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DECOMPOSITION OF SEMILATTICES WITH APPLICATIONS TO TOPOLOGICAL LATTICES

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Every element with finite extent in a meet-continuous semilattice with complete chains is the meet of a finite number of meet irreducibles. This includes both semilattices with the ascending chain condition and compact topological semilattices with finite breadth. By applying this decomposition to topological lattices on an n-cell, the following results are obtained: If L and M are topological lattices on n and m-cells respectively and there is an order isomorphism between the boundaries of L and M, then L and M are homeomorphic. If, in addition, L and M are distributive, L and M are isomorphic.

1. Finite extent. The most general existence theorem for meet irreducible decompositions in a lattice has been proved for compactly generated (algebraic) lattices by Dilworth and Crawley [4]. While this includes the theory of lattices with ascending chain condition, it does not include the class of topological lattices on an n-cell. The results herein include the latter class and lattices with ascending chain condition. We do not know of existence theorem which includes these topological lattices and compactly generated lattices as well.

Several concepts are needed. A subset A of a poset is called a *factor* of B and we write $A|B$ when for each $x \in B$, there exists $y \in A$ such that $x \geqq y$. A subset of a poset is *independent* when no two elements in it are comparable. It is easy to prove the following.

THEOREM 1.1. *If P is a poset, the factor relation on the subsets of P is reflexive and transitive. The factor relation on the independent subsets of P is a partial order.*

An element x of a semilattice S has *extent* n iff x is the irredundant meet of a set M with n elements but M is not a factor of a finite set with more than n elements whose irredundant meet is x. We say that *x has breadth n* iff x is the irredundant meet of a set with n elements but x is not the irredundant of a finite set witc more than n elements. A *semilattice has breadth n* iff one of its elements has breadth n and the breadth of every other element is less than or equal to n. It should be clear that breadth of elements is a function from a lattice to $Z^+ \cup \{\infty\}$; extent, on the other hand, is not, as the next example shows.

EXAMPLE 1.2. For each positive integer n, let Bn be the Boolean Algebra of all subsets of a set with n elements, chosen so that $Bn \cap Bm = \{\phi\}$ if $n \neq m$. Let $S = \cup_{n=1}^{\infty} Bn$ be the semilattice with operation set intersection. The element ϕ has extent n for each positive integer n. Moreover, ϕ does not have finite breadth, even though every chain in S is finite.

THEOREM 1.3. *If x has breadth n, x has extent no greater than n. The breadth of x is the least positive integer n such that if $x = \wedge M$ and M is finite, then there exists a subset F of M such that card $F \leqq n$ and $\wedge F = x$.*

The proof of Theorem 1.3 is a straightforward application of the definitions.

We shall rely on the following result of Birkhoff [2, p. 182] in the proof of two subsequent theorems. His result is stated dually for our purposes.

THEOREM 1.4. *Let P be a poset with ascending chain condition and let S be the set of finite independent subsets of P. If S is ordered by the factor relation, S has the ascending chain condition also.*

THEOREM 1.5. *Each element of a semilattice with ascending chain condition has finite extent.*

Proof. Suppose a does not have finite extent. Let $M_1 = \{a\}$. Since a does not have extent 1, M_1 is a factor of a finite set M_2 with more than one element such that a is the irredundant meet of M_2. Since the meet is irredundant, M_2 is independent. By an inductive process we may define an infinite ascending chain of finite independent sets

$$M_1 \mid M_2 \mid M_3 \cdots$$

contrary to the preceding theorem. Hence the assumption that a does not have finite extent is incorrect, and this proves the theorem.

THEOREM 1-6. *Let S be a semilattice with ascending chain condition with $x \in S$. The breadth of x is $n = \sup\{j \mid x$ has extent $j\}$ if this supremum exists.*

Proof. Clearly, x has extent n, so there exists a set M with n elements such that x is the irredundant meet of M. Suppose x is the irredundant meet of a finite set M_1 with more than n elements. Since x does not have extent greater than n, there exists a finite set M_2 with more elements than M_1 such that $M_1 \mid M_2$ and x is the irredundant meet of M_2. As in the preceding theorem, this procedure generates an infinite ascending chain of finite independent sets, contrary to Birkhoff's theorem. Hence x is not the irredundant meet of a set with more than n elements, and thus x has breadth n.

