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LOCALIZING THE SPECTRUM

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It is the purpose of this paper to show that the notion of spectrum for linear transformations can be extended to nonlinear transformations. The technique used is localization, as it is applied, for example, to define the local Lipschitz property from the global one. A discussion of two attempts to extend globally the spectral concepts to the nonlinear setting will serve as a preliminary to the main results.

Denote by H a complex Banach, or normed linear complete, space whose point-set is S and whose norm is $||\cdot||$; and denote by T a transformation, not necessarily linear, from a subset $D(T)$ of S into S.

DEFINITION (R. I. Kačurovskiǐ [3, p. 1101] and E. H. Zarantonello [5, 6]). "T has a KZ resolvent set" means that there is a complex number λ with the following properties:

(1) $\lambda I - T$ is $1-1$,

(2) $\lambda I - T$ has range S, and

(3) $(\lambda I - T)^{-1}$ is Lipschitzean.

In this case the *KZ resolvent set of T* is the set of all such λ. "T has a KZ spectrum" means that T has no KZ resolvent set or that the KZ resolvent set fails to exhaust the plane. In this case the *KZ spectrum of T* is either the entire plane or that part of it outside the KZ resolvent set.

This definition specializes to the linear situation. Furthermore, every Lipschitzean transformation has a large KZ resolvent set.

THEOREM 1. *If T is Lipschitzean, then T has a KZ resolvent set containing each complex number λ with the property that $|\lambda| > |T|$.*

Proof. Let λ be a complex number with $|\lambda| > |T|$. If each of x and y is in S, then $||(\lambda I - T)x - (\lambda I - T)y|| = ||\lambda(x-y) - (Tx - Ty)|| \geqq (|\lambda| - |T|)||x - y||$. Since $|\lambda| > |T|$, the above inequality tells us that $\lambda I - T$ is $1-1$ and $(\lambda I - T)^{-1}$ is Lipschitzean on its domain with Lipschitz norm not greater than $1/(|\lambda| - |T|)$. What we need to show now is that the domain of $(\lambda I - T)^{-1}$ is S.

Let y be in S. We need to show that there is an x in S such that $\lambda x - Tx = y$, that is, $x = (1/\lambda)(Tx + y)$. Thus what we need to show is that the transformation A on S, where $A = (1/\lambda)(T + y)$, has a fixed point. But this is so; for, by the fact that $|\lambda| > |T|$, we have that A is a contraction mapping on a complete metric space.

Although it is uncertain whether a Lipschitzean transformation has a KZ spectrum, much of the uncertainly disappears if the local Lipschitz condition replaces the global one. The identity function on the plane is a Lipschitzean, hence locally Lipschitzean, transformation with only the number one in its KZ spectrum. At the other extreme, the square function on the plane is also locally Lipschitzean and yet has a KZ spectrum which exhausts the plane. Observing that each of these examples is differentiable as well as locally Lipschitzean, we can conclude that the global Lipschitz property controls the size of the KZ spectrum to an extent which neither the local Lipschitz property nor differentiability can match.

DEFINITION (J. W. Neuberger [4, p. 157]). "T has an N resolvent set" means that there is a complex number λ such that Properties (1) and (2) hold and such that $(\lambda T - T)^{-1}$ is (Fréchet) differentiable (cf. [1, p. 149]). In this case the *N resolvent set of T* is the set of all such λ. "T has an N spectrum" is defined analogously to the statement, "T has a KZ spectrum."

Originally the satisfaction of the local Lipschitz property by $(\lambda I - T)^{-1}$ was required before λ was allowed into the N resolvent set, but we shall see that that requirement is redundant.

