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AUTOMORPHISMS DEFINABLE BY FORMULAS

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The group of definable automorphisms of a structure $\mathfrak{A}$ is denoted by $\mathscr{H}(\mathfrak{A})$. The following theorem is used to discover the group of definable automorphisms of various structures: If $\mathfrak{A}$ has finite type and $\mathfrak{A} \equiv \mathfrak{B}$ then $\mathscr{H}(\mathfrak{A}) \equiv \mathscr{H}(\mathfrak{B})$. It is also shown that every group may be represented as the group of definable automorphisms of some structure. Definable automorphisms are then investigated in infinitary languages. Finally the notion of normal submodel is introduced in analogy to the notion of normal subgroup with definable automorphisms playing the role of inner automorphisms.

In this paper we consider the automorphisms of a structure which are definable by the formulas of a language. See [1] Chapters 3 and 4 and [3] Chapter 6 for an exposition of the application of first-order languages to the study of mathematical objects. We denote structures as $\mathfrak{A} = \langle A, R_\theta \rangle_{\theta<\xi}$, where each R_θ is an n_θ-ary relation on A, and algebras as $\mathscr{A} = \langle A, F_\theta \rangle_{\theta<\xi}$, where each F_θ is an n_θ-ary function on A (in both cases $0 \leqq n < \omega$ for all $\theta < \xi$). If R_θ (resp. F_θ) is a 0-ary relation (resp. function), it is a distinguished constant and we write it as a_θ. The type of $\mathfrak{A}$ is $\mu = \langle n_\theta \rangle_{\theta<\xi}$. L_μ is the appropriate first-order language for $\mathfrak{A}$; we usually just write it as L. The diagram language of $\mathfrak{A}$, $L(\mathfrak{A})$, is the language L with a symbol added for each element of $\mathfrak{A}$. The diagram of $\mathfrak{A}$, $D(\mathfrak{A})$, is the set of all atomic sentences of $L(\mathfrak{A})$ which hold in $\mathfrak{A}$ together with the negations of all atomic sentences of $L(\mathfrak{A})$ which do not hold in $\mathfrak{A}$. When we write definable we mean definable in the diagram language (i.e. definable by parameters).

We use ϕ, ψ, χ for formulas; x, y, z for variables; and f, g, h for functions: functions are written from right to left. When we write a formula ϕ as $\phi(x_1, \cdots, x_n, a_1, \cdots, a_m)$, it is understood that $x_1, \cdots, x_n$ are all the free variables of ϕ and $a_1, \cdots, a_m$ are all the parameters of $\mathfrak{A}$ in ϕ. A sentence has no free variables. The cardinal of $\mathfrak{A}$ is $\overline{\overline{A}}$ and the cardinal of μ is $\overline{\overline{\xi}}$; we denote the latter by $\overline{\overline{\mu}}$. We write $\mathscr{G}(\mathfrak{A})$ for the group of automorphisms of A; $\mathscr{G}(\mathfrak{A}) = \langle G(\mathfrak{A}), \cdot, {}^{-1}, 1 \rangle$ where $G(\mathfrak{A})$ is the set of automorphisms of $\mathfrak{A}$, $\cdot$ and ${}^{-1}$ are the group operations and 1 is the group identity. Similarly we write $\mathscr{H}(\mathfrak{A})$ for the group of definable automorphisms of $\mathfrak{A}$ and $\mathscr{K}(\mathfrak{A})$ for the group of automorphisms definable in L (i.e. definable without parameters). I stands for the trivial group of one element; $\mathscr{H} \Delta \mathscr{G}$ means that $\mathscr{H}$ is a normal subgroup of $\mathscr{G}$; $\mathscr{Z}(\mathscr{G})$ is the center of $\mathscr{G}$ and $\mathscr{Z}_{\mathscr{G}}(\mathscr{H})$ is the centralizer of $\mathscr{H}$ in $\mathscr{G}$.