2. Meet decompositions. A subset A of a poset is (up) *directed* when it contains an upper bound for each pair of its elements. A semilattice S is *meet-continuous* if for each directed subset A of S for which $\vee A$ exists, we have $\vee (w \wedge A) \in S$ and $w \wedge \vee A = \vee (w \wedge A)$ for all $w \in S$.

We now prove the basic existence theorem for finite meet-decompositions.

THEOREM 2.1. *If S is a meet-continuous semilattice in which all chains have suprema, then each element of S with extent n is the irredundant meet of n irreducibles.*

Proof. Let a be an element of S with finite extent. By definition, there exists a set F with n elements such that a is the irredundant meet of F and F is not a factor of a finite set with more than n elements whose irredundant meet is a. Enumerate F by $F = \{x_1, x_2, \cdots x_n\}$. Let C_1 be a maximal chain in $M_1 = \{x \geqq x_1 | x \wedge x_2 \wedge \cdots \wedge x_n = a\}$ and let $b_1 = \vee C_1$. By meet-continuity $b_1 \in M_1$; hence b_1 is a maximal element in M_1. With a similar argument, a maximal element b_2 in $\{x \geqq x_2 | b_1 \wedge x_2 | x \wedge x_3 \wedge \cdots \wedge x_n = a\}$ may be obtained, and this process is continued until the set $B = \{b_1, b_2 \cdots b_n\}$ is achieved. Clearly, $\wedge B = a$. Suppose $b \in B$ and $b = x \wedge y$. Since F is a factor of $(B \backslash b) \cup \{x, y\}$ and this set has more than n elements, its meet is redundant. Since $\wedge (B \backslash b) > a$, $\wedge [(B \backslash b) \cup \{x\}] = a$ or $\wedge [(B \backslash b) \cup \{y\}] = a$, either of which contradicts the maximality of b unless $x = b$ or $y = b$. Thus the elements of B are irreducible. Moreover, the meet of B is irredundant since $F | B$ and a is the irredundant meet of F. This completes the proof. Notice that the set B is maximal with respect to these properties: $F | B$, card $B =$ card F, and a is the irredundant meet of B.

Stralka and Baker [10] independently of the author and at about the same time proved that a complete meet-continuous lattice with finite breadth has finite irreducible decompositions. This is a special case of Theorem 2.1: however, the proofs are quite similar.

By virtue of the fact that a semilattice with ascending chain condition is meet-continuous [4], we have the well-known corollary:

COROLLARY 2.2. *Every element of a semilattice with ascending chain condition is the irredundant meet of a finite number of irreducibles.*

Finite extent is not a necessary condition for the existence of finite irreducible representations. There are, however, some special cases in which the stronger condition of finite breadth is necessary.

THEOREM 2.3. *Let S be a semilattice with finite irreducible decompositions for each $y \geqq x$. If all irredundant irreducible decompositions of x have n elements, then x has breadth n.*

The straightforward proof of Theorem 2.3 is omitted. Modular semilattices were defined in [8] and the semilattice version of the Kurosh Ore theorem was proved.

COROLLARY 2.4. *If S is a modular semilattice with finite irreducible decompositions for each element, then each element of S has finite breadth.*

THEOREM 2.5. (*Newman* [7, p. 31]) *If every element of a complete distributive lattice L has a finite meet irreducible decomposition, then L is meet-continuous.*

THEOREM 2.6. *Let L be a complete distributive lattice. Every element of L has a unique irredundant decomposition into a finite meet of irreducibles if and only if L is meet-continuous and each element of L has finite breadth.*

The uniqueness of decompositions in a distributive lattice was first shown by Birkhoff.

The next two theorems deal with decompositions of product semilattices.

