

IDEMPOTENTS IN THE BOUNDARY OF A LIE GROUP

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Let S be a locally compact semigroup consisting of a dense connected Lie group G and its boundary L ; and let e be an idempotent in L . This paper is concerned with the proof of three principal results: (1) If $L = Ge$ is simply connected, then S is homeomorphic to $G/G_r(e) \times G_r(e)/G_l(e) \times G_l(e)^-$, where $G_l(e)^-$ is a connected locally compact group with zero. (2) For any connected Lie group G and closed normal subgroup H such that G/H is simply connected and H is the direct product of the multiplicative group of positive real numbers and a connected compact group, there is a locally compact semigroup S which contains a dense subgroup isomorphic to G whose boundary is a group isomorphic to G/H . (3) If $G = V^*G_l(e)$, for some subspace $V \subset G$, then Ge is locally compact if and only if there is a local cross-section to the global orbits of G at e .

We first establish notation and state a few basic facts that will be used later. It is understood that we are discussing topological groups and semigroups. Hence, unless stated otherwise, homomorphisms are continuous functions and isomorphisms are homeomorphisms. All spaces are Hausdorff. By the boundary of a set A , we mean the set $A^- \setminus A$, where A^- is the usual topological closure of A . Let G be a group with subsets $V, W \subset G$ such that at least one is a subgroup. If $V \cap W = \{1\}$, and the multiplication map of G restricted to $V \times W$ is a homeomorphism onto G , then we write $G = V^*W$. This generalization of the semidirect product will be extremely useful.

Let G be a transformation group acting on the left in the space X , and let $x \in X$. The left isotropy subgroup of x , $\{g \in G \mid gx = x\}$ is a closed subgroup of G and is denoted by $G_l(x)$. $G_r(x)$ is similarly defined if G acts on the right. The G -orbit through x , Gx , is the set $\{gx \mid g \in G\}$ if G acts on the left, and a similar definition is made for right actions. The map $h: G/G_l(x) \rightarrow Gx$, defined by $h[gG_l(x)] = gx$ is always a continuous bijection. If G is connected and locally compact, and Gx is locally compact, then h is a homeomorphism [3, p. 7]. If, in addition, $G = V^*G_l(x)$, then V is homeomorphic to Gx by the map $v \rightarrow vx$. (Merely note that V is homeomorphic to $G/G_l(x)$ via the projection.)

Suppose now that G is a dense connected Lie subgroup of a locally compact semigroup S . Let L be the boundary of G . G is open in G^- (since G is locally compact), so L is closed and locally compact. G acts in S on the left and on the right by the semigroup multiplication. If $x \in L$, then Gx is homeomorphic to $G/G_l(x)$ if Gx is closed

in L . If Gx is, in addition, simply connected as well as closed, then $G/G_i(x)$ is simply connected, and this implies $G_i(x)$ is connected [1, p. 59]. (One should note that if a space is simply connected in the usual sense, then it is simply connected in the sense of [1], and, for manifolds, the two definitions are equivalent [1, p. 44; 3, p. 51].) Recall that for any semigroup S , $M(S)$ denotes the unique minimal two-sided ideal of S , if it exists. If $L = Ge$, where e is an idempotent, then we invariably write H_L, H_R , respectively, for $G_i(e)$, and $G_r(e)$.

In [4], Hofmann completely describes the locally compact groups with zero on his way to describing the locally compact groups with compact boundary. We will rely heavily on his description of certain of these objects, so we will summarize the pertinent facts and definitions. If S is any semigroup and e is a point of S such that $se = e$, for all $s \in S$, then e is a right zero for S . Left zeros are defined similarly. If $Se = eS = \{e\}$, then e is a zero for S . For any idempotent u of S , $H(u)$ denotes the maximal subgroup of S containing u . If S is a locally compact semigroup consisting of a locally compact group G and a non-isolated point $e \notin G$, then S is a locally compact group with zero. It is immediate that $G = H(1)$ and e is a zero for S [4, p. 22]. If G is connected, then G is isomorphic to $P \times C$, where P is the multiplicative group of positive real numbers and C is a connected compact group [4, pp. 40, 49]. Moreover, S is isomorphic to the quotient semigroup $P^- \times C/\{0\} \times C$, where P^- is the multiplicative semigroup of nonnegative real numbers. We identify P with its image in the quotient semigroup S , and we denote the zero of S by e . It follows that the closure of P in S , $P \cup \{e\}$, is isomorphic to P^- [5, p. 51].

