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DIFFERENTIAL INEQUALITIES AND LOCAL VALENCY

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An entire function $f(z)$ is said to have bounded value distribution (b.v.d.) if there exist constants p, R such that the equation $f(z) = w$ never has more than p roots in any disk of radius R. It is shown that this is the case for a particular p and some $R > 0$ if and only if there is a constant $C > 0$ such that for all z

$$|f^{(p+1)}(z)| \leqq C \max_{\nu=1 \text{ to } p} |f^{(\nu)}(z)| ,$$

so that $f'(z)$ has bounded index in the sense of Lepson.

We consider the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots a_n y = 0 , \tag{1.1}$$

in the disk

$$D_0 = \{z \,|\, |z - z_0| < R\} , \tag{1.2}$$

where $0 < R \leqq \infty$ and the functions a_1 to a_n are supposed to be regular and bounded in D_0. The solutions of (1.1) are regular in D_0 and possess there no zeros of order greater than $n - 1$. This prompted Tijdeman to ask for a disk depending only on z_0, R and the coefficients a_1 to a_n, in which the equation (1.1) is *disconjugate*, i.e. no solution has more than $n - 1$ zeros. He later solved this problem [10] using a method due to Turan [12]. At about the same time a solution was given by Kim [5], who obtained sufficient conditions on a_1 to a_n for (1.1) to be disconjugate in the whole of D_0.

An interesting special case occurs when $R = \infty$, so that D_0 is the open plane. In this case a_1 to a_n must be constant and the solutions take the form

$$y(z) = \sum_{\nu=1}^{k} p_\nu(z) e^{\omega_\nu z} \tag{1.3}$$

where the ω_ν are the roots of the characteristic equation

$$P(\omega) = \omega^n + \sum_{\mu=1}^{n} a_\mu \omega^{n-\mu} = 0 . \tag{1.4}$$

If ω_ν has multiplicity k_ν then $p_\nu(z)$ is a polynomial of degree at most $k_\nu - 1$. With his methods Tijdeman [11] has obtained a number of striking results concerning the distribution of the zeros of (1.3). Setting

$$\Delta = \max_{\nu=1 \text{ to } k} |\omega_\nu| ,$$

he proved that if N_R is the number of zeros of $y(z)$ in D_0, then

$$N_R \leqq 3(n-1) + 4R\Delta .$$

This sharpens considerably an earlier result of Dancs and Turán [2]. He also proved [11, Theorem 2]

THEOREM A. *If $R\Delta \leqq \min\{n^{-2}, (24)^{-2}\}$, then $N_R \leqq n-1$, so that* (1.1) *is disconjugate in D_0.*

We can clearly consider $y(z) - w$ instead of w, where w is a constant, at the expense of replacing the order n of (1.1) by $n+1$. Thus we deduce at once from Tijdeman's results

THEOREM B. *If $N_R(w)$ denotes the number of roots of $y(z) = w$ in D_0, then*

$$N_R(w) \leqq 3n + 4R\Delta , \tag{1.5}$$

and if $R\Delta \leqq \min\{(n+1)^{-2}, (24)^{-2}\}$, we have

$$N_R(w) \leqq n . \tag{1.6}$$

The bounds in (1.5) and (1.6) are independent of w and z_0. We can restate (1.6) by saying that $y(z)$ is at most n-valent in any disk of radius R in the open plane.

We now quote the main result of Kim [5, pp. 721, 722], which may be stated in the following form

THEOREM C. *Suppose that $y(z)$ is regular in D_0, and has n zeros there. Then there exists z_1 in D_0 such that*

$$|y(z_1)| < \frac{1}{n!}(R - |z_1 - z_0|)(R + |z_1 - z_0|)^{n-1}|y^{(n)}(z_1)| , \tag{1.7}$$

$$|y^{(n-k)}(z_1)| < \frac{1}{k!}(R + |z_1 - z_0|)^k |y^{(n)}(z_1)|, \; 1 \leqq k \leqq n-1 . \tag{1.8}$$

Hence if

$$\sum_{k=1}^{n-1} \frac{(R+|z-z_0|)^k}{k!}|a_k(z)| + \frac{(R-|z-z_0|)(R+|z-z_0|)^{n-1}}{n!}|a_n(z)| \leqq 1 \tag{1.9}$$

in D_0 then (1.1) *is disconjugate in D_0.*

Both Kim's and Tijdeman's results can also be applied to equations

(1.1) with polynomial coefficients.

In connection with Theorem B, Turán [3, problem 2.28] considered functions $y(z)$ which satisfy (1.6) for fixed constants R and n and every z_0. He called such $y(z)$ functions of bounded value distribution (b.v.d.). Suppose that in the equation (1.1) the coefficients are integral functions, and that every solution $y(z)$ has b.v.d.. Then he asked if the a_1 to a_n are necessarily constants. This was proved to be the case by Wittich [13] even under the weaker hypothesis, that for every solution $y(z)$ there exists at least one $w \neq 0$ and a constant R, such that $N_R(w)$ is bounded above for all z_0. Turán [3, problem 2.28] also asked whether a b.v.d. function necessarily has at most mean type of order one in the plane.

A related question was raised by Lepson [6]. Let $f(z)$ be an entire function and for each z let $N(z)$ be the least integer such that

$$\sup_{0 \leqq j < \infty} \left| \frac{f^{(j)}(z)}{j!} \right| = \frac{|f^{(N)}(z)|}{N!} . \tag{1.10}$$

If $N(z)$ is bounded above for varying z, then $f(z)$ is said to be of *bounded index*, and the least upper bound N of $N(z)$ is called the *index* of $f(z)$. It was shown by Shah [7] that the solutions of equations (1.1) have bounded index if the $a_n(z)$ are constants. More generally if the $a_n(z)$ are rational functions which remain bounded at ∞, then any solution of (1.1) which is an entire function has bounded index [8]. Shah also showed that any function of bounded index has order 1, type $N+1$ at most [7]. This result is sharp as shown by $f(z) = \exp\{(N+1)z\}$. It is evident that if $f'(z)$ has bounded index N, then $f(z)$ has bounded index at most $N+1$. The converse is however false, as an example at the end of the paper will show. Another example has just been given by Shah [9].