THEOREM 2.7. *If each element of each $S\alpha$ has an irreducible decomposition and precedes a maximal element, then every element of $\pi S\alpha$ has an irreducible decomposition.*

Proof. Let $f \in \pi S\alpha$. Let $f(\alpha) = \wedge x_{i,\alpha}$ be an irreducible decomposition of $f(\alpha)$ in $S\alpha$. Define

$$h_{i,\alpha}(t) = \begin{cases} x_{i,\alpha} \text{ if } t = \alpha \\ m_t \text{ if } t \neq \alpha \end{cases}$$

where m_t is a maximal element containing $f(t)$. Clearly, each $h_{i,\alpha}$ is irreducible and $\wedge_{i,\alpha} h_{i,\alpha} = f$.

THEOREM 2.8. *If every element of $\pi S\alpha$ has an irreducible decomposition, then every element of each $S\alpha$ has an irreducible decomposition. If there is an α such that some element of $S\alpha$ does not precede a maximal element, then every element of $S\beta$ precedes a maximal element if $B \neq \alpha$.*

Proof. We prove only the second part. Suppose there exist S_θ, S_β with $x_\theta \in S_\theta$ and $x_\beta \in S_\beta$ and neither x_θ nor x_β precedes a maximal element. By the Axiom of Choice, there exists $x \in \pi S_\alpha$ such that $x(\beta) = x_\beta$ and $x(\theta) = x_\theta$. Suppose $m \in \pi S_\alpha$ and $m \geqq x$. Since $m(\theta) \geqq x_\theta$ and $m(\beta) \geqq x_\beta$ and neither of these elements precedes a maximal element, there exist $y_\beta > m(\beta)$ and $y_\theta > m(\theta)$. Define

$$h(\alpha) = \begin{cases} m(\alpha) \text{ if } \alpha \neq \beta \\ y_\beta \text{ if } \alpha = \beta \end{cases} \quad \text{and} \quad g(\alpha) = \begin{cases} m(\alpha) \text{ if } \alpha \neq \theta \\ y_\theta \text{ if } \alpha = \theta \ . \end{cases}$$

Then $m = h \wedge g$. This proves that every element greater than x is reducible, contrary to hypothesis, and this contradiction completes the proof.

3. Applications to topological semilattices and lattices.

A semilattice whose operation is continuous in an underlying Hausdorff topology is called a topological semilattice. Continuity of the semilattice operation does not imply meet-continuity generally, but there are exceptions.

THEOREM 3.1. *S be a semilattice and a topological space in which each ascending net converges to the supremum of its range. If the operation of S is continuous, then S is meet-continuous.*

Lawson [5] has shown that ascending nets converge to suprema in a compact topological semilattice. Thus a compact topological semilattice is meet-continuous.

THEOREM 3.2. *Let $\{S_\alpha \mid \alpha \in \Gamma\}$ be a family of compact topological semilattices with finite breadth. Then every element of πS_α has an irreducible decomposition.*

Proof. Combine Theorems 2.1 and 2.7 with the preceding remarks.

THEOREM 3.3. (*Lawson* [6]) *Let S be a locally compact topological semilattice in which $M(x) = \{y \mid y \geqq x\}$ is connected for each $x \in S$. If S has positive codimension n, S has breadth less than or equal to $n + 1$. If each pair of elements of S has an upper bound, then S*

has breadth less than or equal to n.

The special case of Theorem 3.3 for distributive lattices was first proved by Anderson [1].

THEOREM 3.4. *Let S be a compact topological semilattice in which $M(x)$ is connected for each x. If x has positive codimension n, then each element of S has an irredundant decomposition into no more than $n + 1$ irreducibles. If S is a lattice, each element of S has an irredundant decomposition into no more than n irreducibles.*

In particular, topological semilattices on the n-cell have irreducible decompositions.

THEOREM 3.5. *Let S be a topological semilattice on an n-cell ($n \geqq 2$). Then all the irreducibles of S lie on the boundary.*

Proof. Modify the proof of Theorem 1, p. 37, in Brown [3].