1. We begin the proofs with two theorems whose proofs are given in [6, 66. 310–312] and two small lemmas.

THEOREM 1. *Let G be a connected locally compact group of finite dimension embedded in a semigroup, and let e be an idempotent in the closure of G such that Ge is locally compact. Then $e \in G_i(e)^-$.*

THEOREM 2. *Let G be a connected locally compact group embedded in a locally compact semigroup S in such a way that the boundary of G is a single left G -orbit Ge where e is an idempotent. If $G = V * H_L$, for some subspace $V \subset G$, then (i) $H_L^- = H_L \cup \{e\}$, and (ii) the multiplication map of S restricted to $V \times H_L^-$ is a homeomorphism onto G^- .*

LEMMA 3. *Let G be a group and let (\tilde{G}, p) be a group covering of G . Suppose that $\tilde{G} = \tilde{V} * \tilde{H}$, where $\tilde{H} = p^{-1}(H)$ and H is a closed*

subgroup of G . Then $G = V * H$, where $p(\tilde{V}) = V$.

Proof. Clearly $G \subset VH$. We show now that the multiplication map, $m: V \times H \rightarrow G$ is one-to-one. Suppose $v_1 h_1 = v_2 h_2$. Then there exist $\tilde{v}_1, \tilde{v}_2 \in \tilde{V}$ and $\tilde{h}_1, \tilde{h}_2 \in \tilde{H}$ such that $p(\tilde{v}_i) = v_i$ and $p(\tilde{h}_i) = h_i$, for $i = 1, 2$. Thus $p(\tilde{v}_1)p(\tilde{h}_1) = p(\tilde{v}_2)p(\tilde{h}_2)$. This implies, since p is a homomorphism, that $\tilde{v}_1 \tilde{h}_1 = \tilde{v}_2 \tilde{h}_2 \tilde{h}_3$, where $\tilde{h}_3 \in (\text{kernel of } p) \subset \tilde{H}$. Since $\tilde{G} = \tilde{V} * \tilde{H}$, $\tilde{v}_1 = \tilde{v}_2$ and $\tilde{h}_1 = \tilde{h}_2 \tilde{h}_3$. Consequently, $p(\tilde{v}_1) = p(\tilde{v}_2)$ and $p(\tilde{h}_1) = p(\tilde{h}_2)$. Thus we have $v_1 = v_2$ and $h_1 = h_2$. We have now only to show that m is open. Let $v_i h_i \rightarrow x \in G$, and let $\{v_i h_i\} \subset N$, an evenly covered neighborhood of x . Let \tilde{N} be a component of $p^{-1}(N)$. We have $\tilde{v}_i \tilde{h}_i \rightarrow \tilde{x}$ in \tilde{N} , where $p(\tilde{v}_i \tilde{h}_i) = p(\tilde{v}_i)p(\tilde{h}_i) = v_i h_i$. This implies that $p(\tilde{v}_i) = v_i$ and $p(\tilde{h}_i) = h_i$. There is a \tilde{v} and \tilde{h} in \tilde{G} such that $\tilde{x} = \tilde{v} \tilde{h}$, $\tilde{v}_i \rightarrow \tilde{v}$, and $\tilde{h}_i \rightarrow \tilde{h}$, since $\tilde{G} = \tilde{V} * \tilde{H}$. Thus, by continuity of p , $v_i \rightarrow p(\tilde{v})$, $h_i \rightarrow p(\tilde{h})$, and $x = p(\tilde{v})p(\tilde{h})$. This shows that m is open.