This definition, like the Kačurovskiĭ-Zarantonello one, specializes to the case in which T is linear. In addition, it insures the existence of a spectrum when T is nothing more than locally Lipschitzean at zero (cf. [4]). It cannot, however, guarantee the existence of a bound on the moduli of the spectral elements of even a Lipschitzean transformation. For example, the function I^* on the plane, which sends each complex number onto its conjugate, has the property that $(\lambda I - I^*)^{-1}$ exists as a differentiable function for no complex number λ. Moreover, another look at the identity and square functions on the plane reveals that the Neuberger spectrum of a locally Lipschitzean or differentiable function can be as large or small as the Kačurovskiĭ-Zarantonello spectrum.

One conclusion which can be drawn from these observations of Kačurovskiĭ-Zarantonello and Neuberger extensions is that their failure to control the spectrum is not their own fault but perhaps simply a result of the nonuniformity of nonlinear transformations themselves. One way to handle a nonuniform transformation is to study it locally.

DEFINITION. "T has a local resolvent set at the point p" means that p is in $D(T)$ and there are a complex number λ and a positive-number pair (δ, ε) with the following properties:

(4) $\lambda I - T|_{R_p(\delta)}$ is $1 - 1$,
(5) $R_{(\lambda I - T))_p}(\varepsilon) \subseteqq (\lambda I - T)(R_p(\delta))$, and
(6) $(\lambda I - T|_{R_p(\delta)})^{-1}$ is Lipschitzean on $R_{(\lambda I - T)_p}(\varepsilon)$.
In this case the *local resolvent set of T at p*, denoted $\rho_p(T)$, is the set of all such λ. "T has a local spectrum at p" means that p is in $D(T)$ and T has no local resolvent set at p or that $\rho_p(T)$ fails to exhaust the plane. In this case *local spctrum of T at p* is either the plane or that part of it outside $\rho_p(T)$.

THEOREM 2. *If T is continuously differentiable on an open set D containing the point p (that is, T is differentiable on D and T' is continuous on D), then $\rho_p(T)$ contains the (linear) resolvent set $\rho(T'(p))$ of $T'(p)$ (by definition, $T'(p)$ is a continuous linear transformation). Moreover, if H is finite-dimensional, then $\rho_p(T) = \rho(T'(\rho))$.*

Proof. We shall use in this proof a result which is important in its own right. L. A. Harris [2, pp. 16, 17] has shown that a function which is (Fréchet) differentiable on an open set containing zero is locally Lipschitzean at zero, and a slight modification of his argument yields a more general result.

LEMMA 1. *Suppose $\delta > 0$, f is a differentiable function with bounded range from the ball $R_p(\delta)$ into S, and $0 < \delta' < \delta$. Then there is a number M such that $|f'(x)| \leqq M$ for each x in $R_p(\delta')$, and f is Lipschitzean on $R_p(\delta')$.*

(This is the result which permits us to omit the local Lipschitz property from the definition of *N resolvent set*.)

Turning to the proof of Theorem 2, let us denote by λ a member of $\rho(T'(p))$. To show that there is a positive-number pair (δ, ε) such that Properties (4)-(6) are satisfied, we shall first establish that $\lambda I - T$, p, and D satisfy the hypothesis of an inverse function theorem (cf. [1, p. 273]). D is an open set containing p and on which T, hence $\lambda I - T$, is continuously differentiable; and, since λ is in $\rho(T'(p))$, we know that $(\lambda I - T)'(p)$, which is $\lambda I - T'(p)$, is an invertible continuous linear transformation, or homeomorphism, from H onto H. Thus, by the conclusion of the theorem cited, let D' be an open set containing p and contained in D, with the following properties: $\lambda I - T|_{D'}$ is $1-1$, $(\lambda I - T)(D')$ is open, and $(\lambda I - T|_{D'})^{-1}$ is differentiable. We shall choose our δ and ε from D' and $(\lambda I - T)(D')$.

