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POWER SERIES RINGS OVER PRÜFER DOMAINS

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Let R be a commutative ring with identity. R is said to have dimension n, written $\dim R = n$, if there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n+1$ prime ideals of R, where $P_n \subset R$, but no such chain of $n+2$ prime ideals. Seidenberg has shown that if $\dim R = n$ and X is an indeterminate over R, then $n+1 \leqq \dim R[X] \leqq 2n+1$. Moreover, he has shown that dim $R[X] = n+1$ if R is a Prüfer domain. The author has shown that if V is a rank one nondiscrete valuation ring, then dim $V[[X]] = \infty$. The principal result of this paper is that if D is a Prüfer domain with $\dim D = n$, then either $\dim D[[X]] = n+1$ or $\dim D[[X]] = \infty$, and necessary and sufficient conditions are given.

1. NOTATION. Our notation and terminology are essentially that of [4]. Throughout, R denotes a commutative ring with identity and T denotes the total quotient ring of R. By an *overring* S of R, we shall mean a ring S such that $R \subseteqq S \subseteqq T$. The set of natural numbers will be denoted by ω and ω_0 is the set of nonnegative integers. If A is an ideal of R, then we let

$$A[[X]] = \left\{f(X) = \sum_{i=0}^{\infty} a_i X^i / a_i \in A \text{ for each } i \in \omega_0\right\}$$

and we define $AR[[X]]$ to be the ideal of $R[[X]]$ which is generated by A. The ideal A will be called an SFT-*ideal* (an ideal of *strong finite type*) provided there exists a finitely generated ideal $B \subseteqq A$ and $k \in \omega$ such that $a^k \in B$ for each $a \in A$. We say that R is an SFT-*ring* provided each ideal of R is an SFT-ideal.

2. Some properties of SFT-rings. Arnold has shown in [1] that if R is not an SFT-ring, then $\dim R[[X]] = \infty$. In this paper we are primarily concerned with finite-dimensional Prüfer domains which are also SFT-rings, and our main result shows that for such a domain D, if $\dim D = n$, then $\dim D[[X]] = n+1$. Before restricting our attention to Prüfer domains, however, we wish to consider some properties of arbitrary SFT-rings.

LEMMA 2.1. *If A_1, A_2 are SFT-ideals of R and if C is an ideal of R such that $A_1 \cap A_2 \supseteqq C \supseteqq A_1 A_2$, then C is an SFT-ideal.*

Proof. For $i = 1, 2$, there exists a finitely generated ideal

$B_i \subseteq A_i$ and $k_i \in \omega$ such that $a_i^{k_i} \in B_i$ for each $a_i \in A_i$. Set $k = k_1 + k_2$. Then for $c \in C$, we have that $c^k = c^{k_1}c^{k_2} \in B_1B_2 \subseteq A_1A_2 \subseteq C$. Since B_1B_2 is finitely generated, the lemma follows.

PROPOSITION 2.2. *R is an SFT-ring if and only if each prime ideal of R is an SFT-ideal.*

Proof. Suppose that R is not an SFT-ring. It follows from a straight-forward application of Zorn's Lemma that R contains an ideal P which is maximal among those ideals of R which are not SFT-ideals. Thus, if A and B are ideals of R which properly contain P, then A and B are SFT-ideals. It is an immediate consequence of Lemma 2.1 that $P \not\supseteq AB$, for otherwise, P would be an SFT-ideal. Therefore, P is a prime ideal of R which is not an SFT-ideal.

PROPOSITION 2.3. *If the ring S is the homomorphic image of an SFT-ring R, then S is also an SFT-ring.*

The proof of Proposition 2.3 is straightforward and will be omitted.

Before stating our next result, we recall that an overring R_1 of R is called a *flat overring* of R provided R_1 is flat as an R-module. Richman in [8] has studied flat overrings of integral domains and has dubbed them "generalized quotient rings" due to the fact that many of the classical properties of quotient rings also hold for flat overrings. Flat overrings are further considered in [2], where they are shown to be a special class of "generalized transforms." Specifically, if R_1 is a flat overring of R, then there exists a multiplicatively closed set $\mathscr{S}$ of ideals of R such that

$$R_1 = R_{\mathscr{S}} = \{\xi \in T/\xi A \subseteq R \text{ for some } A \in \mathscr{S}\} .$$

Moreover, $\mathscr{S}$ may be chosen so that $AR_1 = R_1$ for each $A \in \mathscr{S}$ [2, Thm. 1.3]. Using this notation and terminology, we now prove the following result.