1. Definable automorphisms. First we give some results of Marsh about definable automorphisms.

DEFINITION 1. ([6]) Let $f \in G(\mathfrak{A})$. Then $f \in H(\mathfrak{A})$ if for some formula $\phi(x, y, a_1, \cdots, a_n)$ *of* $L(\mathfrak{A})$, $a' = f(a)$ iff $\mathfrak{A} \models \phi(a, a', a_1, \cdots, a_n)$.

We say that the automorphism f is defined by the formula $\phi(x, y, a_1, \cdots, a_n)$. A definable automorphism is defined by infinitely many formulas.

The next lemma is proved by induction on the formulas of $L(\mathfrak{A})$.

LEMMA 1. *Let $f \in G(\mathfrak{A})$ and let $\chi(a_1, \cdots, a_n)$ be a sentence of $L(\mathfrak{A})$. Then $\mathfrak{A} \models \chi(a_1, \cdots, a_n)$ iff $\mathfrak{A} \models \chi(f(a_1), \cdots, f(a_n))$.*

PROPOSITION 1. ([6]) $\mathscr{H}(\mathfrak{A}) \Delta \mathscr{G}(\mathfrak{A})$.

Proof. If $f, g \in H$ are defined by $\phi(x, y, a_1, \cdots, a_n)$ and $\psi(x, y, e_1, \cdots, e_m)$ respectively, then f^{-1} is defined by $\phi(y, x, a_1, \cdots, a_n)$ and gf is defined by $(\exists z)[\phi(x, z, a_1, \cdots a_n) \wedge \psi(z, y, e_1, \cdots, e_m)]$ where z is the first variable free for y in ϕ and free for x in ψ. The identity automorphism is defined by $y = x$. Now let $g \in G$ and let $f \in H$ be defined by $\phi(x, y, a_1, \cdots, a_n)$. It follows from Lemma 1 that gfg^{-1} is defined by $\phi(x, y, g(a_1), \cdots, g(a_n))$. Thus $gfg^{-1} \in H$.

LEMMA 2. *$\phi(x, y, a_1, \cdots, a_n)$ defines an automorphism of $\mathfrak{A}$ iff the following $1 + \xi$ formulas of $L(\mathfrak{A})$ hold in $\mathfrak{A}$:*

$$(P) \qquad (\forall x)(\exists y)[\phi(x, y, a_1, \cdots, a_n) \wedge (\forall z)(\phi(x, z, a_1, \cdots, a_n) \rightarrow z = y] \wedge (\forall y)(\exists x)[\phi(x, y, a_1, \cdots, a_n) \wedge (\forall z(\phi(z, y, a_1, \cdots, a_n) \rightarrow z = x] ,$$

$$(P_\theta) \qquad \phi(a_\theta, a_\theta, a_1, \cdots, a_n) \quad if \quad n_\theta = 0 ,$$

$$(P_\theta) \qquad (\forall x_1, \cdots, x_{n_\theta}, y_1, \cdots, y_{n_\theta})\{[\phi(x_1, y_1, a_1, \cdots, a_n) \wedge \cdots \wedge \phi(x_{n_\theta}, y_{n_\theta}, a_1, \cdots, a_n)] \rightarrow [R_\theta(x_1, \cdots, x_{n_\theta}) \leftrightarrow R_\theta(y_1, \cdots, y_{n_\theta})]\} \quad if \quad n_\theta > 0 .$$

PROPOSITION 2. *Suppose $\mathfrak{A} \prec \mathfrak{B}$.*
(a) *(Marsh) Every $f \in H(\mathfrak{A})$ can be extended to an $\bar{f} \in H(\mathfrak{B})$.*
(b) *If a formula of $L(\mathfrak{A})$ defines $g \in H(\mathfrak{B})$ then g is the extension of some $f \in H(A)$, i.e., $g = \bar{f}$.*
(c) *The map $m: f \rightarrow \bar{f}$ embeds $\mathscr{H}(\mathfrak{A})$ into $\mathscr{H}(\mathfrak{B})$.*

Proof. (a) Let $\bar{f}$ be defined by the formula which defines f.