Lepson in conversation with me raised the question as to whether the functions of bounded index and b.v.d. were related. This problem was the basis of the present paper. We can settle the question in one direction very simply by quoting the following form of a classical result on p-valent functions. We denote as usual positive absolute constants by A, constants depending on p, q etc. by $A(p)$, $A(p, q)$, particular constants by $A_1, A_2, \cdots$ etc.

THEOREM D. *Suppose that $f(z)$ is p-valent in D_0,* i.e. *that $f(z)$ is regular in D_0 and assumes no value more than p times there; then for $j > p$*

$$\frac{|f^{(j)}(z_0)| R^j}{j!} \leqq A_1(p) j^{2p} \max_{\nu=1 \text{ to } p} \frac{|f^{(\nu)}(z_0)| R^\nu}{\nu!} . \tag{1.11}$$

The result (with j^{2p-1} instead of j^{2p}) is due to Biernacki [1]. The best known numerical value of $A_1(p)$ is due to Jenkins and Oikawa [4], who proved a bound for the maximum modulus of p-valent functions from which Theorem D follows with $A_1(p) = (A_1/p)^{2p}$. If $R \geqq 2$ we shall deduce from (1.11) that $f'(z)$ has index at most $A_3 p$. If $R < 2$, simple geometrical considerations show that $f(z)$ is at most p_1-valent in $|z - z_0| < 2$, for every z_0, where $p_1 < A_4 p/R^2$. Thus in this case $f(z)$ has index at most $A_5 p/R^2$. Thus we have the following

THEOREM 1. *If $f(z)$ is p-valent in $|z - z_0| < R$, for every z_0 and a fixed R, then $f'(z)$ has index at most*[1] $Ap \max(1, R^{-2})$. *If $R \geqq A_6^p$, then $f'(z)$ has index at most $p - 1$. In particular if $f(z)$ has b.v.d. then $f'(z)$ has bounded index.*

2. **Statement of further results.** We shall in this paper prove a local converse to Theorem D, i.e. we shall show that if $f(z)$ satisfies an inequality such as (1.11) in a disk, then $f(z)$ is p-valent in a suitable smaller disk. The result will also enable us to improve the Theorems of Tijdeman and Kim in certain cases. It is convenient to define

$$f_n(z) = \max_{0 \leqq \nu \leqq n-1} |f^{(\nu)}(z)| . \tag{2.1}$$

Our first result is

THEOREM 2. *If $f(z)$ is regular in D_0 and satisfies*

$$|f^{(n)}(z)| \leqq f_n(z) \tag{2.2}$$

there, then we have in D_0

$$f_n(z) \leqq f_n(z_0) \exp(|z - z_0|) . \tag{2.3}$$

COROLLARY 1. *If $f(z)$ is an integral function satisfying* (2.2) *in the whole open plane, then*

$$|f(z)| \leqq f_n(0)e^{|z|} \tag{2.4}$$

there, so that $f(z)$ has at most order 1, *type one.*

COROLLARY 2. *If $f(z)$ is an integral function of bounded index N, then*

[1] It will follow at once from (2.8), that this estimate can be replaced by $Ap \max(1, R^{-1})$.

$$|f(z)| \leqq e^{(N+1)|z|} \max_{0 \leqq \nu \leqq N} \frac{|f^{(\nu)}(0)|}{(N+1)^{\nu}} \leqq f_{N+1}(0) e^{(N+1)|z|} . \tag{2.5}$$

The Corollary 2 sharpens the theorem of Shah [7] previously referred to. The function $f(z) = e^{(N+1)z}$, which has index N, shows that (2.5) cannot be further sharpened except possibly for the constant $f_{N+1}(0)$. On the other hand the inequality (2.3) is sharp, as $f(z) = e^z$ shows. This is the reason why (2.2) is more convenient as a normalisation than (1.10) for the purpose of this paper. Our main result is an analogue of Theorem C. This is

THEOREM 3. *Suppose that $f(z)$ is regular in $|z| < 2n$ and satisfies (2.2) there. Then $f(z)$ possesses at most $n - 1$ zeros in*

$$|z| \leqq n^{1/2}/(e\sqrt{20}) < n^{1/2}/12.2.$$

Let us compare this result with the conclusion to be drawn from Theorem C.

THEOREM E. *Suppose that $f(z)$ is regular and satisfies (2.2) in $|z| \leqq R$, where $R \leqq 1/2$. Then $f(z)$ has at most $n - 1$ zeros there.*

Theorem E is stronger than Theorem 3 for $n \leqq 37$, and we only have to assume that $f(z)$ satisfies the hypotheses in $|z| \leqq R$. On the other hand for $n \geqq 38$, Theorem 3 yieds a larger disk in which there are at most $n - 1$ zeros, at the cost of assuming that $f(z)$ satisfies the hypotheses in a still larger disk.

The order of magnitude in Theorem 3 is the correct one for large n. We set

$$f(z) = \frac{1}{n!}(z^2 - 3(n-1))^{(1/2)n} , \quad \text{if } n \text{ is even} ;$$

$$f(z) = \frac{1}{n!} z(z^2 - 3n)^{(1/2)(n-1)} , \quad \text{if } n \text{ is odd} .$$

Then $f(z)$ has n zeros in $|z| \leqq \sqrt{(3n)}$. On the other hand

$$f^{(n)}(z) = 1 , \quad f^{(n-1)}(z) = z , \quad f^{(n-2)}(z) = \frac{1}{2}z^2 - \frac{3}{2} ,$$

so that we have for all z

$$1 = |f^{(n)}(z)| \leqq \max(|f^{(n-1)}(z)|, |f^{(n-2)}(z)|) \leqq f_n(z) .$$

Thus we cannot replace $n^{1/2}/(12.2)$, by $n^{1/2}\sqrt{3}$ in Theorem 3, even if (2.2) is satisfied in the whole plane.

By applying Theorem 3 to $f(z) - w$, we obtain the desired converse to Theorem D. This is

THEOREM 4. *Suppose that $f(z)$ is regular in D_0 and satisfies there*

$$(CR)^{p+1}\left|\frac{f^{(p+1)}(z)}{(p+1)!}\right| \leqq \max_{1\leqq\nu\leqq p} (CR)^{\nu}\frac{|f^{(\nu)}(z)|}{\nu!} ,$$

with $C \leqq 1/2$. Then $f(z)$ is p-valent in $|z - z_0| \leqq CR/\{12\cdot 2(p+1)^{1/2}\}$.

