = f$.

Proof. Consider the pullback diagram

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] \arrow[d, "h_1"'] & C \arrow[d, "f-ig"] \\ A \arrow[r, "i"] & B \end{tikzcd} .$$

Then $(f\text{-}ig)p_1 = ih_1$ implies $fp_1 = i(gp_1 + h_1)$. Since P is a pullback, this induces a factorization $s: P_1 \to P$.

$$\begin{tikzcd}
P_1 \arrow[dr, "s"] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "gp_1+h_1"'] & & \\
& P \arrow[r, "p"] \arrow[d, "h"'] & C \arrow[d, "f"] \arrow[dl, "g"'] \\
& A \arrow[r, "i"'] & B
\end{tikzcd}$$

But then $i(gp_1 + h_1) = ihs = igps = igp_1$. This shows that $ih_1 = 0$, and then $h_1 = 0$ since i is a monomorphism. But $i: A \to B$ is an essential monomorphism and P_1 is a pullback diagram, so $h_1 = 0$ implies $f - ig = 0$, and thus $f = ig$.

2. (ρ, σ)-injective modules. A module ${}_RQ$ is injective if (i) for each module ${}_RN$, (ii) each submodule N_0 of N, and (iii) each homomorphism $f: N_0 \to Q$, there exists an extension of f to N. (By an extension of f to N we mean a homomorphism $g: N \to Q$ such that $gi = f$, for the inclusion $i: N_0 \to N$.) This definition can be generalized by suitably restricting the class of modules in (i), the class of submodules in (ii), or the class of homomorphisms in (iii). Azumaya has studied the first of these in a recent paper [1], in which a module ${}_RQ$ is called M-injective if each homomorphism $f: M_0 \to Q$ from a submodule M_0 of the fixed module M can be extended to M. The second of these possible generalizations has been studied extensively in connection with rings of quotients ([5] and [6]), where the class of submodules is restricted to those submodules which are dense with respect to a fixed torsion radical. Finally, a condition placing restrictions on the class of allowable homomorphisms in (iii) has been used by Fuchs [3, Lemma 2] to characterize quasi-injective modules. The following definition combines these approaches.

DEFINITION 2.1. Let ρ and σ be preradicals of ${}_R\underline{M}$. A module ${}_RQ$ will be called (ρ, σ)-injective if each homomorphism $f: N_0 \to Q$ such that (i) N_0 is a ρ-dense submodule of ${}_RN$ and (ii) $\ker(f)$ is a σ-dense submodule of N can be extended to N.

It is immediate from the definition that for preradicals $\rho_1 \geqq \rho_2$ and $\sigma_1 \geqq \sigma_2$, any (ρ_1, σ_1)-injective module is also (ρ_2, σ_2)-injective. The standard argument can be used to show that a direct product of modules is (ρ, σ)-injective if and only if each factor is (ρ, σ)-injective.

DEFINITION 2.2. Let ρ be a preradical of ${}_R\underline{M}$, and $M \in {}_R\underline{M}$.

A module $_RQ$ will be called (ρ, M)-injective if each homomorphism $f: M_0 \rightarrow Q$ such that M_0 is a ρ-dense submodule of M can be extended to M.

THEOREM 2.3. *Let ρ and σ be preradicals of $_R\underline{M}$. A module $_RQ$ is (ρ, σ)-injective $\Leftrightarrow$ Q is (ρ, M)-injective for all $M \in {}_R\underline{M}$ such that 0 is σ-dense in M.*

Proof. $\Rightarrow$). If the zero submodule of M is σ-dense in M, then by Proposition 1.3 every submodule of M is σ-dense in M.

$\Leftarrow$). Assume that Q is (ρ, M)-injective for all M such that 0 is σ-dense in M, and that $f: N_0 \rightarrow Q$ is a homomorphism with N_0 a ρ-dense submodule of N and $\ker(f)$ a σ-dense submodule of N. Then f induces a homomorphism $f_1: N_0/\ker(f) \rightarrow Q$. Since N_0 is ρ-dense in N it follows from Proposition 1.3 that $N_0/\ker(f)$ is ρ-dense in $N/\ker(f)$. Furthermore, since $\ker(f)$ is σ-dense in N it follows that 0 is σ-dense in $N/\ker(f)$ Then, by assumption, f_1 can be extended to $g_1: N/\ker(f) \rightarrow Q$, and it is clear that $g_1 p$ gives the desired extension of f, where p is the projection $p: N \rightarrow N/\ker(f)$.