Since $\mathfrak{A} \prec \mathfrak{B}$ it follows from Lemma 2 that $\bar{f}$ is an extension of f.
(b) Let f be defined by the formula of $L(\mathfrak{A})$ which defines g.
(c) It follows from (a) and Proposition 1 that m is an embedding.

It follows that the map $r: \mathfrak{A} \to \mathscr{H}(\mathfrak{A})$ is a functor from a category of models (the maps being elementary embeddings) to a category of groups (the maps being group embeddings).

The next two theorems follow from Example 10 and Theorems 1 and 2 of [2].

THEOREM 1. *If* $\bar{\bar{\mu}} < \omega$ *and* $\mathfrak{A} \equiv \mathfrak{B}$ *then* $\mathscr{H}(\mathfrak{A}) \equiv \mathscr{H}(\mathfrak{B})$.

THEOREM 2. *If* $\bar{\bar{\mu}} < \omega$ *and* $\mathfrak{A}$ *is elementarily embeddable in* $\mathfrak{B}$ *then* $\mathscr{H}(\mathfrak{A})$ *is elementarily embeddable in* $\mathscr{H}(\mathfrak{B})$.

It follows that if $\bar{\bar{\mu}} < \omega$ the map $r: \mathfrak{A} \to \mathscr{H}(\mathfrak{A})$ is a functor in a category of models (the maps being elementary embeddings).

EXAMPLE 1. The complete theory of densely ordered sets without first or last element is known to have models $\mathfrak{A}$ such that $\mathscr{G}(\mathfrak{A}) = I$ ([7]). Thus $\mathscr{H}(\mathfrak{A}) = I$ and so by Theorem 1 if $\mathfrak{B}$ is any model of this theory, then $\mathscr{H}(\mathfrak{B}) = I$. The same holds also for the three other complete theories of densely ordered sets. This result was proved earlier by Marsh in a different way.

EXAMPLE 2. Let $\mathfrak{A}$ be the real-closed ordered field of real numbers. Then $\mathscr{G}(\mathfrak{A}) = I$ and so $\mathscr{H}(\mathfrak{A}) = I$. Since the theory of real-closed ordered fields is complete ([9] page 105), it follows from Theorem 1 that if $\mathfrak{B}$ is any real-closed ordered field, then $\mathscr{H}(\mathfrak{B}) = I$.

PROPOSITION 3. (*Marsh*) *If* $\mathfrak{A}$ *is finite then* $\mathscr{H}(\mathfrak{A}) = \mathscr{G}(\mathfrak{A})$.

Proof. Let $A = \{a_1, \cdots, a_m\}$ and let $f \in G(\mathfrak{A})$. Then f is defined by the formula $\phi(x, y, a_1, \cdots, a_m)$ where ϕ is $[x = a_1 \wedge y = f(a_1)] \vee \cdots \vee [x = a_m \wedge y = f(a_m)]$.

2. Automorphisms definable in L. In this section we investigate both $\mathscr{H}(\mathfrak{A})$ and $\mathscr{K}(\mathfrak{A})$ for various $\mathfrak{A}$.

DEFINITION 2. Let $f \in G(\mathfrak{A})$. Then $f \in K(\mathfrak{A})$ if for some formula $\phi(x, y)$ of L, $a' = f(a)$ iff $\mathfrak{A} \models \phi(a, a')$.

PROPOSITION 4. $\mathscr{K}(\mathfrak{A})$ *is a subgroup of* $\mathscr{Z}(\mathscr{G}(\mathfrak{A}))$.

Proof. This follows from the proof of Proposition 1.

COROLLARY. *$\mathscr{K}(\mathfrak{A})$ is an abelian group.*

The next lemma is proved using Lemma 2.