COROLLARY. *An integral function $f(z)$ has b.v.d. if and only if $f'(z)$ has bounded index. More particularly if $p(R)$ is the upper bound of the valencies of $f(z)$ in $|z - z_0| < R$, for varying z_0, and N is the index of $f'(z)$ we have*

$$p\left\{\frac{1}{12\cdot 2}(N+2)^{-1/2}\right\} \leqq N + 1 \leqq p(A_6^N) .$$

We can also prove

THEOREM 5. *Suppose that $f'(z)$ is a function of bounded index N and let $p(R)$ be defined as above. Then*

$$A_7(N+1) \leqq p(1) \leqq A_8(N+1) . \tag{2.6}$$

Furthermore for $R \geqq 1$, we have

$$p(R) < (N+1)e(R+2) . \tag{2.7}$$

COROLLARY. *If $0 < R_1 < R_2 < \infty$ then we have*

$$\frac{p(R_2)}{R_2} < A_9\frac{p(R_1)}{R_1} . \tag{2.8}$$

We now turn to applications to the disconjugacy problem of the equation (1.1). We write

$$\alpha_\nu = \sup_{z \in D_0} a_\nu(z)|, \ \alpha_0 = \sup_{1\leqq\nu\leqq n} \alpha_\nu^{1/\nu} \tag{2.9}$$

and let t_0 be the positive root of the equation

$$\sum_{\nu=1}^{n} \alpha_\nu t_0^\nu = 1 . \tag{2.10}$$

Then we have evidently

$$\frac{1}{2\alpha_0} < t_0 \leqq \frac{1}{\alpha_0} .$$

We suppose that $y(z)$ is a solution of (1.1) and set $f(z) = y(z_0 + tz)$ where $t \leqq t_0$. Then

$$f^{(\nu)}(z) = t^\nu y^\nu(z_0 + tz) \ , \quad \nu = 0 \text{ to } n \ ,$$

so that $f(z)$ satisfies for $|z| \leqq R/t$ the differential equation

$$f^{(n)}(z) = -\sum_1^n a_\nu(z_0 + tz) t^\nu f^{(n-\nu)}(tz) \ .$$

In view of (2.9), (2.10) this leads to (2.2). Thus by applying Theorem 3 to $f(z)$, with $t = t_1 = \min\{t_0, R/2n\}$ we obtain

THEOREM 6. *If $y(z)$ is a solution of the equation* (1.1) *in D_0, then $y(z)$ has at most $n - 1$ zeros in*

$$|z - z_0| \leqq R_1 = \min\{t_0 n^{1/2}/(e\sqrt{20}), R/[4e(5n)^{1/2}]\}$$

i.e. *the equation is disconjugate in $|z - z_0| < R_1$.*

To compare Theorem C and Theorem 6, we take $|a_k|$ constant in (1.1) and $R_1 = 1$. Then the condition for Theorem C is certainly satisfied if

$$\sum_{k=1}^n \frac{2^k |a_k|}{k!} \leqq 1 \ ,$$

while the condition for Theorem 6 is that

$$\sum_{k=1}^n |a_k| \frac{(12.2)^k}{n^{(1/2)k}} \leqq 1 \ .$$

Thus Kim's condition is weaker and so his Theorem is stronger unless n is very large, and the a_k for small k relatively large compared with the others, when Theorem 6 gives a better result.

Finally we return to the exponential polynomials (1.3). We shall prove

THEOREM 7. *If $y(z)$ is the exponential polynomial* (1.3) *then $y(z)$ has at most $n - 1$ zeros in $|z - z_0| < \Delta^{-1} \max\{0.025, 0.15n^{-1/2}\}$.*

The result with $n > 36$ will be deduced from Therem 6 and when $n \leqq 36$ from Theorem E. Both the results are somewhat sharper than the conclusion of Tijdeman's Theorem A.

3. *Proof of Theorem* 2. Suppose that $f(z)$ is regular in D_0 and satisfies (2.2) there. We set for a fixed real θ

$$g(t) = \max_{0 \leqq \nu \leqq n-1} |f^{(\nu)}(z_0 + te^{i\theta})| \ , \quad 0 \leqq t < R \ .$$

The interval $[0, R)$ can be divided into a finite number of subintervals in which one of the derivatives $|f^{(\nu)}(z_0 + te^{i\theta})|$ is maximal. Thus $g(t)$ is continuous and except for a finite number of points we have in $(0, R)$ for some ν, $0 \leqq \nu \leqq n-1$

$$g'(t) \leqq |f^{(\nu+1)}(z + te^{i\theta})| \leqq g(t)$$

in view of (2.2). Thus $e^{-t}g(t)$ is nonincreasing in $[0, R)$ and this yields (2.3), when we set $z = z_0 + te^{i\theta}$. We deduce (2.4), on setting $z_0 = 0$.

Finally to deduce Corollary 2, we suppose that $f(z)$ has index bounded by N in the whole plane, so that in view of (1.10) we have

$$\frac{|f^{(N+1)}(z)|}{(N+1)!} \leqq \max_{0 \leqq \nu \leqq N} \frac{|f^{(\nu)}(z)|}{\nu!} .$$

We set $g(z) = f(tz)$, where $t = (N+1)^{-1}$, and deduce that

$$|g^{(N+1)}(z)| \leqq \max_{0 \leqq \nu \leqq N} \frac{t^{(N+1-\nu)}(N+1)!}{\nu!} |g^{(\nu)}(z)| \leqq \max_{0 \leqq \nu \leqq N} |g^{(\nu)}(z)| .$$

Thus $g(z)$ satisfies (2.2), with $n = N+1$, and we deduce from (2.4) that

$$|g(z)| \leqq g_{N+1}(0)e^{|z|} = \max_{0 \leqq \nu \leqq N} |g^{(\nu)}(0)|e^{|z|} = \max_{0 \leqq \nu \leqq N} t^{\nu}|f^{(\nu)}(0)|e^{|z|} \leqq f_{N+1}(0)e^{|z|} .$$

Since $f(z) = g[(N+1)z]$, we deduce (2.5).

4. *Proof of Theorem* 3. To prove Theorem 3 we need, apart from Theorem 2, two subsidiary results.