Since $(\lambda I - T|_{D'})^{-1}$ is differentiable on the open set $(\lambda I - T)(D')$, which contains $(\lambda I - T)p$, it is also continuous, hence locally bounded, at $(\lambda I - T)p$. Therefore, let (ε', B) be a number-pair such that $R_{(\lambda I - T)p}(\varepsilon') \subseteqq (\lambda I - T)(D')$ and such that, if y is in S and $\|y - (\lambda I - T)p\| <$

ε', then $||(\lambda I - T|_{D'})^{-1}y|| \leqq B$. Thus ε', $\varepsilon'/2$, H, $(\lambda I - T)p$, $(\lambda I - T|_{D'})^{-1}$, and the ball $R_{(\lambda I-T)p}(\varepsilon')$ satisfy the hypothesis of Lemma 1; so $(\lambda I - T|_{D'})^{-1}$ is Lipschitzean on $R_{(\lambda I-T)p}(\varepsilon'/2)$. Denote by M a positive Lipschitz constant for $(\lambda I - T|_{D'})^{-1}$ on $R_{(\lambda I-T)p}(\varepsilon'/2)$. Denote by δ a positive number such that $R_p(\delta) \subseteqq D'$, and let $\varepsilon = \min\{\varepsilon'/2, \delta/M\}$. We need to show that (δ, ε) is a pair of the type we desire.

Since $\lambda I - T|_{D'}$ is $1 - 1$ and $R_p(\delta) \subseteqq D'$ we know that $\lambda I - T|_{R_p}(\delta)$ is also $1 - 1$. Additionally, since $(\lambda I - T|_{D'})^{-1}$ is Lipschitzean on $R_{(\lambda I-T)p}(\varepsilon'/2)$ and $\varepsilon \leqq \varepsilon'/2$, we know that $(\lambda I - T|_{R_p(\delta)})^{-1}$, which is a restriction of $(\lambda I - T|_{D'})^{-1}$, is Lipschitzean on $(\lambda I - T)(R_p(\delta))$, its domain.

To show that $R_{(\lambda I-T)p}(\varepsilon) \subseteqq (\lambda I - T)(R_p(\delta))$, let us suppose that y is in $R_{(\lambda I-T)p}(\varepsilon)$. Thus y is in $(\lambda I - T)(D')$, so let $x = (\lambda I - T|_{D'})^{-1}y$. Then

$$\begin{aligned} ||x - p|| &= ||(\lambda I - T|_{D'})^{-1}y - (\lambda I - T|_{D'})^{-1}(\lambda I - T)p|| \\ &\leqq M||y - (\lambda I - T)p|| < M\cdot\varepsilon \leqq M\cdot\frac{\delta}{M} \\ &= \delta\ , \end{aligned}$$

so x is in $R_p(\delta)$. Thus y is in $(\lambda I - T)(R_p(\delta))$, and λ is in $\rho_p(T)$.

Assume now that H is finite-dimensional. A second lemma will reverse the containment.

LEMMA 2. *Suppose that f is a function from a subset of S containing p into S with the following properties*:

(7) *there is a positive-number pair (r, K) such that, if each of x and y is in S and* $\max\{||x - p||, ||y - p||\} < r$, *then* $||f(x) - f(y)|| \geqq K||x - y||$; *and*

(8) *f is differentiable at p.*

Then $f'(p)$ is $1 - 1$.

Proof. Since f is differentiable at p, there is a positive number b' with the property that, if x is in S and $0 < ||x - p|| < b'$, then $||f(x) - f(p) - (f'(p))(x - p)||/||x - p|| < K/2$. Let b' be such a number, and let $b = \min\{b', r\}$. We want to show that, if x is in the ball $R_p(b)$ and $(f'(p))x = (f'(p))p$, then $x = p$. Since $f'(p)$ is linear, we shall then have that $f'(p)$ is $1 - 1$.

Suppose that x is a point of $R_p(b)$ different from p but for which $(f'(p))x = (f'(p))p$. Thus the following inequality holds.

$$\begin{aligned} 0 &= ||(f'(p))(x - p)|| \\ &= ||f(x) - f(p) + (f'(p))(x - p) - (f(x) - f(p))|| \\ &\geqq ||f(x) - f(p)|| - ||f(x) - f(p) - (f'(p))(x - p)|| \\ &> K||x - p|| - (K/2)||x - p|| \quad \text{since} \quad 0 < ||x - p|| < b \leqq b' \\ &= (K/2)||x - p||. \end{aligned}$$

This means that, since $K/2 > 0$, the quantity $||x - p|| < 0$. This contradiction implies that our assumption is false, and the lemma stands.