PROPOSITION 2.4. *Let R be an SFT-ring. If R_1 is a flat overring of R, then R_1 is an SFT-ring.*

Proof. Let $R_1 = R_{\mathscr{S}}$ as described above, and let Q be a prime ideal of R. If we set $P = Q \cap R$, then $Q = P_{\mathscr{S}}$ [2, Thm. 1.1]; thus, for $q \in Q$, there exists $A \in \mathscr{S}$ such that $qA \subseteq P$. But P is an SFT-ideal, so there is a finitely generated ideal $B \subseteq P$ and $k \in \omega$ such that $p^k \in B$ for each $p \in P$. In particular, $q^k a^k \in B$ for each $a \in A$. Let $\mathscr{A} = \{\xi \in R_1 / q^k \xi \in BR_1\}$. Then $\mathscr{A}$ is an ideal of R_1 and $a^k \in \mathscr{A}$ for each $a \in A$. Consequently, we have that $\sqrt{\mathscr{A}} = \sqrt{AR_1} = R_1$, from which it is immediate that $\mathscr{A} = R_1$. This shows that $q^k \in BR_1$ for each $q \in Q$, and hence, that Q is an SFT-ideal in R_1.

PROPOSITION 2.5. *If R is an SFT-ring, then R satisfies the ascending chain condition for radical ideals, i.e., the prime spectrum of R is Noetherian.*

Proof. Clearly, each radical ideal of R is the radical of a finitely generated ideal. But this is equivalent to the ascending chain condition for radical ideals [7, p. 633].

If R satisfies the ascending chain condition for radical ideals, then it is shown in [6, p. 59] that each ideal of R has only finitely many minimal prime divisors. As an immediate consequence we have

COROLLARY 2.6. *Each ideal of an SFT-ring has only finitely many minimal prime divisors.*

We conclude this section with the following lemma.

LEMMA 2.7. *Let D be an integral domain which is an SFT-ring. If P is a nonzero prime ideal of D, then $P \neq P^2$.*

Proof. Let V be a valuation overring of D for which $PV \neq V$. Since P is an SFT-ideal, there exists a finitely generated ideal $B \subseteqq P$ and $k \in \omega$ such that $p^k \in B$ for each $p \in P$. If $P_1 = PV$ and $B_1 = BV$, then we also have $\xi^k \in B_1$ for each $\xi \in P_1$. Since V is a valuation ring, it follows that $P_1^k \subseteqq B_1 \subseteqq P_1$. If $B_1 = P_1$, then P_1 is principal, so $P_1 \neq P_1^2$. If $B_1 \subset P_1$, then $P_1^k \neq P_1$, and again it follows that $P_1 \neq P_1^2$. Consequently, $P \neq P^2$ as we wished to show.

3. Prüfer domains which are SFT-rings. Throughout this section D will denote a Prüfer domain. We begin by giving a characterization of those Prüfer domains which are also SFT-rings.

PROPOSITON 3.1. *In order that the Prüfer domain D be an SFT-ring, it is necessary and sufficient that for each nonzero prime ideal P of D, there exists a finitely generated ideal A such that*

$$P^2 \subseteqq A \subseteqq P .$$

Proof. In view of Proposition 2.2, it is clear that the given conditions are sufficient to insure that an arbitrary ring is an SFT-ring. To show that they are also necessary for the Prüfer domain D, suppose that D is an SFT-ring and let P be a nonzero prime ideal of D. Since P is an SFT-ideal, $P = \sqrt{B}$ for some finitely generated ideal B of D. By Lemma 2.7 there exists $p \in P - P^2$. If we set $A = B + (p)$, then A is finitely generated, $P = \sqrt{A}$, and $P^2 \not\supseteqq A$. Let M be a maximal ideal of D which contains P. Since P^2 is $P-$ primary [4, 19.3], we have $P^2 = P^2 D_M \cap D$. It follows that $P^2 D_M \not\supseteqq AD_M$; hence $P^2 D_M \subseteqq AD_M$. Consequently, $P^2 \subseteqq A \subseteqq P$.

COROLLARY 3.2. *Suppose that D is an SFT-ring, let P be a nonzero prime ideal of D and let $p \in P - P^2$. For each $n \in \omega$ there exists $s_n \in D - P$ such that $s_n P^{n+1} \subseteqq (p^n)$.*

Proof. Let $A = (a_1, \cdots, a_m)$ be a finitely generated ideal of D such that $P^2 \subseteqq A \subseteqq P$. Then $AD_P \subseteqq PD_P = (p)D_P$, so we may find $s \in D - P$ such that $sa_i \in (p)$ for $1 \leqq i \leqq m$. For each $n \in \omega$, set $s_n = s^n$. For $n = 1$ we get $s_1 P^2 \subseteqq s_1 A \subsetneqq (p)$, and for $n > 1$ we get $s_n P^{n+1} = (s_1 P^2)(s_{n-1} P^{n-1}) \subseteqq (p)(s_{n-1} P^{n-1}) \subseteqq s_{n-1} P^n$. The corollary follows by induction on n.