LEMMA 1. *Let* z_ν, $\nu = 1$ *to* n *be complex numbers such that* $\max_{1 \leqq \nu \leqq n} |z_\nu| = \rho_0$. *If further*

$$\phi(z) = \left\{\prod_{\nu=1}^{n} (1 - z_\nu z)\right\}^{\varepsilon} = \sum_{0}^{\infty} b_k z^k ,$$

where $b_1 = 0$, *and* $\varepsilon = 1$ *or* -1, *then*

$$|b_k| < n^{(1|2)k}\rho_0^k, \ k > 1 .$$

The Lemma is false, without the hypothesis $b_1 = 0$. Thus if $\phi(z) = (z+1)^n$, then

$$\rho_0 = 1, \ b_k = n(n-1) \cdots (n-k+1)/k! \sim \frac{n^k}{k!} ,$$

as $n \to \infty$ for fixed k.

With the hypotheses of Lemma 1, we can set

$$\psi(z) = \sum_{\nu=1}^{n} \log\left(\frac{1}{1 - z_\nu z}\right) = \sum_{k=1}^{\infty} \frac{S_k z^k}{k},$$

where $S_1 = 0$ and

$$|S_k| = \left|\sum_{\nu=1}^{n} z_\nu^k\right| \leqq n\rho_0^k \leqq (n^{1/2}\rho_0)^k, \quad k \geqq 2.$$

Since $\phi(z) = \exp\{\mp\psi(z)\}$, we deduce that the coefficents of $\phi(z)$ cannot exceed those of $\exp\{\sum_1^\infty (|S_k|/k)z^k\}$ which are smaller than those of

$$\exp\left\{\sum_1^\infty \frac{(n^{1/2}\rho_0 z)^k}{k}\right\} = (1 - n^{1/2}\rho_0 z)^{-1}.$$

This prove Lemma 1.

We have next

LEMMA 2. *Suppose that* $g(z) = \sum_0^\infty g_\nu z^\nu$ *is regular in* $|z| \leqq \rho$, *and that*

$$\sum_{\nu=n}^{\infty} |g_\nu|^2 \rho^{2\nu} \leqq \sum_{\nu=0}^{n-1} |g_\nu|^2 \rho^{2\nu}. \tag{4.1}$$

Further let z_ν, $1 \leqq \nu \leqq n$ *be zeros of* $g(z)$ *in* $|z| \leqq \rho$, *such that*

$$\sum_{\nu=1}^{n} z_\nu = 0. \tag{4.2}$$

Here multiple zeros may be counted according to their multiplicity. Then

$$\rho_0 = \max_{1 \leqq \nu \leqq n} |z_\nu| > \rho(5n)^{-1/2}. \tag{4.3}$$

Here also the condition (4.2) is essential. For without this we may take

$$g(z) = (z + a)^n,$$

which has n zeros at $z = -a$, and certainly satisfies (4.1) if

$$\rho^{2n} \leqq n^2|a|^2\rho^{2n-2}, \quad \text{i.e.} \quad \rho \leqq n|a|.$$

Thus without (4.2) we can certainly not assert $\rho_0 > \rho/n$ instead of (4.3).

To prove (4.3) we assume without loss of generality that $\rho = 1$, since otherwise we may consider $g(\rho z)$ instead of $g(z)$. We now set

$$h(z) = \prod_{\nu=1}^{n} \frac{z - z_\nu}{1 - \bar{z}_\nu z} = \sum_0^\infty h_\nu z^\nu.$$

It follows from Lemma 1 that

$$\prod_{\nu=1}^{n} \frac{1}{1 - z_\nu z} = \sum_0^\infty b_k z^k\,, \quad \text{where} \quad b_1 = 0,\, |b_k| < (\rho_0 n^{1/2})^k,\, k > 1\,.$$

Similarly if

$$\prod_{\nu=1}^{n} (z - z_\nu) = \sum_{k=0}^{n} c_k z^{n-k}$$

then

$$\prod_{\nu=1}^{n} \left(1 - \frac{z_\nu}{z}\right) = \sum_0^n c_k z^{-k}\,, \quad \prod_{\nu=1}^{n} (1 - z_\nu z) = \sum_{k=0}^{n} c_k z^k\,,$$

and now Lemma 1 shows that

$$c_1 = 0,\, |c_k| < (\rho_0 n^{1/2})^k\,, \quad k \geqq 2\,.$$

We set $\rho_0 n^{1/2} = t$, $(1 - t^2)^{-1} = \tau$. Thus we have for $k \geqq 1$

$$|h_{n-k}| = \left|\sum_{p=k}^{n} c_p b_{p-k}\right| < \sum_{p=k}^{n} t^{2p-k} < \frac{t^k}{1 - t^2} = \tau t^k\,.$$

We now set

$$g(z) = G(z)h(z)\,,$$

and assume that $\rho_0 < \rho$, since otherwise there is nothing to prove. Then $G(z)$ is regular in $|z| \leqq 1$, and $|G(z)| = |g(z)|$ for $|z| = 1$. Thus if

$$G(z) = \sum_{\nu=0}^{\infty} G_\nu z^\nu\,,$$

we have

$$\sum_{\nu=0}^{\infty} |G_\nu|^2 = \sum_0^\infty |g_\nu|^2 = \sigma$$

say. Now for $p < n$, we have

$$\begin{aligned}|g_p|^2 = \left|\sum_{k=0}^{p} G_k h_{p-k}\right|^2 &\leqq \sum_{k=0}^{p} |G_k|^2 \sum_{k=0}^{p} |h_{p-k}|^2 \\ &< \sigma \sum_{k=0}^{p} \tau^2 t^{2(n+k-p)} < \frac{\sigma\tau^2 t^{2n-2p}}{1 - t^2} = \sigma\tau^3 t^{2n-2p}\,.\end{aligned}$$

This yields in turn

$$\sum_{p=0}^{n-1} |g_p|^2 < \sigma\tau^3 \sum_{p=0}^{n-1} t^{2n-2p} < \sigma\tau^4 t^2 < \frac{1}{2}\sigma\,,$$

if $t^2 \leqq 1/5$, so that $\tau \leqq 5/4$. This contradicts (4.1). Thus $t > 5^{-1/2}$, and Lemma 2 is proved.