LEMMA 4. *If G is a connected Lie group and H is a closed normal subgroup of G such that G/H is simply connected, then there is a subspace V of G such that $G = V * H$.*

Proof. Let (\tilde{G}, p) be a simply connected group covering of G , and let $p^{-1}(H) = \tilde{H}$. Since G/H is simply connected, and G/H is homeomorphic to \tilde{G}/\tilde{H} , it follows that \tilde{H} is connected [1, p. 59]. Consequently, there is a subspace $\tilde{V} \subset \tilde{G}$ such that $\tilde{G} = \tilde{V} * \tilde{H}$ [3, p. 135]. By Lemma 3, $G = V * H$, where $V = p(\tilde{V})$.

THEOREM 5. *Let G be a connected Lie group which is dense in a locally compact semigroup. Suppose that L , the boundary of G , is a single left G -orbit generated by an idempotent e , and let $S = G \cup L$. Then*

- (i) $H_L^- = H_L \cup \{e\}$ is a locally compact group with zero.
- (ii) $eG = H(e)$ is a closed topological subgroup of L .
- (iii) $L = M(S)$.
- (iv) L is a group if and only if $H_L = H_R$.
- (v) S is simply connected if and only if L is simply connected.
- (vi) If L is simply connected, then S is homeomorphic to $G/H_R \times H_R/H_L \times H_L^-$, and H_L^- is connected.
- (vii) Let L be simply connected. Then S is a manifold with boundary if and only if H_L is isomorphic to the multiplicative group of positive real numbers; which happens if and only if $\dim L = \dim G - 1$.

Proof. (i) e is in H_L^- , by Theorem 1. Since e is a right identity for L and a right zero for H_L^- it follows that $H_L^- \cap L = \{e\}$. H_L is

closed in G so $H_L^- = H_L \cup \{e\}$. Thus H_L^- is a locally compact group with zero.

(ii) That eG is an algebraic subgroup of L follows easily from the fact that $eG \subset Ge$. Now, $eS = e(G^-) = e(G \cup Ge) = eG \cup eGe = eG$. Similarly, $Se = Ge$. Thus $eS \cap Se = eG$, and this implies $eG = H(e)$. Clearly, eS and Se are closed sets, thus eG is closed. Since eG is locally compact, a theorem due to Ellis [2] implies that eG is a topological group.

(iii) The argument for (ii) shows that L is an ideal in S . If I is any ideal of S , then, for $x \in I \cap L$, there exist elements g and h of G such that $x = ge$ and $eg = he$. Hence $Gex = Gege = Ghe = Ge = L \subset I$. Thus $L = M(S)$.

(iv) Assume $H_L = H_R$. We will show that e is in the center of S . Fix $g \in G$. Since $eG \subset Ge$, $geg^{-1} \in Ge$ so $geg^{-1} = he$ for some $h \in G$. Now $e = egeg^{-1} = egeg^{-1} = ehe = eh$. Thus $h \in H_R$. This implies $e = he = geg^{-1}$ and $eg = ge$. Thus $Ge = eG$ and, by (ii), L is a group. Conversely, if $L = M(S)$ is a group, then e is in the center of S so $H_L = H_R$.

(v) The map $s \rightarrow se$ is a retraction of S onto L , thus if S is simply connected, L is simply connected. Assume now that L is simply connected. By (i) H_L^- is a locally compact group with zero. Since L is simply connected, G/H_L is simply connected and so H_L is connected. Consequently, $H_L = P \times C$, where P is a subgroup of G isomorphic to the multiplicative group of positive real numbers and C is a connected compact subgroup of G . Moreover, $P^- = P \cup \{e\}$. Let $[1, e]$ denote the closed arc of P^- that connects 1 to e . Define $T: [1, e] \times S \rightarrow S$ by $T(p, s) = sp$, for $p \in [1, e]$, and $s \in S$. T is a deformation retraction of S onto L . Thus S is simply connected.