LEMMA 3. *If $\mathfrak{A} \equiv \mathfrak{B}$ then $\mathscr{K}(\mathfrak{A}) = \mathscr{K}(\mathfrak{B})$.*

EXAMPLE 3. One model for the complete theory of divisible torsion free abelian groups is the additive group of rationals, $\mathscr{R}$ ([1] page 180). We treat $\mathscr{R}$ as an algebra rather than as a structure. $\mathscr{G}(\mathscr{R}) \cong \mathscr{M}$ where $\mathscr{M}$ is the multiplicative group of rationals; for if $f \in G(\mathscr{R})$ then $f(x) = rx$ for some $r \in R$. Since $y = rx$ can be expressed as a formula of the appropriate L, it follows that $\mathscr{G}(\mathscr{R}) = \mathscr{H}(\mathscr{R}) = \mathscr{K}(\mathscr{R})$. Let $\mathscr{S}$ be any divisible torsion free abelian group. By Lemma 3, $\mathscr{K}(\mathscr{S}) \cong \mathscr{M}$ and by Theorem 1, $\mathscr{H}(\mathscr{S}) \equiv \mathscr{M}$.

EXAMPLE 4. Let $\mathfrak{A}_p$ be the algebraic closure of the prime field of characteristic p. If $f \in G(\mathfrak{A}_p)$ then $f(x) = x^{p^k}$ for some integer k ([8] page 614). Thus $\mathscr{G}(\mathfrak{A}_p) \cong \mathscr{C}$ where $\mathscr{C}$ is the cyclic infinite group. Just as in Example 3, $\mathscr{G}(\mathfrak{A}_p) = \mathscr{H}(\mathfrak{A}_p) = \mathscr{K}(\mathfrak{A}_p)$. Let $\mathfrak{B}_p$ be any algebraically closed field of characteristic p. Since the theory of algebraically closed fields of characteristic p is complete ([1] page 179), it follows by Lemma 3 that $\mathscr{K}(\mathfrak{B}_p) \cong \mathscr{C}$ and by Theorem 1 that $\mathscr{H}(\mathfrak{B}_p) \equiv \mathscr{C}$.

Next we consider theories categorical in power ω. Note that the models of such theories are assumed to have $\bar{\bar{\mu}} \leqq \omega$. We use the following result.

PROPOSITION 5. (*Ryll-Nardzewski*) ([11]) *A theory T is categorical in power ω iff for each n there are only finitely many formulas with n free variables which are inequivalent T.*

The next result follows from Proposition 5.

PROPOSITION 6. *If $\mathfrak{A}$ is a model of a theory categorical in power ω then $\mathscr{K}(\mathfrak{A})$ is finite.*

PROPOSITION 7. *If $\mathfrak{A}$ is a model of a theory categorical in power ω then $\mathscr{H}(\mathfrak{A})$ is periodic.*

Proof. Recall that a group is periodic if every element has finite order. Now let $f \in H(\mathfrak{A})$ be defined by the formula $\phi_1(x, y, a_1, \cdots, a_n)$. Then f^m is defined by a formula $\phi_m(x, y, a_1, \cdots, a_n)$. By Proposition 5 for some k, $\mathfrak{A} \models \phi_1(x, y, z_1, \cdots, z_n) \leftrightarrow \phi_k(x, y, z_1, \cdots, z_n)$. This implies that $f^k = f$.

Recall that a group is torsion free if every element, except the identity, has infinite order.

LEMMA 4. *If $\mathfrak{A}$ is a totally ordered set, $\mathfrak{A} = \langle A, < \rangle$, then $\mathscr{G}(\mathfrak{A})$ is torsion free.*

COROLLARY. *If $\mathfrak{A}$ is a model of a theory of totally ordered sets which is categorical in power ω then $\mathscr{H}(\mathfrak{A}) = I$.*

Each of the four complete theories of densely ordered sets of Example 1 is categorical in power ω ([1] pages 176–177). Thus the Corollary gives another proof of Marsh's result. This result may also be proved in a more direct way by showing, using induction on formulas, that if $\mathfrak{A}$ is a model of such a theory and $\phi(x, y, a_1, \cdots, a_n)$ defines an automorphism of $\mathfrak{A}$, then $\mathfrak{A} \models \phi(x, y, a_1, \cdots, a_n) \leftrightarrow y = x$. The elimination of quantifiers ([4] pages 51–52) simplifies the proof.