4.1. We can now complete the proof of Theorem 3. We assume that $f(z)$ has n zeros, z_1 to z_n, in $|z| \leqq \rho_0$, and set

$$z_0 = \frac{1}{n}\sum_{\nu=1}^{n} z_\nu , \quad z'_\nu = z_\nu + z_0 ,$$

$$F(z) = f(z_0 + z) .$$

Then $F(z)$ has the n zeros z'_ν with $|z'_\nu| \leqq 2\rho_0$, and $\sum z'_\nu = 0$. Further $F(z)$ satisfies (2.2) in $|z| < 2n - \rho_0$. From this we proceed to obtain a lower bound for ρ_0. We assume $\rho_0 < (1/2)n$, since otherwise there is nothing to prove, set

$$\mu = F_n(0) = \max_{0 \leqq \nu \leqq n-1} |F^{(\nu)}(0)| ,$$

and write

$$F(z) = \sum_0^\infty F_\nu z^\nu .$$

Then it follows from (2.3) that

$$M_n(R) = \max_{|z| \leqq R} |F^{(n)}(z)| \leqq \mu e^R , \quad 0 \leqq R \leqq n .$$

Further

$$F^{(n)}(z) = \sum_{\nu=n}^\infty \frac{\nu!}{(\nu - n)!} F_\nu z^{\nu-n} .$$

Hence Cauchy's inequality yields for $0 < R < n$

$$\frac{\nu!}{(\nu - n)!}|F_\nu| \leqq \frac{M_n(R)}{R^{\nu-n}} \leqq \frac{\mu e^R}{R^{\nu-n}} , \quad \nu \geqq n . \tag{4.4}$$

We now set $\nu = n + t$, $R = \inf(t, n - 1)$ in (4.4) and take $\rho = n/e$ in Lemma 2. Then (4.4) may be written as

$$|F_{n+t}|\rho^{n+t} \leqq \frac{\mu\rho^n}{n!}\frac{n!t!}{(n + t)!}\left(\frac{\rho}{R}\right)^t e^R = \frac{\mu\rho^n}{n!} U_t , \tag{4.5}$$

say, where

$$\begin{aligned} U_0 &= 1 , \\ U_t &= \frac{n!t!}{(n + t)!}\left(\frac{n}{e}\right)^t\left(\frac{e}{t}\right)^t , \quad 1 \leqq t \leqq n - 1 \\ U_t &= \frac{n!t!}{(n + t)!}\cdot\left(\frac{n}{e}\right)^t \frac{e^{n-1}}{(n - 1)^t} , \quad t \geqq n . \end{aligned}$$

We proceed to prove that

$$(4.6) \qquad U_t \leqq t^{-1}, \quad t \geqq 1.$$

Suppose first that $1 \leqq t \leqq n-1$. Then

$$U_t = \frac{n!n^t}{(n+t)!} \cdot \frac{t!}{t^t} \leqq \frac{1}{t} \cdot \frac{2}{t} \cdots \frac{t}{t} \leqq \frac{1}{t}$$

as required. Next for $n = 2$, $t > 1$ we have

$$U_t = \frac{2}{(t+1)(t+2)} e\left(\frac{2}{e}\right)^t \leqq \left(\frac{2}{e}\right)^{t-1} \frac{1}{t+1} < \frac{1}{t}.$$

If $n > 2$, $t = n$, we have

$$\begin{aligned} U_t &= \left(1 + \frac{1}{n-1}\right)^n e^{-1} \frac{(n!)^2}{(2n)!} < \frac{n}{n-1} \frac{1 \cdot 2 \cdots n}{(n+1)(n+2) \cdots 2n} \\ &\leqq \frac{n}{n-1} \frac{1 \cdot 2}{2n(2n-1)} < \frac{1}{2n-1} < \frac{1}{t}. \end{aligned}$$

Finally if $n > 2$, $t > n$, then

$$\begin{aligned} U_t &= \frac{n!t!}{(n+t)!} \left(\frac{n}{n-1}\right)^t e^{n-1-t} \\ &\leqq \frac{n!t!}{(n+t)!} e^{t/(n-1)+n-1-t} = \frac{n!t!}{(n+t)!} \exp\left\{n - 1 - t\frac{n-2}{n-1}\right\} \\ &\leqq \frac{n!t!}{(n+t)!} \leqq \frac{1}{n+t} < \frac{1}{t}. \end{aligned}$$

Thus (4.6) holds in all cases.

We deduce from (4.5) and (4.6) that

$$(4.7) \qquad \begin{aligned} \sum_{t=0}^{\infty} |F_{n+t}|^2 \rho^{2n+2t} &\leqq \frac{\mu^2 \rho^{2n}}{(n!)^2} \sum_{t=0}^{\infty} U_t^2 \\ &\leqq \frac{\mu^2 \rho^{2n}}{(n!)^2} \left(1 + \sum_{t=1}^{\infty} t^{-2}\right) = \frac{\mu^2 \rho^{2n-2}}{((n-1)!)^2} \cdot \frac{\rho^2}{n^2} \left(1 + \frac{\pi^2}{6}\right) \\ &= e^{-2}\left(1 + \frac{\pi^2}{6}\right) \frac{\mu^2 \rho^{2n-2}}{((n-1)!)^2} < \left\{\frac{\mu \rho^{n-1}}{(n-1)!}\right\}^2. \end{aligned}$$

On the other hand

$$(4.8) \qquad \begin{aligned} \sum_{\nu=0}^{n-1} |F_\nu|^2 \rho^{2\nu} &\geqq \max_{0 \leqq \nu \leqq n-1} |F_\nu|^2 \rho^{2\nu} \\ &= \max_{0 \leqq \nu \leqq n-1} (\nu! |F_\nu|)^2 \frac{\rho^{2\nu}}{(\nu!)^2} \geqq \max_{0 \leqq \nu \leqq n-1} (\nu! |F_\nu|)^2 \inf_{0 \leqq \nu \leqq n-1} \frac{\rho^{2\nu}}{(\nu!)^2}. \\ &= \frac{\mu^2 \rho^{2n-2}}{((n-1)!)^2}. \end{aligned}$$

In fact for $\rho > 1$, $\rho^\nu/\nu!$ first increases and then decreases, as ν

increases from $\nu = 0$, and for $\rho \leqq 1$, $\rho^\nu/\nu!$ is steadily decreasing in this range. In either case $\rho^\nu/\nu!$ attains its minimum in the range $0 \leqq \nu \leqq n-1$ either at $\nu = 0$ or $\nu = n-1$. Since

$$\frac{\rho^{n-1}}{(n-1)!} = \left(\frac{n}{e}\right)^{n-1} \frac{1}{(n-1)!}$$

decreases with increasing n and is equal to 1 for $n = 1$, we deduce that the minimum of $\rho^\nu/\nu!$ is attained at $\nu = n-1$, and (4.8) follows.