Hereafter, we assume that D has finite dimension; $\Pi = \{P_\alpha\}_{\alpha \in A}$ is the set of minimal prime ideals for D, and $\mathscr{M} = \{M_\beta\}_{\beta \in \Gamma}$ is the set of maximal ideals of D.

If D is an SFT-ring, then as an immediate consequence of Lemma 2.7, we see that D_Q is a discrete valuation ring for each prime ideal Q of D [4, p. 177]. In particular, D_{P_α} is a rank one discrete valuation ring for each $P_\alpha \in \Pi$. Dedekind domains and discrete valuation rings with finite dimension provide immediate examples of Prüfer domains wich are SFT-rings. In fact, if $\dim D = 1$, then it follows from [4, 30.2] that D is an SFT-ring if and only if D is a Dedekind domain. If we set $D' = \bigcap_\alpha D_{P_\alpha}$, then from [4, 22.1], we see that D' is a Prüfer domain. Richman shows in [8] that each overring of a Prüfer domain is a flat overring, so by Proposition 2.4 D' is an SFT-ring. It is immediate from Corollary 2.6 that D' has finite real character [cf. 4, p. 505], so by [4, 35.8] we have $\dim D' = 1$. Our preceding remarks now imply that D' is a Dedekind domain. By [4, 36.11], $D'[[X]]$ is a Krull domain, and since D' is Noetherian we have that $\dim D'[[X]] = 2$ [3, p. 603]. But the maximal ideals of $D'[[X]]$ are of the form $P + (X)$, where P is a maximal ideal of D', so $J' = (D'[[X]])_{D'-(0)}$ is a one-dimensional Krull domain—that is, J' is a Dedekind domain [4, 35.16]. Set $J = (D[[X]])_{D-(0)}$ and let L denote quotient field of $D[[X]]$.

LEMMA 3.3. *If D is an SFT-ring, then $J = J' \cap L$.*

Proof. Clearly, $J \subseteqq J' \cap L$, so let $\xi(X) = f(X)/g(X) \in J' \cap L$, where $f(X)$, $g(X) \in D[[X]]$, $f(X) = \sum_{i=0}^{\infty} f_i X^i$ and $g(X) = \sum_{i=0}^{\infty} g_i X^i$. Since $\xi(X) \in J'$, there exist $\lambda \in D' - (0)$ and $h(X) \in D'[[X]]$ such that $\lambda f(X) = g(X)h(X)$. Consequently, if there exists $m \in \omega_0$ such that $g_i = 0$ for $i \leqq m$, then we also have that $f_i = 0$ for $i \leqq m$. Therefore, in our representation $\xi(X) = f(X)/g(X)$, we may assume that $g_0 \neq 0$. If $M_\beta \in \mathscr{M}$ is such that $g_0 \notin M_\beta$, then $g(X)$ is a unit in $D_{M_\beta}[[X]]$. Thus, $\xi(X) \in D_{M_\beta}[[X]]$. Let Q be a minimal prime divisor of $(g_0)D$ and let P be the minimal prime ideal of D contained in Q. Clearly, $\xi(X) \in (D_P[[X]])_{D_P-(0)}$ and by [3, p. 602], $(D_P[[X]])_{D_P-(0)} = (D_Q[[X]])_{D_Q-(0)}$. Hence, there exists $d \in D_Q - (0)$ such that $d\xi(X) \in D_Q[[X]]$. In fact, we may assume that $d \in D$. If $M_\beta \in \mathscr{M}$ is such that $M_\beta \supseteqq Q$, then $QD_Q \subseteqq D_{M_\beta}$ [4, 14.6]. Consequently, for $q \in Q - (0)$, we have

$$qd\xi(X) \in D_{M_\beta}[[X]] .$$

By Corollary 2.6, $(g_0)D$ only finitely many minimal prime divisors, so it follows that we may find $r \in D - (0)$ such that

$$r\xi(X) \in \bigcap_\beta D_{M_\beta}[[X]] = D[[X]] .$$

Therefore, $\xi(X) \in J$ as we wished to show.