(vi) The argument for (v) shows that H_L is connected, thus H_L^- is connected. Using the notation of the proof of (v), define $T': [1, e] \times S \rightarrow S$ by $T'(p, s) = ps$. T' is a deformation retraction of S onto eG which implies eG is simply connected. Now, the map $f: G \rightarrow eG$, defined by $f(g) = eg$, is a homeomorphism and H_R is the kernel of f . Thus H_R is a closed normal subgroup of G . By Lemma 4, there is a subspace V of G such that $G = V * H_R$. We will show that the multiplication map, $m: V \times H_R^- \rightarrow S$ is a homeomorphism onto S . First we show that $H_R^- \cap L = H_R e$. Clearly $H_R e \subset H_R^- \cap L$ since the closure of a subgroup is a semigroup, and L is an ideal. Let $x \in H_R^- \cap L$. Then, for some $g \in G$, $x = ge$, and $e = ex = ege = eg$. Thus $g \in H_R$, and $x \in H_R e$. Since $G = VH_R$ and $L = Ge = VH_R e$, it is clear that m is onto. We show next that m is one-to-one. Since m is one-to-one on $V \times H_R$, we have only to show that if $v_1 h_1 e = v_2 h_2 e$, for $v_1, v_2 \in V$ and $h_1, h_2 \in H_R$, then $v_1 = v_2$ and $h_1 e = h_2 e$. Let $v_1 h_1 e = v_2 h_2 e$. Then $v_1 h_1 = v_2 h_2 h$, where $h \in H_L \subset H_R$. (Notice that $H_L \subset H_R$ since, by (i),

$H_L^- = H_L \cup \{e\}$.) This implies $v_1 = v_2$ and $h_1 = h_2h$, since $G = V * H_R$. Thus $h_1e = h_2e$. To show that m is open we have only to show the following two statements:

(a) $v_i h_i e \rightarrow v h e \Rightarrow v_i \rightarrow v$ and $h_i e \rightarrow h e$, where $v_i v \in V$ and $h_i, h \in H_R$; and

(b) $v_i h_i \rightarrow v h e \Rightarrow v_i \rightarrow v$ and $h_i \rightarrow h_i e$, where $v_i, v \in V$ and $h_i, h \in H_R$.

Now, (b) follows easily from (a) (postmultiplication by e) so we will prove only (a). Let $v_i h_i e \rightarrow v h e$. Then $ev_i = ev_i h_i e \rightarrow ev h e = ev$. Since eG is a group, this implies $ev_i^{-1} \rightarrow ev^{-1}$. Since $G = H_R * V^{-1}$, this implies $v_i^{-1} \rightarrow v^{-1}$. Thus $v_i \rightarrow v$ and $h_i e \rightarrow h e$. What has been shown to this point is that G is homeomorphic to $G/H_R \times H_R^-$. In the next paragraph we show that H_L^- is homeomorphic to $H_R/H_L \times H_L^-$.

By restricting the maps T and T' to $[1, e] \times H_R^-$, respectively, one sees that H_{Re} is a deformation retraction of H_R^- , and H_R^- is contractible. Hence, H_{Re} is contractible. Since H_{Re} is closed, this implies H_R/H_L is contractible. Consequently, there is a subspace $W \subset H_R$ such that $H_R = W * H_L$ [7, p. 53]. It is worth noticing that H_R^- is a locally compact semigroup with a dense connected Lie group whose boundary is a single left orbit generated by a left zero. A description of these semigroups would be a big step toward completely describing the semigroups we are concerned with in this theorem. We now show that the multiplication map, $m: W \times H_L^- \times H_R^-$ is a homeomorphism. To show that m is onto, let $k \in H_R^- \cap L$. We have seen that $H_R^- \cap L = H_{Re}$. Hence there exists $w \in W, h \in H_L$ such that $(wh)e = we = k$. Since $H_R = W * H_L$, this shows that m is onto. To show that m is open, we have only to check that $w_i h_i \rightarrow we$, where $w_i, w \in W$ and $h_i \in H_L \Rightarrow w_i \rightarrow w$ and $h_i \rightarrow e$. If $w_i h_i \rightarrow we$, then $w_i e \rightarrow we$. Since W is homeomorphic to W via the map $w \rightarrow we$, we have $w_i \rightarrow w$ and thus $h_i \rightarrow e$.