EXAMPLE 5. Consider the complete theory T of discretely ordered sets without first or last element ([4] page 53). Let $\mathfrak{A}$ be a model of T. We can prove by induction on formulas that if $\phi(x, y, a_1, \cdots, a_n)$ defines an automorphism of $\mathfrak{A}$, then $\mathfrak{A} \models \phi(x, y, a_1, \cdots, a_n) \leftrightarrow y = s^m x$ for some integer m. Thus $\mathscr{H}(\mathfrak{A}) = \mathscr{K}(\mathfrak{A}) \cong \mathscr{C}$.

3. Representation theorems. In [3] pages 68–69 it is proved that for every group $\mathscr{P}$ there is an algebra $\mathscr{A}$ such that $\mathscr{P} \cong \mathscr{G}(\mathscr{A})$. We prove representation theorems for $\mathscr{H}$ and $\mathscr{K}$. We denote by $\mathscr{S}_P$ the symmetric group on the elements of $\mathscr{P}$ and by $\mathscr{I}(\mathscr{P})$ the group of inner automorphisms of P.

LEMMA 5. *$\mathscr{I}(\mathscr{P})$ is a subgroup of $\mathscr{H}(\mathscr{P})$.*

THEOREM 3. *For every group $\mathscr{P}$ there is an algebra $\mathscr{A}$ such that $\mathscr{P} \cong \mathscr{G}(\mathscr{A}) = \mathscr{H}(\mathscr{A})$.*

Proof. When $\mathscr{P}$ is finite the construction in [3] page 68 gives a finite algebra. By Proposition 3 we can take this finite algebra as our $\mathscr{A}$. Therefore it suffices to give a construction when $\mathscr{P}$ is infinite. Actually our construction works for all $\mathscr{P}$ except if $\overline{\overline{P}} = 2$ or 6.

Embed $\mathscr{P}$ in $\mathscr{S}_P$ in the usual way ([5] page 90), and denote the regular subgroup of $\mathscr{S}_p$ so obtained by $\mathscr{P}^*$. Well order $\mathscr{P}^*$, $P^* = \langle p_0, \cdots, p_\iota, \cdots \rangle_{\iota < \delta}$.

From [5] pages 92–95 and [12] page 314 it follows that $\mathscr{G}(\mathscr{S}_P) = \mathscr{I}(\mathscr{S}_P) \cong \mathscr{S}_P$. By Lemma 5, $\mathscr{G}(\mathscr{S}_P) = \mathscr{H}(\mathscr{S}_P) = \mathscr{I}(\mathscr{S}_P)$. We let $\mathscr{A} = \langle P_P, \cdot, ^{-1}, p_0, \cdots, p_\iota, \cdots \rangle_{\iota < \delta}$, i.e. $\mathscr{A}$ is obtained from $\mathscr{S}_P$ by adding the elements of $\mathscr{P}^*$ as distinguished constants.

Note that $f \in G(\mathscr{A})$ iff $f \in G(\mathscr{S}_p)$ and f leaves all p_ι, $\iota < \delta$, fixed. Since $\mathscr{G}(\mathscr{S}_p) = \mathscr{I}(\mathscr{S}_p)$, $f \in G(\mathscr{A})$ iff $f(x) = h^{-1}xh$ for some $h \in S_P$ and $h^{-1}p_\iota h = p_\iota$ for all $p_\iota \in P^*$. So $\mathscr{G}(\mathscr{A}) = \mathscr{Z}_{\mathscr{S}_P}(\mathscr{P}^*)$. But $\mathscr{Z}_{\mathscr{S}_P}(\mathscr{P}^*) \cong \mathscr{P}$ ([5] page 91). Observe that if $f \in H(\mathscr{S}_p)$ then f is definable in $\mathscr{A}$. Also since $\mathscr{G}(\mathscr{A})$ is a subgroup of $\mathscr{G}(\mathscr{S}_p)$, we obtain $\mathscr{P} \cong \mathscr{G}(\mathscr{A}) = \mathscr{H}(\mathscr{A})$.