The next theorem extends Baer's condition for injective modules to (ρ, σ)-injective modules.

THEOREM 2.4. *Let ρ and σ be torsion preradicals of $_R\underline{M}$. A module $_RQ$ is (ρ, σ)-injective $\Leftrightarrow$ each homomorphism $f: A \rightarrow Q$ such that A is a ρ-dense left ideal of R and* $\ker(f)$ *is σ-dense in R can be extended to R.*

Proof. Our proof follows very closely the standard proof for injectivity. Given a ρ-dense submodule N_0 of $_RN$ and a homomorphism $f: N_0 \rightarrow Q$ with $\ker(f)$ σ-dense in N, there exists a maximal extension $f_1: N_1 \rightarrow Q$, with $N_0 \subseteqq N_1 \subseteqq N$. If there exists $x \in N$, $x \notin N_1$, then let $A = \{r \in R: rx \in N_1\}$ and define $g: A \rightarrow Q$ by $g(r) = f_1(rx)$. An extension of g to R gives rise to an extension of f_1 to $N_1 + Rx$, a contradiction, which then shows that $N_1 = N$. Thus to complete the proof we must show that the homomorphism $g: A \rightarrow Q$ defined above can be extended to R. If we assume that Q satisfies the condition of the theorem, then it is sufficient to show that A is ρ-dense in R and $\ker(g)$ is σ-dense in R. By assumption N_0 is ρ-dense in N, so N_1 is also ρ-dense in N, and since ρ is a torsion preradical, N/N_1 is ρ-torsion. From the definition of A it follows that R/A is isomorphic to a submodule of N/N_1, and again since ρ is a torsion preradical, it follows that R/A is ρ-torsion and A is ρ-dense in R.

From the definition of g it follows that $\ker(g) = \{r \in R: rx \in \ker(f_1)\}$, and so $R/\ker(g)$ is isomorphic to a submodule of $N/\ker(f_1)$. But $\ker(f_1) \supseteqq \ker(f)$, and by assumption $\ker(f)$ is σ-dense in N. It follows as before that $\ker(g)$ is σ-dense in R, using the assumption that σ is a torsion preradical.

The following theorem, with $\sigma = \infty$, extends the known results for ρ-injective modules to the case in which ρ is only a preradical, rather than a torsion radical.

THEOREM 2.5. *Let ρ and σ be preradicals of ${}_R\underline{M}$ and $Q \in {}_R\underline{M}$. The following conditions* (a) – (c) *are equivalent and imply* (d). *If ρ is a torsion preradical, then all four conditions are equivalent.*

(a) *Q is a direct summand in each extension ${}_RM \supseteqq Q$ such that Q is a ρ-dense submodule of M and $Q + M \cap \sigma(E(M)) = M$.*

(b) $Q \supseteqq C_\rho(Q) \cap \sigma(E(Q))$.

(c) *Each homomorphism $f: N_0 \to Q$ such that N_0 is a ρ-dense submodule of ${}_RN$ and $N_0 + N \cap \sigma(E(N)) = N$ can be extended to N.*

(d) *Q is (ρ, σ)-injective.*

Proof. (a) $\Rightarrow$ (b). Let $M = Q + C_\rho(Q) \cap \sigma(E(Q))$. Then since $M \subseteqq C_\rho(Q)$ it follows that Q is a ρ-dense submodule of M. Furthermore, since $Q \subseteqq M \subseteqq E(Q)$, it follows that $E(M) = E(Q)$, and consequently $M = Q + M \cap \sigma(E(M))$. If Q satisfies condition (a) then it must be a direct summand of M, and since Q is an essential submodule of M this implies that $Q = M$, or equivalently, that $Q \supseteqq C_\rho(Q) \cap \sigma(E(Q))$.