THEOREM 3.6. (*Lawson* [5, p. 89], *and Strauss* [11].) *A compact metrizable topological lattice has the order topology.*

THEOREM 3.7. *Let L and M be topological lattices on n and m cells respectively. If there is an order isomorphism between the boundaries of L and M, then $n = m$ and L and M are homeomorphic.*

Proof. The case $m = 1$ is straightforward. Assume $m > 1$. By Theorem 3.6, L and M have the order topology and by Theorem 3.3, L and M have finite breadth. In lattices of finite breadth the order topology coincides with the interval topology [2, p. 250]. Thus the closed intervals $[a, b]$ constitute a subbasis for L and M.

Let $f: B(L) \to B(M)$ be an order isomorphism from the boundary of L onto the boundary of M. We must show f is continuous, that is, $f^{-1}([a, b] \cap B(M))$ is closed in $B(L)$. Let x be a sequence in $f^{-1}([a, b] \cap B(M))$ with x converging to some $x_0 \in B(L)$. Clearly $f(x_0)$ is defined and $f(x_0) \in B(M)$. By Theorem 3.4, b has an irreducible decomposition $b = \wedge B$ and by Theorem 3.5, $B \subseteqq B(M)$. Since $f(x_n) \leqq b$ for every n, $f(x_n) \leqq y$ for every $y \in B$. Since f is an isomorphism, $x_n \leqq f^{-1}(y)$ for every $y \in B$. By continuity of the semilattice operation, $x_0 \leqq f^{-1}(y)$ for every $y \in B$. Hence $f(x_0) \leqq \wedge B = b$. Using join irreducible decompositions we may show $a \leqq f(x_0)$. Thus $x_0 \in f^{-1}([a, b] \cap B(M))$ and this set is closed. This proves that f is continuous and since $B(L)$ and $B(M)$ are compact Hausdorff spaces, f is a homeomorphism. Since the boundaries are homeomorphic, $n - 1 =$

$m - 1$. Hence $n = m$ and L and M are homeomorphic.

Shields [9] has shown that if there is a homeomorphism φ from the boundary of a topological semigroup S on an n-cell onto the boundary of the product semigroup T on the same cell, and φ is an isomorphism, then S and T are isomorphic. The above theorem shows that order isomorphism is sufficient in the lattice case when one is the product lattice. Theorem 3.8 below shows that the requirement that one be the product lattice may be dropped if both lattices are distributive. We note that distributive lattices on an n-cell are not, in general, isomorphic.

THEOREM 3.8. *If L and M are distributive topological lattices on an n-cell, and $f: B(L) \to B(M)$ is an order isomorphisms, then there exists a lattice isomorphism $g: L \to M$ that is an extension of f.*

Proof. The case $n = 1$ is trivial. Assume $n > 1$. Define $g: L \to M$ by

$$g(x) = f(x_1) \wedge f(x_2) \wedge \cdots \wedge f(x_m)$$

where $x = x_1 \wedge x_2 \wedge \cdots \wedge x_m$ is the unique irredundant decomposition of x into irreducibles. Since this representation is unique and irreducibles lie in the boundary, g is well-defined. The proof is carried out in a sequence of lemmas.

1. *If $\wedge R = \wedge T$ and R and T are finite sets of irreducibles in L (in M), then $\wedge f(R) = \wedge f(T) [\wedge f^{-1}(R) = \wedge f^{-1}(T)]$.*

Proof. We prove only the first part. Suppose $x \in R$. Since x is irreducible, L is distributive, and $x \geqq \wedge T$, $x \geqq y$ for some $y \in T$. Thus $T | R$ and since f is an order isomorphism, $f(T) | f(R)$. Similarly, $f(R) | f(T)$; hence $\wedge f(R) = \wedge f(T)$.