THEOREM 4. *For every abelian group $\mathscr{P}$ there is an algebra $\mathscr{A}$ such that $\mathscr{P} \cong \mathscr{G}(\mathscr{A}) = \mathscr{K}(\mathscr{A})$.*

Proof. Construct an algebra $\mathscr{A}'$ such that $\mathscr{P} \cong \mathscr{G}(\mathscr{A}')$, say $\mathscr{A}' = \langle A, F_0, \cdots, F_\iota, \cdots\rangle_{\iota<\eta}$. Well order the elements of $\mathscr{G}(\mathscr{A}')$, $G(\mathscr{A}') = \langle f_0, \cdots, f_\sigma, \cdots\rangle_{\sigma<\tau}$. We let $\mathscr{A} = \langle A, F_0, \cdots, F_\iota, \cdots, f_0, \cdots, f_\sigma, \cdots\rangle_{\iota<\eta,\sigma<\tau}$. This can be done since each f_σ is a unary operation on A. It follows that $\mathscr{G}(\mathscr{A}) = \mathscr{Z}(\mathscr{G}(\mathscr{A}')) = \mathscr{G}(\mathscr{A}') \cong \mathscr{P}$. Since $\mathscr{G}(\mathscr{A}) = \mathscr{K}(\mathscr{A})$, $\mathscr{P} \cong \mathscr{G}(\mathscr{A}) = \mathscr{K}(\mathscr{A})$.

Note that by the corollary to Proposition 4, Theorem 4 is the best representation theorem possible for $\mathscr{K}$.

4. Automorphisms definable in infinitary languages. In this section we consider the infinitary languages $L_{\alpha\beta}$ ([1] Chapter 14). We denote by $L_{\infty\infty}$ the language in which arbitrarily long connectives and quantifiers are allowed. We define $\mathscr{H}_{\alpha\beta}(\mathfrak{A})$ as $\mathscr{H}(\mathfrak{A})$ with L replaced by $L_{\alpha\beta}$ in the definition. $\mathscr{K}_{\alpha\beta}(\mathfrak{A})$ is defined similarly. Then statements analogous to ones in §§1 and 2 may be proved. In particular,

(1) $\mathscr{H}_{\alpha\beta}(\mathfrak{A}) \Delta \mathscr{G}(\mathfrak{A})$;
(2) If $\bar{\bar{\mu}} < \alpha$ and $\mathfrak{A} \equiv_{\alpha\alpha} \mathfrak{B}$, then $\mathscr{H}_{\alpha\alpha}(\mathfrak{A}) \equiv_{\alpha\alpha} \mathscr{H}_{\alpha\alpha}(\mathfrak{B})$;
(3) $\mathscr{H}_{\max(\bar{\bar{A}}^+,\omega)\omega}(\mathfrak{A}) = \mathscr{G}(\mathfrak{A})$;
(4) $\mathscr{K}_{\alpha\beta}(\mathfrak{A})$ is a subgroup of $\mathscr{Z}(\mathscr{G}(\mathfrak{A}))$.

EXAMPLE 6. For every symmetric group $\mathscr{S}_\lambda$ where λ is any cardinal $\neq 2$ or 6, $\mathscr{I}(\mathscr{S}_\lambda) = \mathscr{G}(\mathscr{S}_\lambda) \cong \mathscr{S}_\lambda$ and $\mathscr{Z}(\mathscr{S}_\lambda) = I$([5] pages 92–95 and [12] page 314). Thus $\mathscr{Z}(\mathscr{G}(\mathscr{S}_\lambda)) = I$. This last statement is also true for $\lambda = 2, 6$. By (4) $\mathscr{K}_{\infty\infty}(\mathscr{S}_\lambda) = I$ for every cardinal λ.