(b) $\Rightarrow$ (c). Let $f: N_0 \to Q$ be a homomorphism which satisfies the conditions of (c). Then f can be extended to $g: E(N) \to E(Q)$, and since $N \subseteqq N_0 + C_\rho(N_0: E(N)) \cap \sigma(E(N))$ it follows that

$$\begin{aligned} g(N) &\subseteqq g(N_0) + g(C_\rho(N_0: E(N))) \cap g(\sigma(E(N))) \\ &\subseteqq Q + C_\rho(Q) \cap \sigma(E(Q)) \subseteqq Q \end{aligned}$$

and so g restricted to N gives the desired extension of f to N.

(c) $\Rightarrow$ (a). This is immediate.

(c) $\Rightarrow$ (d). This follows from Theorem 2.3, since if 0 is σ-dense in N, then $N \cap \sigma(E(N)) = N$, and N satisfies the conditions of (c) for every ρ-dense submodule.

(d) $\Rightarrow$ (b). It is necessary to assume that ρ is a torsion preradical. In this case, since Q is ρ-dense in $C_\rho(Q)$, it follows that $Q \cap \sigma(E(Q))$ is ρ-dense in $C_\rho(Q) \cap \sigma(E(Q))$. If Q is (ρ, σ)-injective, then the inclusion $i: Q \cap \sigma(E(Q)) \to Q$ extends to $f: C_\rho(Q) \cap \sigma(E(Q)) \to Q$, since $\ker(i) = 0$ is σ-dense in $C_\rho(Q) \cap \sigma(E(Q))$. In fact f extends to an endomorphism g of $E(Q)$, and then $g(\sigma(E(Q))) \subseteqq \sigma(E(Q))$ implies that $f(C_\rho(Q) \cap \sigma(E(Q))) \subseteqq Q \cap \sigma(E(Q))$. Thus fi is the identity on $Q \cap \sigma(E(Q))$, and since $Q \cap \sigma(E(Q))$ is essential in $C_\rho(Q) \cap \sigma(E(Q))$, this implies that $Q \cap \sigma(E(Q)) = C_\rho(Q) \cap \sigma(E(Q))$. Therefore $Q \supseteqq C_\rho(Q) \cap \sigma(E(Q))$, and the proof is complete.

COROLLARY 2.6. *Let ρ and σ be preradicals of ${}_R\underline{M}$ and let $M, N \in {}_R\underline{M}$, with N a submodule of M. If M is (ρ, σ)-injective and $N \supseteqq C_\rho(M) \cap \sigma(E(M))$, then N is (ρ, σ)-injective. The converse is true if ρ is a torsion radical and N is an essential ρ-dense submodule of M.*

Proof. If N is a submodule of M then $C_\rho(M) \supseteqq C_\rho(N)$ and $\sigma(E(M)) \supseteqq \sigma(E(N))$. Hence if

$$N \supseteqq C_\rho(M) \cap \sigma(E(M)) \supseteqq C_\rho(N) \cap \sigma(E(N)) ,$$

then N is (ρ, σ)-injective by Theorem 2.5. Conversely, if N is essential in M, then $E(N) = E(M)$ and $\sigma(E(N)) = \sigma(E(M))$. If ρ is a radical, then since N is ρ-dense in M it follows that $C_\rho(N) = C_\rho(M)$, and then the result follows from Theorem 2.5.

These results are simplified considerably if σ is a torsion preradical. In this event, for any module ${}_RN$ we have $N \cap \sigma(E(N)) = \sigma(N)$ and $C_\rho(N) \cap \sigma(E(N)) = \sigma(C_\rho(N))$. The next corollary can be used to show the existence of a "(ρ, σ)-injective envelope" when ρ is a torsion radical.

COROLLARY 2.7. *Let ρ be a torsion radical of ${}_R\underline{M}$ and let σ be a preradical of ${}_R\underline{M}$. Then for any module ${}_RN$, $N + C_\rho(N) \cap \sigma(E(N))$ is (ρ, σ)-injective and is contained in every (ρ, σ)-injective submodule of $E(N)$ which contains N.*

Proof. Let $Q = N + C_\rho(N) \cap \sigma(E(N))$. Then $N \subseteqq Q \subseteqq C_\rho(N)$, and since ρ is a radical it follows that $C_\rho(Q) = C_\rho(N)$. Furthermore, $E(Q) = E(N)$, and so $Q \supseteqq C_\rho(Q) \cap \sigma(E(Q))$. Theorem 2.5 implies that Q is (ρ, σ)-injective. If $N \subseteqq M \subseteqq E(N)$, with M (ρ, σ)-injective, then by Theorem 2.5, $M \supseteqq C_\rho(M) \cap \sigma(E(M)) \supseteqq C_\rho(N) \cap \sigma(E(N))$, and so $M \supseteqq Q$.