(vii) We saw above that $V \times H_R^-$ is homeomorphic to S and $W \times H_L^-$ is homeomorphic to H_R^- ; each by a suitable restriction of the multiplication map of S . Thus S is homeomorphic to $U \times H_L^-$ by a restriction of the multiplication map of S , where $U = VW \subset G$. Moreover, $G = U * H_L$, and $L = Ue$ is homeomorphic to U . Since U, P, C and L are manifolds, we will mean by the dimension of a space its dimension as a manifold. If $H_L^- = P^-$, then, clearly, S is a manifold with boundary. Also, in this case, $\dim G = \dim U + 1$. Thus $\dim L = \dim G - 1$. We will show now that if $C \neq \{1\}$, then S cannot be a manifold with boundary, and, moreover, $\dim L \leq \dim G - 2$. This suffices to show the equivalence of the three statements of the theorem. It might be well to keep in mind that S could be a manifold without boundary if, say, C were the circle group; for then H_L^- would be the multiplicative semigroup of complex numbers.

Let $C \neq \{1\}$. Then $\dim G = \dim U + 1 + \dim C$. Since $\dim C \geq 1$

(C is connected), we have $\dim L = \dim U \leq \dim G - 2$. We have only to show now that S cannot be a manifold with boundary. We assume that S is a manifold with boundary and arrive at a contradiction. G is an open homogeneous subset of S , hence $G \subseteq \text{Int } S$. Consequently, $\phi \neq \text{bd}S \subset L$. Let $u_1e, u_2e \in L$. U is a homogeneous space so there is a homeomorphism $h: U \rightarrow U$ and $h(u_1) = u_2$. Let $x \in U, y \in H_L^-$. The following composition of homeomorphisms is a homeomorphism of S onto S that carries u_1e to u_2e : $xy \in S \rightarrow (x, y) \in U \times H_L^- \rightarrow (h(x), y) \in U \times H_L^- \rightarrow h(x)y \in S$. This implies that all of L must be in the boundary of S and thus $L = \text{bd}S$. Consequently, we have the four statements, (i) $L = \text{bd}S$, (ii) $G = \text{Int } S$, (iii) $\dim L \leq \dim G - 2$ and (iv) S is a manifold with boundary. This is impossible.

THEOREM 6. *Let G be a connected Lie group with a closed normal subgroup H such that G/H is simply connected. Let $H = C \times P$ where C is a connected compact group and P is the multiplicative group of positive real numbers. Then G can be embedded in a locally compact semigroup as a dense subgroup whose boundary is a group isomorphic to G/H .*

Proof. Let H^- denote the locally compact semigroup $P^- \times C / \{0\} \times C$, where P^- is the multiplicative semigroup of nonnegative real numbers, and $P = P^- \setminus \{0\}$. We identify P with its image in H^- , and we denote the zero of H^- by e . The closure of P in H^- , $P \cup \{e\}$, is isomorphic to P^- . By Lemma 4 we have $G = V * H$. For $g \in G$, there exist unique $v \in V, h \in H$ such that $g = vh$. We define $[g]$ to be v . Since H is normal and multiplication in G is associative, we have $[g_1[g_2g_3]] = [[g_1g_2]g_3]$. Also, $[g_1g_2] = [v_1v_2]$ where $g_1 = v_1h_1$ and $g_2 = v_2h_2$; and $g_i \rightarrow g = [g_i] \rightarrow [g]$.