Combining (4.7) and (4.8) we deduce that

$$\sum_0^{n-1} |F_\nu|^2 \rho^{2\nu} > \sum_{\nu=n}^{\infty} |F_\nu|^2 \rho^{2\nu} .$$

Thus we can apply Lemma 2. Since $F(z)$ has the n zeros z'_ν with $|z'_\nu| \leqq 2\rho_0$ and $\sum z'_\nu = 0$ we deduce from Lemma 2 that

$$2\rho_0 > \rho(5n)^{-1/2} = n^{1/2} e^{-1} 5^{-1/2},\ \rho_0 > n^{1/2}/(20^{1/2} e) .$$

This proves Theorem 3.

5. **Exponential polynomials.** We have already seen how to deduce Theorem 6 from Theorem 3. We proceed now to deduce Theorem 7. Let

$$y(z) = \sum_{\nu=1}^{k} p_\nu(z) e^{\omega_\nu(z)}$$

be an exponential polynomial (1.3), and set

$$\omega_0 = \frac{1}{n} \sum_1^n \omega_\nu,\ \omega'_\nu = \omega_\nu - \omega_0 ,$$

where each ω_ν is counted with multiplicity k_ν, and $k_\nu - 1$ is the degree of $p_\nu(z)$. Then $f(z) = y(z) e^{-\omega_0 z}$ satisfies the differential equation

$$P(D) f = 0 ,$$

where

$$P(D) = \prod_{\nu=1}^{n} (D - \omega'_\nu) = D^n + b_1 D^{n-1} + \cdots b_n$$

say. Also if

$$\Delta = \max_{1 \leqq \nu \leqq n} |\omega_\nu| , \quad \text{then} \quad \Delta' = \max_{1 \leqq \nu \leqq n} |\omega'_\nu| \leqq 2\Delta$$

and since $\Sigma \omega'_\nu = 0$, we deduce from Lemma 1 that

$$|b_k| \leqq \Delta'^k n^{(1/2)k} , \quad k \geqq 2;\ b_1 = 0 .$$

We choose

$$t_0 = \alpha/(2\Delta n^{1/2}) \ , \quad \text{where} \quad \alpha = \frac{\sqrt{5}-1}{2} \ , \quad \text{so that} \quad \alpha^2 + \alpha = 1 \ ,$$

and consider $g(z) = f(t_0 z)$. Then $g(z)$ satisfies differential equation

$$\Big(D^n + \sum_{\nu=2}^{n} t_0^\nu b_\nu D^{n-\nu}\Big) g(z) = 0 \ ,$$

and

$$\sum_2^n t_0^\nu |b_\nu| \leqq \sum_2^n (2t_0 \Delta n^{1/2})^k < \sum_2^\infty \alpha^k = \frac{\alpha^2}{1-\alpha} = 1 \ . \tag{5.1}$$

It now follows from Theorem 6 that $f(z)$ and so $y(z)$ has at most $n-1$ zeros in

$$|z| \leqq n^{1/2} t_0 / e\sqrt{20} = \alpha/(4e\sqrt{5}\Delta) \doteqdot (39\cdot 3\Delta)^{-1}$$

This proves one of the inequalities of Theorem 7.

To obtain the other we first note that Theorem E follows at once from Theorem C. For if $f(z)$ has n zeros in $|z| < R$, where $R \leqq 1/2$, let z_1 be a point such that (1.7), (1.8) hold with $f(z)$ instead of $y(z)$. Then

$$|f^{(n-k)}(z_1)| < \frac{(2R)^k}{k!} |f^{(n)}(z_1)| \leqq |f^{(n)}(z_1)| \ , \quad 1 \leqq k \leqq n \ ,$$

and this contradicts (2.2). Thus Theorem E is proved. It follows from (5.1) that $g(z)$ satisfies (2.2) and so has at most $n-1$ zeros in $|z| < 1/2$, so that $f(z)$ has at most $n-1$ zeros in $|z| < (1/2)t_0$. This completes the proof of Theorem 7.

6. Index and local valency. It remains to prove Theorems 1, 4 and 5 and to this we now turn. We start by applying Theorem D with

$$A_1(p) = (A_2/p)^{2p} \ .$$

To see how this result follows from the Theorem of Jenkins and Oikawa [4] we assume that $z_0 = 0$, $R = 1$. Then the above authors proved that if $f(z) = \sum_0^\infty a_n z^n$ is p-valent in $|z| < 1$, and $0 < r < 1$, we have

$$M(r, f(z) - a_0) \leqq \mu_p A_0(p)(1-r)^{-2p} \ ,$$

where

$$\mu_p = \max_{1 \leqq \nu \leqq p} |a_\nu|, \ A_0(p) = A_0^p \ .$$

Now Cauchy's inequality yields for $n > p$

$$|a_n| \leqq A_0(p)\mu_p r^{-n}(1-r)^{-2p} ,$$

and choosing $r = 1 - p/(2n)$, we deduce that

$$|a_n| \leqq A_0^p \mu_p \left(\frac{2n}{p}\right)^{2p} \left(1 - \frac{p}{2n}\right)^{-n} \leqq A_0^p \mu_p 2^{3p}(n/p)^{2p} .$$

This proves (1.11) with $A_1(p) = (8A_0/p^2)^p$, when $R = 1$, $z_0 = 0$. We deduce the general case of Theorem D, by considering $f(z_0 + Rz)$ instead of $f(z)$.