The next theorem gives a condition equivalent to (ρ, M)-injectivity. It generalizes Theorem 15 of [1] and a theorem of [8]. Its application in Corollary 2.9 shows the connection between (ρ, σ)-injectivity and M-injectivity.

THEOREM 2.8. *Let ρ be a torsion preradical of ${}_R\underline{M}$, and let $M, Q \in {}_R\underline{M}$. Then Q is (ρ, M)-injective* $\Leftrightarrow f(M) \subseteqq Q$, *for all*

$$f \in \mathrm{Hom}_R(M, C_\rho(Q)) .$$

Proof. $\Rightarrow$). Let $f\colon M \to C_\rho(Q)$ and consider the pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & M \arrow[d, "f"] \arrow[dl, dashed, "g"] \\ Q \arrow[r, "i"'] & C_\rho(Q) \end{tikzcd} .$$

Since $Q \to C_\rho(Q)$ is a monomorphism, so is $P \to M$. Furthermore, viewing P as a submodule of M, M/P is isomorphic to a submodule of $C_\rho(Q)/Q$, and so P is ρ-dense in M since Q is ρ-dense in $C_\rho(Q)$. If we assume that Q is (ρ, M)-injective, then $h\colon P \to Q$ extends to $g\colon M \to Q$. Lemma 1.8 implies that $f = ig$, and so $f(M) \subseteqq Q$.

$\Leftarrow$). If $g\colon M_0 \to Q$, with M_0 a ρ-dense submodule of M, then g extends to $f\colon E(M) \to E(Q)$, and $f(M) \subseteqq f(C_\rho(M_0\colon E(M))) \subseteqq C_\rho(Q)$. By assumption, $f(M) \subseteqq Q$, and so f yields the required extension of g to M.

COROLLARY 2.9. *The following conditions are equivalent for any modules ${}_RQ$ and ${}_RM$.*

(a) *Q is M-injective.*

(b) *Q is (∞, rad^M)-injective.*

(c) *Q is (∞, Rad^M)-injective.*

Proof. Theorem 2.8 implies that Q is M-injective (equivalently, Q is (∞, M)-injective) if and only if $Q \supseteqq \mathrm{rad}^M(E(Q)) = \mathrm{Rad}^M(E(Q))$. The three conditions are then equivalent as a consequence of Theorem 2.5.

A module ${}_RQ$ is quasi-injective if and only if it is Q-injective, so Theorem 2.3 and 2.4 imply Lemma 1 and 2 of Fuchs [3]. (A module N is Rad^Q-torsion if and only if $\bar{\Omega}(N) \subseteqq \bar{\Omega}(Q)$.) Theorem 2.5 is closely related to Theorem 11 of [1], which can be seen by taking $\rho = \infty$ and $\sigma = \mathrm{rad}^M$. Corollary 2.6 shows the existence of an "M-injective envelope" ([9]). If N is a submodule or factor module of M, then $\mathrm{Rad}^N \leqq \mathrm{Rad}^M$, and every M-injective module is N-injective ([1, Proposition 10]). If M is cofaithful, then $\mathrm{Rad}^M = \infty$, and every M-injective module is injective ([1, Theorem 14]). If $A = \mathrm{Ann}(M)$, and $A \in \bar{\Omega}(M)$, then M is a cofaithful R/A-module, and hence any R/A-module

Q is $\underline{M}$-injective if and only if it is injective as an R/A-module (compare [3, Theorem 1]).

3. $(\underline{E}, \sigma)$-projective objects. The results in § 2 depend on the existence of injective envelopes. The dual of this condition does not hold in many categories of modules, so the earlier results cannot simply be dualized. We can, however, give results dual to those obtained when ρ is a torsion preradical and σ is an idempotent preradical.