We wish to show that J is, in fact, a Dedekind domain. In order to do this, we first need to consider the domain

$$U = D'[[X]] \cap L .$$

LEMMA 3.4. *If A is an ideal of U such that $AD'[[X]]$ is contained in no minimal prime ideal of $D'[[X]]$, then $X^n \in A$ for some $n \in \omega$.*

Proof. The only possible minimal prime divisors for $AD'[[X]]$ are of the form $P + (X)$, where P is a maximal ideal of D'. Consequently, $X \in \sqrt{AD'[[X]]}$ — that is there exists $n \in \omega$ such that $X^n \in AD'[[X]]$. Let $\lambda_1(X), \cdots, \lambda_k(X) \in D'[[X]]$ and $a_1(X), \cdots, a_k(X) \in A$ be such that $X^n = \sum_{i=1}^{k} \lambda_i(X)a_i(X)$. If $\lambda_i(X) = \sum_{j=0}^{\infty} \beta_{ij}X^j$, set $\gamma_i(X) = \sum_{j=0}^{n} \beta_{ij}X^j$ and $\zeta_i(X) = (\lambda_i(X) - \gamma_i(X))/X^{n+1}$. For $1 \leqq i \leqq k$ we have that $\gamma_i(X) \in D'[[X]] \cap L = U$, and hence,

$$\begin{aligned}\sum_{i=1}^{k} \gamma_i(X)\, a_i(X) &= \sum_{i=1}^{k} \lambda_i(X)a_i(X) - X^{n+1}\Big(\sum_{i=1}^{k} \zeta_i(X)a_i(X)\Big)\\ &= X^n - X^{n+1}\sum_{i=1}^{k} \zeta_i(X)a_i(X) \in A .\end{aligned}$$

But $u(X) = 1 - X(\sum_{i=1}^{k} \zeta_i(X)a_i(X)) \in U$ is a unit in $D'[[X]]$, so it is also a unit in U. Therefore, $X^n \in A$.

Assuming that D is an SFT-ring, we know that D_{P_α} is a rank one discrete valuation ring for each $P_\alpha \in \Pi$. If v is a valuation associated with D_{P_α}, then we may define a "trivial extension" v^* of v to L by setting

$$v^*\left(\sum_{i=0}^{\infty} h_i x^i\right) = \min_{i \in \omega_0} \{v^*(h_i) \mid h_i \neq 0\}$$

for $\sum_{i=0}^{\infty} h_i X^i \in D[[X]]$ [5, p. 380]. If V^* is the valuation overring of $D[[X]]$ associated with v^*, then V^* is rank one discrete and is centered on $P_\alpha[[X]]$ in $D[[X]]$. Since $D' \subseteqq D_{P_\alpha}$, we may also extend D_{P_α} to a rank one discrete valuation overring V_1^* of $D'[[X]]$. If $P_1 = P_\alpha D_{P_\alpha} \cap D'$, then V_1^* is the essential valuation overring of $D'[[X]]$ associated with the minimal prime ideal $P_1[[X]]$ [5, p. 380]. It follows from [4, 36.10] that $U = D'[[X]] \cap L$ is a Krull domain and each minimal prime ideal of U has the form $Q \cap U$, where Q is a minimal prime ideal of $D'[[X]]$. Moreover, whenever Q is a minimal prime ideal of $D'[[X]]$ such that $Q \cap U \neq (0)$, then $Q \cap U$ is a minimal prime ideal of U and $(D'[[X]])_Q \cap L$ is the essential valuation overring of U associated with $Q \cap U$. In particular, $V^* = V_1^* \cap L$ is the essential valuation overring of U associated with $P_1' = P_1[[X]] \cap U$. We are now in a position to prove the following key result.

PROPOSITION 3.5. *Suppose that D is an SFT-ring and let $P_\alpha \in \Pi$. Then $P_\alpha[[X]]$ is a minimal prime ideal of $D[[X]]$.*

Proof. Let Q be a nonzero prime ideal of $D[[X]]$, $Q \subseteqq P_\alpha[[X]]$. If $(0) \subset Q \cap D \subseteqq P_\alpha = P_\alpha[[X]] \cap D$, then $Q \cap D = P_\alpha$. Consequently, $Q \supseteqq P_\alpha D[[X]]$. But $P_\alpha[[X]] = \sqrt{P_\alpha D[[X]]}$ [1, Thm. 1], so it follows that $Q = P_\alpha[[X]]$. Thus, we may assume that $Q \cap D = (0)$. Let W be a valuation overring of $D[[X]]$ with prime ideals $Q_1 \supset Q_2$ such that Q_1 is maximal in W, $Q_1 \cap D[[X]] = P_\alpha[[X]]$ and $Q_2 \cap D[[X]] = Q$. If $p \in P_\alpha - P_\alpha^2$, then $p \in Q_1 - Q_2$, so we may assume that $Q_1 = \sqrt{pW}$. We wish to show that $U \subseteqq W$. Thus, let $\xi(X) \in U$. Since