Suppose now that $R \geqq 2$. Then (1.11) shows that for $j > p$

$$\frac{|f^{(j)}(z_0)|}{j!} \leqq \max_{1 \leqq \nu \leqq p} (8A_0 j^2/p^2)^p R^{\nu - j} \frac{|f^{(\nu)}(z_0)|}{\nu!} . \tag{6.1}$$

We write $F(z) = f'(z)$, and deduce that for $R \geqq 2$, $n > p$

$$\frac{|F^{(n-1)}(z_0)|}{(n-1)!} \leqq \max_{0 \leqq \nu \leqq p-1} \frac{|F^{(\nu)}(z_0)|}{\nu!}$$

provided that

$$\frac{|f^{(n)}(z_0)|}{(n-1)!} \leqq \max_{1 \leqq \nu \leqq p} \frac{|f^{(\nu)}(z_0)|}{(\nu - 1)!} . \tag{6.2}$$

In view of (6.1) this condition is satisfied for $n > 2p$ if

$$n(8A_0 n^2/p^2)^p 2^{-(1/2)n} < 1 ,$$

i.e. provided that $n \geqq Ap$. Thus if the hypotheses of Theorem 1 hold with a fixed $R \geqq 2$, and $n \geqq Ap$ then (6.2) holds for all z_0 and so $F(z) = f'(z)$ has index at most Ap. If $R < 2$, then disks of radius 2 can be covered by almost AR^{-2} disks of radius R, so that $f(z)$ is p_1-valent in disks of radius 2, where $p_1 < ApR^{-2}$. Thus we obtain the first statement of Theorem 1 also in this case.

Next we deduce similarly from (6.1) that $F(z)$ has index less than p, provided that (6.2) holds for $n > p$, i.e. provided that

$$j(8A_0 j^2/p^2)^p R^{p-j} < 1 , \quad j > p .$$

This is equivalent to

$$(j - p) \log R > 2p\{(\log j/p) + A\} + \log j ,$$

which is satisfied provided that $\log R > Ap$. This proves the second part of Theorem 1, and completes the proof of that Theorem.

6.1. We next prove Theorem 4. For this purpose we apply Theorem 3 to $F(z) = f(tz) - w$, where w is any complex number, $f(z)$ is the function of Theorem 4, and $t = CR/(p+1)$. Our hypotheses imply that

$$\left(\frac{CR}{t}\right)^{p+1} \frac{F^{p+1}(z)}{(p+1)!} \leqq \max_{1 \leqq \nu \leqq p} \left(\frac{CR}{t}\right)^{\nu} \frac{|F^{(\nu)}(z)|}{\nu!} \tag{6.3}$$

for $|tz - z_0| \leqq R$, i.e. $|z - t^{-1}z_0| \leqq t^{-1}R$, and since $C \leqq 1/2$, $t^{-1}R \geqq 2(p+1)$. Also (6.3) shows that

$$\begin{aligned} |F^{(p+1)}(z)| &\leqq \max_{1 \leqq \nu \leqq p} \left(\frac{t}{CR}\right)^{p+1-\nu} \frac{(p+1)!}{\nu!} |F^{(\nu)}(z)| \\ &\leqq \max_{1 \leqq \nu \leqq p} |F^{(\nu)}(z)| \leqq \max_{0 \leqq \nu \leqq p} |F^{(\nu)}(z)| \end{aligned}$$

since $CR = (p+1)t$. Thus $F(z)$ satisfies the hypothesis of Theorem 3, with $t^{-1}R$ instead of R, and $n = p + 1$, and we deduce that $F(z)$ has at most p zeros in $|z - t^{-1}z_0| \leqq (p+1)^{1/2}/12 \cdot 2$, so that the equation $f(z) = w$ has at most p roots in

$$|z - z_0| < t(p+1)^{1/2}/12 \cdot 2 = CR/\{12 \cdot 2(p+1)^{1/2}\} ,$$

i.e. $f(z)$ is p-valent in this disk. This proves Theorem 4.

We next prove the Corollary. Suppose first that $f(z)$ is an integral function, which has b.v.d., so that for some R, $f(z)$ assumes no value more than p times in any disk of radius R. Then it follows from Theorem 1, that $f'(z)$ has bounded index. Furthermore if $R \geqq A_6^p$, then the index N of $f'(z)$ is at most $p - 1$. Suppose now that with $R = A_6^N$, where N is the index of $f(z)$, we have $p(R) \leqq N$, so that $f(z)$ assumes no value more than N times in any disk of radius R. Then Theorem 1 shows that $f'(z)$ has index at most $N - 1$, which gives a contradiction. This proves the second inequality of the Corollary.

Next if $f'(z)$ has bounded index N, it follows that we have for all z

$$\frac{|f^{(N+2)}(z)|}{(N+1)!} \leqq \max_{1 \leqq \nu \leqq N+1} \frac{|f^{(\nu)}(z)|}{(\nu - 1)!}$$

and hence à fortiori

$$\frac{|f^{(N+2)}(z)|}{(N+2)!} \leqq \max_{1 \leqq \nu \leqq N+1} \frac{|f^{(\nu)}(z)|}{\nu!} .$$

Thus we may apply Theorem 4, with $p = N + 1$, $C = 1/2$, $R = 2$, and deduce that $f(z)$ is $(N+1)$-valent in every disk of radius

$$r = 1/\{12 \cdot 2(N+2)^{1/2}\} .$$

Thus $f(z)$ has b.v.d. and $p(r) \leqq N + 1$. This completes the proof of the Corollary of Theorem 4.

6.2. It remains to prove Theorem 5. The first inequality in (2.6) follows at once from Theorem 1. The other inequality in (2.6) will follow at once from (2.7), which we now proceed to prove. The method is similar to that employed by Tijdeman [10]. We write

$$f(z) = \sum_0^\infty a_n z^n ,$$

and suppose, as we may do without loss of generality, that

$$\max_{1 \leqq \nu \leqq N+1} |a_\nu| = 1 . \tag{6.4}$$

Thus

$$\frac{|f^{(\nu+1)}(0)|}{(N+1)^\nu} = \frac{|a_{\nu+1}|(\nu+1)!}{(N+1)^\nu} \leqq 1 , \quad 0 \leqq \nu \leqq N .$$