Let $\underline{A}$ be an abelian category and let σ be a radical of $\underline{A}$. These will remain fixed throughout, along with a nonempty class $\underline{E}$ of epimorphisms of $\underline{A}$ which satisfies the following two conditions: (i) if $A \to B \in \underline{E}$ and

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & Q \end{tikzcd}$$

is a pushout diagram in $\underline{A}$, then $C \to Q \in \underline{E}$ and (ii) if $A \to B \in \underline{E}$ and

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is a pullback diagram in $\underline{A}$, then $Q \to C \in \underline{E}$. (Note that the dual of (i) and (ii) is satisfied by the class of monomorphisms $A \to B$ such that A is ρ-dense in B, where ρ is a torsion preradical.)

DEFINITION 3.1. An object $P \in \underline{A}$ is called $(\underline{E}, \sigma)$-projective if each diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed] \\ A \arrow[r] & B \end{tikzcd}$$

such that $A \to B \in \underline{E}$ and $\ker(A \to B/f(P))$ is σ-torsionfree can be completed to a commutative diagram.

Here we use $f(P)$ for the image of f and $B/f(P)$ rather than cokernel (f). Note that $\ker(A \to B/f(P))$ is σ-torsionfree if

$$\ker(A \to B/f(P)) \cap \sigma(A) = 0 .$$

An object $P \in \underline{A}$ is called $(\underline{E}, A)$-projective, for $A \in \underline{A}$, if each diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

such that $A \to B \in \underline{E}$ can be completed to a commutative diagram.

THEOREM 3.2. *The following conditions are equivalent for an object* $P \in \underline{A}$.

(a) *P is $(\underline{E}, \sigma)$-projective.*

(b) *P is $(\underline{E}, A)$-projective for all $A \in \underline{A}$ such that A is σ-torsion-free.*

(c) *There exists a lifting $h: P \to A$ in all diagrams*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

such that (i) $A \to B \in \underline{E}$ *and* (ii) $\ker(g) \cap \sigma(A) = 0$.

(d) *Each epimorphism $p: C \to P$ in $\underline{E}$ such that* $\ker(p) \cap \sigma(C) = 0$ *has a splitting morphism $i: P \to C$ with $pi = 1_P$.*

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious. We prove (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a), to complete the argument.

(b) $\Rightarrow$ (c). Given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

such that $A \to B \in \underline{E}$ and $\ker(g) \cap \sigma(A) = 0$, form the pushout diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "f"] \arrow[ddl, dashed] \\ A \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ A/\sigma(A) \arrow[r] & D \end{tikzcd}.$$

Then $A \to B \in \underline{E}$ implies $A/\sigma(A) \to D \in \underline{E}$, and $A/\sigma(A)$ is σ-torsionfree since σ is a radical, so pf lifts by assumption to $P \to A/\sigma(A)$. But $\ker(g) \cap \sigma(A) = 0$ implies that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/\sigma(A) \arrow[r] & D \end{tikzcd}$$

is also a pullback diagram, and this gives the desired factorization $P \to B = P \to A \to B$.

(d) $\Rightarrow$ (a). Given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

with $A \to B \in \underline{E}$ and $\sigma(\ker(A \to B/f(P))) = 0$, consider the pullback

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "h"'] & P \arrow[d, "f"] \arrow[l, dashed, bend right] \\ A \arrow[r, "g"'] & B \end{tikzcd} .$$

Then $C \to P \in \underline{E}$. Let $D = \ker(p) \cap \sigma(C)$. Now $h(D) \subseteqq h(\sigma(C)) \subseteqq \sigma(A)$, and $gh(C) = fp(C) \subseteqq f(P)$ implies $h(C) \subseteqq \ker(A \to B/f(P))$. Hence $h(D) \subseteqq \sigma(\ker(A \to B/f(P))) = 0$, and so because C is a pullback and $h(D) = p(D) = 0$, it follows that $D = 0$. Therefore $p\colon C \to P$ satisfies the conditions of (d), and the splitting morphism $P \to C$ induces the required lifting $P \to C \to A$ of $P \to B$.