It follows from (3) that $\mathscr{H}_{\infty\infty}(\mathscr{A}) = \mathscr{G}(\mathscr{A})$. The next theorem is the converse of (4); it is a special case of a theorem of Rogers (Theorem 7 of [10]).

THEOREM 5. $\mathscr{K}_{\infty\infty}(\mathfrak{A}) = \mathscr{Z}(\mathscr{G}(\mathfrak{A}))$.

Proof. By (4) it suffices to prove that $\mathscr{Z}(\mathscr{G}(\mathfrak{A}))$ is a subgroup of $\mathscr{K}_{\infty\infty}(\mathfrak{A})$. So suppose that $f \in Z(\mathscr{G}(\mathfrak{A}))$. We find a formula $\phi(x, y)$ of $L_{\infty\infty}$ which defines f. Well order the elements of $\mathfrak{A}$, $A = \langle a_0, \cdots, a_\iota, \cdots \rangle_{\iota < \overline{\overline{A}}}$. We write $D(X)$ for $D(\mathfrak{A})$ with each a_ι changed to x_ι. Let $\phi(x, y)$ be $(\exists_{\iota < \overline{\overline{A}}} x_\iota)\{(\forall z)(\bigvee_{\iota < \overline{\overline{A}}} z = x_\iota) \wedge D(X) \wedge (\bigvee_{\iota < \overline{\overline{A}}} x = x_\iota \wedge y = f(x_\iota))\}$.

Now we show that $\phi(x, y)$ defines f. Suppose that $a' = f(a)$. Then for some $\rho, \sigma < \overline{\overline{A}}$, $a = a_\rho$ and $a' = a_\sigma$. Thus $a_\sigma = f(a_\rho)$. Interpreting the x_ι as a_ι we obtain $\mathfrak{A} \models \phi(a_\rho, a_\sigma)$. Conversely suppose that $\mathfrak{A} \models \phi(a, a')$. This means that there is a well ordering of $\mathfrak{A}$, say $A = \langle e_0, \cdots, e_\iota, \cdots \rangle_{\iota < \overline{\overline{A}}}$ such that for some $\rho, \sigma < \overline{\overline{A}}$, $a = e_\rho$ and $a' = e_\sigma$. Therefore in the original well ordering of $\mathfrak{A}$, $a_\sigma = f(a_\rho)$. Consider the map s: $a_\iota \to e_\iota$, $\iota < \overline{\overline{A}}$; $s \in G(\mathfrak{A})$. Also $sf(a_\rho) = e_\sigma$ and $fs(a_\rho) = f(e_\rho)$. Since $fs = sf$, we obtain $e_\sigma = f(e_\rho)$.

COROLLARY. (a) *If* $\overline{\overline{A}} < \omega$ *and* $\overline{\overline{\mu}} < \omega$ *then* $\mathscr{K}(\mathfrak{A}) = \mathscr{Z}(\mathscr{G}(\mathfrak{A}))$. (b) *If* $\overline{\overline{A}} \geqq \omega$ *or* $\overline{\overline{\mu}} \geqq \omega$ *then* $\mathscr{K}_{\max(\overline{\overline{A}}^+, \overline{\overline{\mu}}^+)\max(\overline{\overline{A}}^+, \omega)}(\mathfrak{A}) = \mathscr{Z}(\mathscr{G}(\mathfrak{A}))$.

In case $\overline{\overline{A}} = \omega$ and $\overline{\overline{\mu}} \leqq \omega$, $\mathscr{K}_{\omega_1\omega}(\mathfrak{A}) = \mathscr{Z}(\mathscr{G}(\mathfrak{A}))$. This can be proved using a theorem of D. Scott as explained in the footnote on pages 197–198 of [10].