Consider the function $m: (V \times H^-) \times (V \times H^-) \rightarrow V \times H^-$ defined by (i) m restricted to $(V \times H) \times (V \times H)$ is the multiplication of G . (ii) $m((v_1, e), (v_2, h)) = m((v_1, h), (v_2, e)) = m((v_1, e), (v_2, e)) = ([v_1v_2], e)$. It is straightforward to show that m is an associative multiplication. To show that m is a continuous multiplication on $V \times H^-$ it suffices to show the following statements:

$$(1) \quad (v_i, e) \rightarrow (v, e) \text{ and } (v_{1i}, h_i) \rightarrow (v_1, h_1) \Rightarrow ([v_i v_{1i}], e) \rightarrow ([v v_1], e) \text{ and } ([v_{1i} v_{1i}], e) \rightarrow ([v_1 v_1], e).$$

$$(2) \quad (v_i, e) \rightarrow (v, e) \text{ and } (v_{1i}, h_i) \rightarrow (v_1, e) \Rightarrow ([v_i v_{1i}], e) \rightarrow ([v v_1], e) \text{ and } ([v_{1i} v_{1i}], e) \rightarrow ([v_1 v_1], e).$$

$$(3) \quad (v_i, e) \rightarrow (v_1, e) \text{ and } (v'_i, e) \rightarrow (v', e) \Rightarrow ([v_i v'_i], e) \rightarrow ([v v'], e).$$

$$(4) \quad (v_{1i}, h_{1i}) \rightarrow (v_1, h_1) \text{ and } (v_{2i}, h_{2i}) \rightarrow (v_2, h_2) \Rightarrow ([v_{1i} v_{2i}], h'_i h'_i h_{2i}) \rightarrow ([v_1 v_2], e) \text{ where } h'_i = v_{2i}^{-1} h_{1i} v_{2i} \text{ and } v_{1i} v_{2i} = [v_{1i} v_{2i}] h'_i.$$

$$(5) \quad (v_{1i}, h_{1i}) \rightarrow (v_1, h_1) \text{ and } (v_{2i}, h_{2i}) \rightarrow (v_2, h_2) \Rightarrow ([v_{1i} v_{2i}], h'_i h'_i h_{2i}) \rightarrow ([v_1 v_2], e) \text{ where } h'_i = v_{2i}^{-1} h_{1i} v_{2i} \text{ and } v_{1i} v_{2i} = [v_{1i} v_{2i}] h'_i.$$

(6) $(v_{1i}, h_{1i}) \rightarrow (v_1, e)$ and $(v_{2i}, h_{2i}) \rightarrow (v_2, e) \Rightarrow ([v_{1i}v_{2i}], h_i''h_i'h_{2i}) \rightarrow ([v_1v_2], e)$ where $h_i' = v_{2i}^{-1}h_{1i}v_{2i}$ and $v_{1i}v_{2i} = [v_{1i}v_{2i}]h_i''$.

(1)-(4) are immediate, and (5)-(6) will follow if we show:

(*) $v_i \rightarrow v$ and $h_i \rightarrow e \Rightarrow v_i h_i v_i^{-1} \rightarrow e$, where $v_i \in V, h_i \in H$. We now show (*).

For each $g \in G$, let Ag be the automorphism of H induced by the inner automorphism of $G, x \rightarrow gxg^{-1}$. Let k be the projection of H onto P , and let $kAg|P$ be denoted by kg . Notice that C is the maximal compact subgroup of H , so C is invariant under Ag for all $g \in G$. Clearly, for any $g \in G, kg$ is an automorphism of P . It follows from this that the map $g \rightarrow kg$ is a continuous homomorphism of G into the connected component of the identity of the automorphism group of P . We assume now that P is the group of additive reals. Thus $kg \in A(P)_0$, which we identify with the multiplicative positive reals. We may assume that $p_i \rightarrow e$ in H if and only if $p_i \rightarrow +\infty$ in the usual orientation of P . Thus, for each $g \in G$, there is a positive real number $a(g)$ such that $kg(p) = a(g) \cdot p$ for all $p \in P$. Consequently, $p_i \rightarrow e$ if and only if $kg(p_i) \rightarrow e$ for all $g \in G$. Consider $\{c_i p_i\} \subset H$, where $c_i \in C$ and $p_i \in P$. Let $Ag(c_i p_i) = c_{i1} c_i' p_i'$, where $Ag(c_i) = c_{i1}$ and $Ag(p_i) = c_i' p_i'$. Clearly, $Ag(c_i p_i) \rightarrow e \Leftrightarrow p_i' \rightarrow e \Leftrightarrow kg(p_i) \rightarrow e \Leftrightarrow p_i \rightarrow e \Leftrightarrow c_i p_i \rightarrow e$.