Thus, applying (2.5) to $f'(z)$, we obtain for $|z| \leqq R$

$$|f'(z)| \leqq e^{(N+1)R} . \tag{6.5}$$

We deduce that for $|z| = R > 0$

$$|f(z) - a_0| \leqq \int_0^R e^{(N+1)t} dt < e^{(N+1)R} .$$

We proceed to estimate the number $n(\rho)$ of zeros of $f(z)$ in $|z| < \rho$. Suppose first that

$$|a_0| > e^{(N+1)\rho} .$$

Then for $|z| \leqq \rho$

$$|f(z)| > |a_0| - |f(z) - a_0| > e^{(N+1)\rho} - e^{(N+1)\rho} = 0 ,$$

so that is this case $n(\rho) = 0$. Thus we may assume that

$$|a_0| \leqq e^{(N+1)\rho}$$

and we deduce that

$$|f(z)| \leqq 2e^{(N+1)R} , \quad R \geqq \rho .$$

It now follows from the normalisation (6.4) and Cauchy's inequality that

$$M(1, f) \geqq 1 , \quad \text{where} \quad M(r, f) = \max_{|z|=r} |f(z)| .$$

Thus there exists z_0, such that $|z_0| = 1$, and $|f(z_0) \geqq 1$. We apply Jensen's inequality to

$$\phi(z) = f(z_0 + z)$$

in the circle $|z| \leqq R$, and deduce that the number n_1, of zeros of $\phi(z)$ in $|z| \leqq \rho + 1$, satisfies

$$n_1 \leqq \frac{\log \{M(R, \phi)/|\phi(0)|\}}{\log (R/(\rho + 1))} \leqq \frac{(N + 1)(R + 1) + \log 2}{\log \{R/(\rho + 1)\}},$$

since

$$M(R, \phi) = \max_{|z|=R} |\phi(z)| \leqq 2e^{(N+1)(R+1)} .$$

Choosing $R = e(\rho + 1)$, we deduce that

$$n(\rho) \leqq n_1 \leqq (N + 1)(e(\rho + 1) + 1) + \log 2 < (N + 1)e(\rho + 2) .$$

The same argument can be applied to $f(z) - w$, whose derivative is the same as that of $f(z)$ and we deduce that the equation $f(z) = w$ has at most $(N + 1)e(\rho + 2)$ roots in any disk of radius $\rho \geqq 1$. This proves (2.7). In particular setting $R = 1$ we deduce the right hand inequality of (2.6) with $A_8 = 3e$.

6.3. It remains to prove the Corollary. We first take $R_1 = 1$, and assume that $f'(z)$ has index N. Then it follows from (2.6) that

$$p(1) \geqq A_7(N + 1) .$$

On the other hand in view of (2.7) we have for $R_2 > 1$

$$p(R_2) \leqq 3e(N + 1)\, R_2 \leqq \frac{3e}{A_7} R_2 p(1) ,$$

so that (2.8) holds if $R_1 = 1$. The general case now follows, since we may apply this result to $f(R_1 z)$.

The result of the Corollary shows in particular that for any b.v.d. function $f(z)$, $p(R) = O(R)$ as $R \to \infty$, and also by Theorem 2, Corollary 2, $f'(z)$ and so $f(z)$ has at most exponential type. These two result answer affirmatively a previous conjecture [3, problem 2.28]. In conclusion it is worth pointing out that since the index is rather easy to deal with we can obtain various applications for functions of bounded value distribution. One example of such an application is (2.8). Also if $f'(z)$ has index $N, f(z)$ has index at most $N + 1$, as we have already pointed out. Thus if $f(z)$ has b.v.d. the same is true of successive integrals of $f(z)$. The converse is false in general. However we can prove

THEOREM 8. *If $f(z)$ is p-valent in $|z - z_0| < R$, then $f'(z)$ has at most $p - 1$ zeros in $|z - z_0| < A(p)R$.*

To see this we note that Theorem D, with $j = p + 1$, $A_1(p) = (A_2/p)^{2p}$, shows that for $|z - z_0| \leqq (1/2)R$

$$|f^{(p+1)}(z)| \leqq (p + 1)!\,(A_2(p + 1)/p)^{2p} \max_{1 \leqq \nu \leqq p} \frac{\left(\frac{1}{2}R\right)^{\nu-(p+1)} |f^{(\nu)}(z)|}{\nu!} .$$

If $R = R_p = 2(p + 1)!\,\{A_2(p + 1)/p\}^{2p} < A_{10}^p$, this yields

$$|f^{(p+1)}(z)| \leqq \max_{1 \leqq \nu \leqq p} |f^{(\nu)}(z)|,\ |z - z_0| \leqq \frac{1}{2}R .$$

It now follows from Theorem 3, that $f'(z)$ has at most $p - 1$ zeros in $|z - z_0| < p^{1/2}/12\cdot 2$. Thus we deduce the desired conclusion with $A(p) = A_{11}^p$, where A_{11} is a small absolute constant. If $R \neq R_p$, we obtain our conclusion by considering $f(R_p z/R)$ instead of $f(z)$.

6.4. We complete the paper by giving an example of a function $f(z)$ such that $f(z)$ has bounded index but $f'(z)$ does not.

Let

$$F(z) = e^z \prod_{n=1}^{\infty} \left(1 + \frac{z}{2^n}\right)^n .$$

Then $F(z)e^{-z}$ is an integral function of order zero with zeros of arbitrarily high order. Thus $F(z)$ cannot have bounded index. Also

$$(6.6) \qquad F(z) = O\{e^{-C|z|}\} \text{ and so } F'(z) \longrightarrow 0 , \quad \text{as} \quad |z| \longrightarrow \infty ,$$

for $\pi/2 + \delta \leqq \arg z \leqq 3\pi/2 - \delta$, where C is a positive constant depending on δ.

Next

$$(6.7) \quad \frac{F'(z)}{F(z)} = 1 + \sum_{n=1}^{\infty} \frac{n}{z + 2^n} \longrightarrow 1,\ \text{as } |z| \longrightarrow \infty,\ \text{for } |\arg z| \leqq \pi - \delta .$$

We now set

$$f(z) = B + \int_0^z F(z)dz .$$

It follows from (6.6) that by a suitable choice of B we can make sure that

$$|f(z)| \geqq 1 , \quad \text{for} \quad \frac{3\pi}{4} \leqq |\arg z \leqq \frac{5\pi}{4}$$

so that

$$|F'(z)| \leqq A|f(z)|$$

in this range in view of (6.6). Also for $|\arg z| \leqq 3\pi/4$, we have

$$|F'(z)| \leqq A|F(z)|$$

in view of (6.7). Thus we have in the whole plane

$$|f''(z)| \leqq A \max \{|f(z), |f'(z)|\}$$

where A is a suitable constant, and so by Theorem 4

$$\int f(z) dz$$

has b.v.d. in the plane and so by the Corollary of Theorem 4 $f(z)$ has bounded index, and in fact if ε is a sufficiently small positive number $f(\varepsilon z)$ has index one. However the derivatives of $f(\varepsilon z)$ do not have bounded index.

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