Returning to the proof of the theorem, let us assume that λ is in $\rho_p(T)$. Suppose that (r_1, c) is a positive-number pair of the type which the definition of *local resolvent set* says must exist for λ at p. Since $\lambda I - T$ is continuous at p by the fact that T is differentiable there, let r_2 be a positive number with the property that, if $||x - p|| < r_2$, then $||(\lambda I - T)x - (\lambda I - T)p|| < c$. Denote by r the $\min\{r_1, r_2\}$. We shall show that $\lambda I - T|_{R_p(r)}$ satisfies the hypothesis of Lemma 2.

Since $(\lambda I - T|_{R_p(r_1)})^{-1}$ is Lipschitzean on $R_{(\lambda I - T)p}(c)$, let K be a positive number with property that, if each of x and y is in $R_{(\lambda I - T)p}(c)$, then

$$||(\lambda I - T|_{R_p(r_1)})^{-1}x - (\lambda I - T|_{R_p(r_1)})^{-1}y|| \leqq K||x - y|| ,$$

that is,

$$||x - y|| \geqq (1/K)||(\lambda I - T|_{R_p(r_1)})^{-1}x - (\lambda I - T|_{R_p(r_1)})^{-1}y|| .$$

Thus, if each of u and v is in $R_p(r)$, then each of $(\lambda I - T)u$ and $(\lambda I - T)v$ is in $R_{(\lambda I - T)p}(c)$, and

$$\begin{aligned}
&||(\lambda I - T|_{R_p(r)})u - (\lambda I - T|_{R_p(r)})v|| \\
&\quad = ||(\lambda I - T|_{R_p(r_1)})u - (\lambda I - T|_{R_p(r_1)})v|| \quad \text{since} \quad r \leqq r_1 \\
&\quad \geqq (1/K)||(\lambda I - T|_{R_p(r_1)})^{-1}(\lambda I - T|_{R_p(r_1)})u \\
&\qquad - (\lambda I - T|_{R_p(r_1)})^{-1}(\lambda I - T|_{R_p(r_1)})v|| \\
&\quad = (1/K)||u - v|| .
\end{aligned}$$

Finally, since T is differentiable at p, $\lambda I - T|_{R_p(r)}$ is also. Thus $\lambda I - T|_{R_p(r)}$ satisfies the hypothesis of Lemma 2, so $(\lambda I - T|_{R_p(r)})'(p)$ is $1 - 1$. But, since H is finite-dimensional, this means that $(\lambda I - T|_{R_p(r)})'(p)$ is regular (that is, its inverse exists as a continuous linear transformation on H). Since $(\lambda I - T|_{R_p(r)})'(p) = \lambda I - (T|_{R_p(r)})'(p) = \lambda I - T'(p)$, this means that $\lambda I - T'(p)$ is regular, or that λ is in $\rho(T'(p))$.

Theorem 2 reveals a strong connection between the local spectra of a continuously differentiable function and the spectra of its Fréchet derivatives. Theorem 3 will give us an analog to the resolvent-set-existence theorem for a continuous linear transformation, an example of which can be obtained from Theorem 1 by making T continuous and linear.

THEOREM 3. *If T is locally Lipschitzean at the point p, then there is a number B with the property that $\rho_p(T)$ contains each complex number λ such that $|\lambda| \geqq B$.*

Proof. Denote by (r, M) a positive-number pair with the property that, if each of x and y is in S and $\max\{\|x - p\|, \|y - p\|\} < r$, then each of x and y is in $D(T)$ and $\|Tx - Ty\| \leqq M\|x - y\|$. Denote by δ a positive number less than $\min\{r, 1\}$, and let $B = 2M/\delta$. Suppose that λ is a complex number with $|\lambda| \geqq B$. We want to show that λ qualifies as a member of $\rho_p(T)$, with $(r, \delta|\lambda|/2)$ as a number-pair of the desired type.