COROLLARY 3.3. *If $P \in \underline{A}$ is $(\underline{E}, \sigma)$-projective and $p\colon P \to A \in \underline{E}$, with* $\ker(p) \subseteqq \sigma(P)$, *then A is $(\underline{E}, \sigma)$-projective.*

Proof. We will use Theorem 3.2 (b). If $B \to C \in \underline{E}$ and $\sigma(B) = 0$, then for any morphism $A \to C \in \underline{A}$, $P \to A \to C$ lifts to $g\colon P \to B$, since P is $(\underline{E}, \sigma)$-projective. But $\sigma(B) = 0$ implies $g(\sigma(P)) = 0$, so $\ker(g) \supseteqq \sigma(P) \supseteqq \ker(p)$. Therefore g factors through A, and since p is an epimorphism this induces the desired lifting of $A \to C$ to $A \to B$.

THEOREM 3.4. *Let $p\colon C \to P \in \underline{E}$ and p be a coessential epimorphism. If P is $(\underline{E}, A)$-projective, for $A \in \underline{A}$, then* $\ker(f) \supseteqq \ker(p)$ *for all morphisms $f\colon C \to A$ in $\underline{A}$.*

Proof. Consider the pushout diagram

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "f"'] & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ A \arrow[r] & D \end{tikzcd} .$$

Then $C \to P \in \underline{E}$ implies $A \to D \in \underline{E}$. Since P is $(\underline{E}, A)$-projective, g lifts to $h: P \to A$. But then the dual of Lemma 1.8 implies that $f = hp$ and so $\ker(f) = \ker(hp) \supseteqq \ker(p)$.

By an $\underline{E}$-projective cover $p: C \to P$ of an object $P \in \underline{A}$ we mean a coessential epimorphism $p \in \underline{E}$ and an object $C \in \underline{A}$ such that C is $(\underline{E}, A)$-projective for all $A \in \underline{A}$.

THEOREM 3.5. *If $P \in \underline{A}$ has an $\underline{E}$-projective cover $p: C \to P$, then the following conditions are equivalent.*

(a) *P is $(\underline{E}, \sigma)$-projective.*

(b) *P is $(\underline{E}, C/\sigma(C))$-projective.*

(c) $\ker(p) \subseteqq \sigma(C)$.

Proof. (a) $\Rightarrow$ (b). This follows from Theorem 3.2 (b) and the fact that $C/\sigma(C)$ is σ-torsionfree.

(b) $\Rightarrow$ (c). Apply Theorem 3.4 to the projection $C \to C/\sigma(C)$.

(c) $\Rightarrow$ (a). This follows from Corollary 3.3 since C is certainly $(\underline{E}, \sigma)$-projective.

We now assume that $\underline{A} = {}_R\underline{M}$ is a category of modules. Let $\underline{E}$ be the class of all epimorphisms of ${}_R\underline{M}$. If σ is a radical of ${}_R\underline{M}$ and ${}_RP$ is $(\underline{E}, \sigma)$-projective, we will simply say that P is σ-projective. If P is $(\underline{E}, M)$-projective, for $M \in {}_R\underline{M}$, we say that P is M-projective. Our final corollary to these results is in essence Theorem 8 of Azumaya [1]. From this it follows immediately that if ${}_RP$ has a projective cover and is M-projective for a faithful module ${}_RM$, then $\mathrm{Rad}_M = 0$ and P is projective ([1, Theorem 9]). In general, our results on M-projective modules are not as good as those of Azumaya, since our characterization of M-projective modules holds only for those modules with projective covers.

COROLLARY 3.6. *Let $P, M \in {}_R\underline{M}$. If P has a projective cover, then the following conditions are equivalent.*

(a) *P is M-projective.*

(b) *P is* rad_M*-projective.*

(c) *P is* Rad_M*-projective.*

Proof. Let $p: C \to P$ be the projective cover of P.

(a) $\Rightarrow$ (b). By Theorem 3.4, if P is M-projective then $\mathrm{rad}_M(C) \supseteqq \ker(p)$, and therefore by Theorem 3.5 (*c*) it follows that P is rad_M-projective.

(b) $\Rightarrow$ (c). This is immediate from Theorem 3.5, since C is projective and therefore $\mathrm{rad}_M(C) = \mathrm{Rad}_M(C)$.

(c) $\Rightarrow$ (a). Because M is Rad_M-torsionfree, this follows from Theorem 3.2 (b).

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NORTHERN ILLINOIS UNIVERSITY