2. *g is a semilattice homomorphism.*

Proof. Suppose $x = \wedge X$ and $y = \wedge Y$ are irredundant irreducible decompositions in L. There is a set $T \subseteqq X \cup Y$ such that $x \wedge y = \wedge T$ is an irredundant irreducible decomposition of $x \wedge y$. Then by part(1),

$$g(x \wedge y) = g(\wedge R) = \wedge f(R) = \wedge f(X \cup Y) = g(x) \wedge g(y) .$$

3. *The image and preimage of irreducibles under f are irreducible.*

Proof. Suppose x is irreducible in L. Let $f(x) = x_1 \wedge x_2 \wedge \cdots \wedge x_m$ be an irreducible decomposition of $f(x)$ in M. Since $x_1, x_2, \cdots x_n \in$

$B(M)$ and f is an order isomorphism, $x = f^{-1}(x_1) \wedge f^{-1}(x_2) \wedge \cdots \wedge f^{-1}(x_m)$; this implies that $x = f^{-1}(x_i)$ or $f(x) = x_i$ for some i. Thus the image of an irreducible is irreducible and a similar argument holds for preimages as well.

4. *g is an injection.*

Proof. Suppose $g(x) = g(y)$ with $x = x_1 \wedge x_2 \wedge \cdots \wedge x_m$ and $y = y_1 \wedge \cdots \wedge y_p$ the irredundant irreducible decompositions of x and y in L. Then $f(x_1) \wedge \cdots \wedge f(x_m) = f(y_1) \wedge \cdots \wedge f(y_p)$. These meets are irredundant, for if $\bigwedge_{i \neq j} f(x_i) = \wedge f(x_i)$, then by part 1, $\bigwedge_{i \neq j} x_i = \wedge x_i$, contrary to the irredundancy of the decomposition of x. Since each $f(x_i)$ and $f(y_i)$ is irreducible (by part 3) and M is distributive, $\{f(x_i)\} = \{f(y_i)\}$. Thus $\{x_i\} = \{y_i\}$ because f is an isomorphism; and it follows that $x = y$.

5. *g is a surjection.*

Proof. Suppose $y \in M$. Let $y = y_1 \wedge \cdots \wedge y_m$ be the irredundant irreducible decomposition of y in M and let $x = f^{-1}(y_1) \wedge \cdots \wedge f^{-1}(y_m)$. Clearly $g(x) = y$.

This proves that g is a semilattice isomorphism and since L and M are lattices, g is a lattice isomorphism.

COROLLARY 3.9. *If L and M are distributive topological lattices on n and m cells respectively, and the boundaries of L and M are isomorphic, then L and M are homeomorphic and isomorphic.*

The preceding results, with the exception of Theorems 3.7, 3.8 and 3.9, were part of a paper presented to the American Mathematical Society in January, 1969, under the title "Chain Conditions in Topological Semilattices".

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REFERENCES

1. L. W. Anderson, *On the breadth and co-dimension of topological lattice*, Pacific J. Math., **9** (1959), 327-333.

2. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc., Providence, Rhode Island, 1967.

3. D. R. Brown, *Topological semilattices on the two-cell*, Pacific J. Math. **15** (1965), 35-46.

4. R. P. Dilworth and P. Crawley, *Decomposition theory for lattices without chain conditions*, Trans. Amer. Math. Soc., **96** (1960), 1-22.

5. J. D. Lawson, *Vietoris Mappings and Embeddings of Topological Semilattices*, unpublished doctoral dissertation, The University of Tennessee, Knoxville, Tennessee, 1967.

6. J. D. Lawson, *The relation of breadth and codimension in topological semilattices*, Duke Math. J. **37** (1970), 207-212.
7. T. Newman, *Homomorphic Images of Compactly Generated Lattices*, unpublished doctoral dissertation, The University of Texas, Austin, Texas, 1967.
8. Joe B. Rhodes, *Modular and Distributive Semilattices*. (to appear in Trans. Amer. Math. Soc.)
9. A. Shields, *The n-Cube as a Product Semigroup*, Michigan Math. J. **4** (1957), 165-166.
10. A. R. Stralka and K. A. Baker, *Compact, distributive lattices of finite breadth*, Pacific J. Math., **34** (1970), 311-320.
11. D. P. Strauss, *Topological lattices*, Proc. London Math. Soc., (3) **18** (1968), 217-30.

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