Now let $v_i \rightarrow v, v_i, v \in V$, and $h_i = c_i p_i \rightarrow e$. Then $v_i(c_i p_i)v_i^{-1} = c_i'(a(v_i) \cdot p_i)$, so $v_i(c_i p_i)v_i^{-1} \rightarrow e \Leftrightarrow a(v_i) \cdot p_i \rightarrow e$. Since $v_i \rightarrow v$ and $p_i \rightarrow \infty$, it follows that $a(v_i) \rightarrow a(v)$, and thus $a(v_i) \cdot p_i \rightarrow \infty$. Thus $v_i h_i v_i^{-1} \rightarrow e$, and we are through.

Clearly, $V \times H^-$ is locally compact and contains a dense subgroup isomorphic to G . The map $f: G \rightarrow V \times \{e\}$ defined by $f(g) = ([g], e)$ induces an isomorphism from G/H onto $V \times \{e\}$, the boundary of $V \times H$ in $V \times H^-$.

REMARKS AND EXAMPLES. If G is an abelian Lie group, then it is not too difficult to see that we may pick V to be a vector group and that S is the product semigroup $V \times H^-$. In this case, our result is a corollary of a theorem of Stepp's [8, Theorem 3, p. 404]. The semigroups on a three-dimensional half-space without radical [5, pp. 45-48] provide a variety of examples of semigroups that satisfy the hypotheses of Theorem 5. It is worth noticing here that if G is the noncommutative nilpotent group on E^3 and S is the half-space semigroup $G \cup G/P$, where P is the center of G , then P has no complementary group in G [6]. In [5, pp. 46-47] Horne constructs in a completely different way a half-space semigroup $G \cup G/P$, where the boundary of G is the group G/P for any group G on E^3 which has a normal one-parameter subgroup P . It is clear that the construction given in the argument of Theorem 6 will provide an n -dimensional

half-space semi-group for any group on E^n which has a normal one-parameter subgroup.

Our next result is a theorem concerning the action of G at e that is an improvement of Theorem 2. Following Definition 1.10, page 315 (and the Remark on the page) of Hofmann and Mostert's book, *Elements of Compact Semigroups* [Charles E. Merrill Books, Inc. (1966)], we will define a local cross section to the local orbits and a local cross section to the global orbits of a transformation group at a point. For connected locally compact groups G of finite dimension action on completely regular spaces X , our definition of a *local* cross section to the *local* orbits is equivalent to that given in Hofmann and Mostert's book.

DEFINITION Let G be a transformation group acting on the right on a space X . Let $x \in X$, and let $H = G_r(x)$. We call the triple (C, K, U) a local cross section to the local orbits of G at x if

- (1) U is a neighborhood of x ;
- (2) C is a closed subset of U containing x ;
- (3) K is a closed subset of G containing 1 that is mapped homeomorphically, via the projection map, onto a neighborhood of H in $G/H = \{Hg | g \in G\}$;
- (4) the action of G , restricted to $C \times K$, is a homeomorphism onto U .

If G acts on X on the left, then $H = G_l(x)$, and (C, K, U) is a local cross section to the local orbits of G at x where C, K are as above except that $G/H = \{gH | g \in G\}$ in (3), and $K \times C$ is the domain of the action in (4). If, in addition to the above, K maps onto G/H in (3), then (C, K, U) is a local cross section to the global orbits of G at x . If then $U = X$, we have a global or complete cross section to the action of G at x .