If each of x and y is in $R_p(r)$, then

$$\begin{aligned}\|(\lambda I - T)x - (\lambda I - T)y\| &= \|\lambda(x - y) - (Tx - Ty)\| \\ &\geqq |\lambda|\cdot\|x - y\| - M\|x - y\| \\ &= (|\lambda| - M)\|x - y\| .\end{aligned}$$

Since $|\lambda| \geqq B = 2M/\delta$ and $\delta < 1$, we have that $|\lambda| - M > 0$; so the above inequality tells us that $\lambda I - T|_{R_p(r)}$ is $1 - 1$ and $(\lambda I - T|_{R_p(r)})^{-1}$ is Lipschitzean on its domain. It remains to show that $R_{(\lambda I - T)p}(\delta|\lambda|/2) \subseteqq (\lambda I - T)(R_p(r))$.

Suppose that q is in $R_{(\lambda I - T)p}(\delta|\lambda|/2)$. We need to show that there is an x in $R_p(r)$ such that $\lambda x - Tx = q$, that is, $x = (1/\lambda)(Tx + q)$. We shall find such a point by the technique of successive approximation, defining a sequence $x_0, x_1, x_2, \cdots$ in the following way.

Denote by x_0 the point p, and let $x_1 = (1/\lambda)(Tx_0 + q)$. Thus

$$\begin{aligned}\|x_1 - p\| &= \|(1/\lambda)(Tx_0 + q) - p\| \\ &= \|(1/\lambda)Tx_0 + (1/\lambda)q - (1/\lambda)(\lambda I - T)p + (1/\lambda)(\lambda I - T)p - p\| \\ &= \|(1/\lambda)Tp + (1/\lambda)[q - (\lambda I - T)p] + p - (1/\lambda)Tp - p\| \\ &= (1/|\lambda|)\|q - (\lambda I - T)p\| \\ &< (1/|\lambda|)(\delta|\lambda|/2) \\ &= \delta/2 ,\end{aligned}$$

so x_1 is in $R_p(\delta/2)$, hence, since $\delta/2 < \delta < r$, in $D(T)$. Denote by x_2 the point $(1/\lambda)(Tx_1 + q)$. Now

$$\begin{aligned}\|x_2 - x_1\| &= \|(1/\lambda)(Tx_1 + q) - (1/\lambda)(Tx_0 + q)\| \\ &= (1/|\lambda|)\|Tx_1 - Tx_0\| \\ &\leqq (M/|\lambda|)\|x_1 - x_0\| = (M/|\lambda|)\|x_1 - p\| \\ &< (M/|\lambda|)(\delta/2) .\end{aligned}$$

Since $|\lambda| \geqq 2M/\delta$, it follows that $\delta/2 \geqq M/|\lambda|$; so $\|x_2 - x_1\| < (\delta/2)^2$, and

$$\begin{aligned}\|x_2 - p\| \leqq \|x_2 - x_1\| + \|x_1 - p\| &\leqq \left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right) \\ &= \sum_{i=1}^{2}\left(\frac{\delta}{2}\right)^i .\end{aligned}$$

For the inductive step, suppose that n is an integer not less than 2 and that $x_0, x_1, \cdots, x_n$ is a sequence with the following properties:

(9) if k is an integer in $[0, n]$, then

$$||x_k - p|| \leqq \sum_{i=1}^{k} \left(\frac{\delta}{2}\right)^i ;$$

(10) if k is an integer in $[1, n]$, then

$$||x_k - x_{k-1}|| < (\delta/2)^k ;$$

and

(11) $x_0 = p$, and $x_k = (1/\lambda)(Tx_{k-1} + q)$ for each integer k in $[1, n]$. Since $||x_n - p|| \leqq \sum_{i=1}^{n} (\delta/2)^i$ and $\delta < 1$, we have $||x_n - p|| < \delta$. Thus x_n is in $D(T)$, so denote by x_{n+1} the point $(1/\lambda)(Tx_n + q)$. Now

$$\begin{aligned} ||x_{n+1} - x_n|| &= (1/|\lambda|)\,||Tx_n - Tx_{n-1}|| \\ &\leqq (M/|\lambda|)\,||x_n - x_{n-1}|| \\ &< (M/|\lambda|)(\delta/2)^n \text{ by (10)} \\ &< (\delta/2)^{n+1} \quad \text{since} \quad |\lambda| \geqq 2M/\delta . \end{aligned}$$

Thus $||x_{n+1} - p|| \leqq ||x_{n+1} - x_n|| + ||x_n - p|| < \sum_{i=1}^{n+1} (\delta/2)^i$ by (9). This completes the inductive step and yields a sequence $x_0, x_1, x_2, \cdots$ with the following properties:

(12) if n is a nonnegative integer, then

$$||x_n - p|| \leqq \sum_{i=1}^{n} \left(\frac{\delta}{2}\right)^i ;$$

(13) if n is a positive integer, then

$$||x_n - x_{n-1}|| < (\delta/2)^n ;$$

and

(14) $x_0 = p$, and $x_n = (1/\lambda)(Tx_{n-1} + q)$ for each positive integer n.

Since picking $\delta < 1$ yields the fact that $\sum_{n=1}^{\infty} (\delta/2)^n$ converges, we have that $x_0, x_1, x_2, \cdots$ is Cauchy. Thus, since H is complete, denote by x the point of S which is the sequential limit of $x_0, x_1, x_2, \cdots$. By (12) we know that $||x - p|| \leqq \delta < r$, so x is in $R_p(r)$. And, since T is Lipschitzean, hence continuous, on $R_p(r)$, we have that

$$\begin{aligned} (1/\lambda)(Tx + q) &= (1/\lambda)(\lim_{n\to\infty} Tx_n + \lim_{n\to\infty} q) \\ &= (1/\lambda) \lim_{n\to\infty} (Tx_{n-1} + q) \\ &= \lim_{n\to\infty} (1/\lambda)(Tx_{n-1} + q) \\ &= \lim_{n\to\infty} x_n = x . \end{aligned}$$

Therefore $\lambda x - Tx = q$; so $R_{(\lambda I - T)p}(\delta|\lambda|/2) \subseteqq (\lambda I - T)(R_p(r))$, and $\rho_p(T)$ exists and contains λ.

Theorems 2 and 3 provoke several questions. Can the finite dimensionality be removed from Theorem 2 without destroying the equality of $\rho_p(T)$ and $\rho(T'(p))$? Does the existence of $\rho_p(T)$ imply that T is locally Lipschitzean at p? Finally, does the existence of a continuous linear transformation A such that $\rho_p(T) = \rho(A)$ imply that $T'(p) = A$? At present I have no strong feelings about an answer to any of these questions.

REFERENCES

1. J. Dieudonné, *Foundations of Modern Analysis*, New York: Academic Press, 1969.
2. L. A. Harris, *The Numerical Range of Holomorphic Functions*, unpublished manuscript.
3. R. I. Kačurovskiĭ, *The reqular points, spectrum, and eigenfunctions of nonlinear Operators*, Soviet Math. Dokl. **10** (1969), 1101-1104.
4. J. W. Neuberger, *Existence of a spectrum for nonlinear transformations*, Pacific J. Math. **31** (1969), 157-159.
5. E. H. Zarantonello, *The closure of the numerical range contains the spectrum*, Bull. Amer. Math. Soc., **70** (1964), 781-787.
6. ———, *The closure of the numerical range contains the spectrum*, Pacific J. Math., **22** (1967), 575-595.

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