5. Analogies between group theory and model theory. In this section we assume that $\mathfrak{A} \prec \mathfrak{B} \prec \mathfrak{C}$, $a_i \in A$, $b_i \in B$, $c_i \in C$, and $\overline{\overline{\mu}} < \omega$. Our analogs for group, subgroup, and inner automorphism are structure, elementary submodel, and definable automorphism respectively.

DEFINITION 3. $\mathfrak{A} \Delta \mathfrak{B}$ if for every $f \in H(\mathfrak{B})$, $f \mid A \in G(\mathfrak{A})$.

DEFINITION 4. If there is an $f \in H(\mathfrak{A})$ such that $f(a_1) = a_2$, then a_1 and a_2 are conjugate elements (in $\mathfrak{A}$). The set of elements of $\mathfrak{A}$ conjugate to $a \in A$ forms a conjugacy class of $\mathfrak{A}$.

DEFINITION 5. The cardinal of the set of conjugacy classes of $\mathfrak{A}$ is denoted by $\kappa(\mathfrak{A})$.

Consider the theory T of Example 5. Then given a cardinal δ, T has a model $\mathfrak{A}_\delta$ such that $\kappa(\mathfrak{A}_\delta) = \delta$. Let $\mathfrak{A}_\delta$ be $(\omega^* + \omega)\delta$. The result follows since each copy of the set of integers, $\omega^* + \omega$, forms a conjugacy class.

PROPOSITION 8. *If* $\mathfrak{A}\Delta\mathfrak{C}$ *then* $\mathfrak{A}\Delta\mathfrak{B}$.

Proof. Let $f \in H(\mathfrak{B})$. Extend f to $\bar{f} \in H(\mathfrak{C})$ as in Proposition 2(a). Since $\bar{f} \mid A = f \mid A$, the result follows.

PROPOSITION 9. *If* $\mathfrak{A}\Delta\mathfrak{B}$ *then* $\mathscr{H}(\mathfrak{A})$ *can be embedded as a normal subgroup in* $\mathscr{H}(\mathfrak{B})$.

Proof. We choose the m of Proposition 2(c) for the embedding. Now let $f \in H(\mathfrak{A})$ be defined by $\phi(x, y, a_1, \cdots, a_n)$ and let $g \in H(\mathfrak{B})$. Since $g\bar{f}g^{-1}$ is defined by $\phi(x, y, g(a_1), \cdots, g(a_n))$, by Proposition 2(b) $g\bar{f}g^{-1} = \bar{h}$ where $h \in H(\mathfrak{A})$.

LEMMA 6. *Two elements of* $\mathfrak{A}$ *are conjugate in* $\mathfrak{A}$ *iff they are conjugate in* $\mathfrak{B}$.

COROLLARY. $\kappa(\mathfrak{A}) \leqq \kappa(\mathfrak{B})$.

THEOREM 6. $\mathfrak{A}\Delta\mathfrak{B}$ *iff every conjugacy class of* $\mathfrak{A}$ *is a conjugacy class of* $\mathfrak{B}$.

Proof. Suppose that $\mathfrak{A}\Delta\mathfrak{B}$. Let K be a conjugacy class of $\mathfrak{A}$. By Lemma 6 $K \subseteqq L$ where L is a conjugacy class of $\mathfrak{B}$ and $L \cap A = K$. Therefore $L = K$. Conversely, if every conjugacy class of $\mathfrak{A}$ is a conjugacy class of $\mathfrak{B}$, then by definition $\mathfrak{A}\Delta\mathfrak{B}$.

We may define a structure $\mathfrak{A}$ to be abelian if $H(\mathfrak{A}) = I$. If $\mathfrak{B}$ is abelian then $\mathfrak{A}\Delta\mathfrak{B}$ and $\mathfrak{A}$ is abelian. By Examples 1 and 2, all densely ordered sets and real-closed ordered fields are abelian.

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