THEOREM 7. *Let G be a connected, locally compact, dense subgroup of a locally compact semigroup S . Let e be an idempotent in the closure of G such that $G = W * H$, where H is the left isotropy subgroup of e relative to the natural action of G on S by the multiplication in S . It follows that Ge is a locally compact subspace of S if and only if (H^-, W, WH^-) is a local cross section to the global orbits of G at e .*

Proof. Assume Ge locally compact. Then W is homeomorphic to $We = Ge$ via the map $w \rightarrow we$. There is a net $w_i h_i \rightarrow e$, so $w_i e \rightarrow e$ and $w_i \rightarrow 1$. This implies $h_i \rightarrow e$ and $e \in H^-$. We now show that the multiplication map, $W \times H^- \rightarrow WH^-$ is a homeomorphism. If $w_1 h_1 = w_2 h_2$, where $w_i \in W, h_i \in H^-$, then $w_1 e = w_2 e$ $w_1 = w_2$. Thus $h_1 = h_2$

and m is one-to-one. Suppose $w_i h_i \rightarrow wh$, where $w_i, w \in W$ and $h_i, h \in H^-$. Then $w_i e \rightarrow we$, so $w_i \rightarrow w$. Hence $h_i \rightarrow h$ and m is open onto WH^- . We have shown that $m: W \times H^- \rightarrow S$ is a homeomorphism into S . Since $e \in H^-$, $G \cup Ge \subset WH^-$. Since $W \times H^-$ is locally compact, WH^- is a locally compact, dense subset of S . Thus WH^- is an open subset of S .

Assume that (H^-, W, WH^-) is a local cross section to the global orbits of G at e . It is immediate that $G \subset WH$ and $e \in H^-$, so $Ge = m(W \times \{e\})$ is locally compact since $W \times \{e\}$ is locally compact.

We conclude with a corollary that has applications to half-space semigroups. First, we mention a fact that will be used in the proof that follows. A reference for the relevant theorems about Lie groups is H. Samelson's survey article, *Topology of Lie groups* [Bull. Amer. Math. Soc., 58 (1952), 2-37]. Let G be a connected Lie group and let H be a maximal compact subgroup of G . A theorem of Iwasawa states that G is homeomorphic to the product space $H \times V$, where V is a euclidean space. Since H is an orientable manifold and consequently not acyclic, it follows that G is a group on euclidean space if and only if G contains no nontrivial compact subgroup. An immediate consequence of this is that a closed connected subgroup of a group on euclidean space is a group on euclidean space.

COROLLARY 8. *Let G be a Lie group on euclidean n -space and a dense subgroup of a locally compact semigroup S . Let e be an idempotent in the boundary of G such that Ge is a locally compact and simply connected subspace of S . Then $G = W * H$, for some $W \subset G$, where H is the left isotropy subgroup of e relative to the natural action of G on S by the multiplication in S . Consequently, (H^-, W, WH^-) is a local cross section to the global orbits of G at e .*

Proof. Since Ge is locally compact, Ge is homeomorphic to G/H , so G/H is simply connected. This implies that H is connected. By the remarks above, H is a group on euclidean m -space, $m \leq n$. Since G is a bundle over G/H with solid fibre H , there is a cross section, $f: G/H \rightarrow G$ [7, p. 55]. Let $f(G/H) = W$. Clearly $p|W$ is the inverse of f , so f is a homeomorphism. If now, $w_i h_i \rightarrow wh$, where $w_i, w \in W$, $h_i, h \in H$, then $p(w_i) \rightarrow p(w)$ which implies $w_i \rightarrow w$. Thus $h_i \rightarrow h$ and we have shown that the multiplication map of G , restricted to $W \times H$ is open onto $WH = G$. Clearly this map is also injective and continuous, so $G = W * H$. Theorem 7 implies now that (H^-, W, WH^-) is a local cross section to the global orbits of G at e .

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