$$U \subseteqq J' \cap L = J\,,$$

we may write $\xi(X) = f(X)/d$, where $f(X) \in D[[X]]$ and $d \in D - (0)$. Suppose that $\xi(X) \notin W$. Then

$$\xi(X)^{-1} = d/f(X) \in Q_1\,, \quad \text{so} \quad w(d) > w(f(X)) \geqq 0\,,$$

where w is a valuation associated with W. Now $w(d) > 0$ implies that $d \in P_\alpha$ and hence, that $v^*(d) > 0$. Since $\xi(X) \in U \subseteqq V^*$, we have that $v^*(\xi(X)) \geqq 0$ — that is, $v^*(f(X)) \geqq v^*(d) > 0$. If $v^*(d) = k$, then $(d)D_{P_\alpha} = P_\alpha^k D_{P_\alpha} = (p^k)D_{P_\alpha}$, so there exists $s, t \in D - P_\alpha$ such that $sd = tp^k$. But then $v^*(f(X)) \geqq k$, so $f(X) \in P_\alpha^k[[X]]$. Since $Q_1 = \sqrt{pW}$, there exists $n \in \omega$ such that $d^n/f(X)^n = p\lambda(X)$ for some $\lambda(X) \in W$. But $f(X)^n \in P_\alpha^{nk}[[X]]$, so by Corollary 3.2, there exists $\tau \in D - P_\alpha$ such that $\tau(f(X))^n = p^{nk-1}f_1(X)$, where $f_1(X) \in D[[X]]$. Since $nk \leqq v^*(\tau f(X)^n) = v^*(p^{nk-1}f_1(X)) = (nk-1) + v^*(f_1(X))$, it follows that $f_1(X) \in P_\alpha[[X]]$. We now have

$$\begin{aligned}\lambda(X) &= d^n/p(f(X))^n = \tau s^n d^n/s^n p\tau(f(X))^n \\ &= \tau t^n p^{nk}/s^n p^{nk} f_1(X) = \tau t^n/s^n f_1(X) .\end{aligned}$$

But $w(\tau t^n/s^n f_1(X)) = -w(f_1(X)) < 0$, so it must be the case that $W \supseteqq U$.

Let P_1' be the center of V^* on U — that is, $P_1' = P_1[[X]] \cap U$, and let Q_1' be the center of W on U. We claim that $Q_1' \supseteqq P_1'$, for let $\xi(X) = f(X)/d \in P_1'$. Then $v^*(f(X)) > v^*(d) \geqq 0$; in particular,

$$f(X) \in P_\alpha[[X]] \subseteqq Q_1 .$$

If $d \notin P_\alpha$, then $w(\xi(X)) = w(f(X)) > 0$ and hence, $\xi(X) \in Q_1 \cap U = Q_1'$. Thus, assume that $d \in P_\alpha$ — say $v^*(d) = k$. Then arguing as above, there exist $s, t \in D - P_\alpha$ such that $sd = tp^k$. Moreover,

$$v^*(f(X)) \geqq k + 1 , \quad \text{so} \quad f(X) \in P_\alpha^{k+1}[[X]] .$$

Consequently, there exists $\tau \in D - P_\alpha$ and $f_1(X) \in P_\alpha[[X]]$ such that $\tau f(X) = p^k f_1(X)$. This yields $\xi(X) = s\tau f(X)/\tau sd = sp^k f_1(X)/\tau tp^k = sf_1(X)/\tau t$ which, as we have just observed, is in Q_1'. Therefore, $P_1' \subseteqq Q_1'$. But we also have that $Q_2 \cap U \subset Q_1'$, and $Q_2 \cap U \not\supseteqq P_1'$ since $(Q_2 \cap U) \cap D[[X]] = Q \subset P_\alpha[[X]] = P_1' \cap D[[X]]$. It follows that Q_1' contains at least two distinct minimal prime ideals of U and hence, $(Q_1')D'[[X]]$ cannot be contained in any minimal prime ideal of $D'[[X]]$. By Lemma 3.4, there exists $n \in \omega$ such that $X^n \in Q_1' \subseteqq Q_1$, contrary to our assumption that $Q_1 \cap D[[X]] = P_\alpha[[X]]$. We conclude that $P_\alpha[[X]]$ is minimal in $D[[X]]$.

We now digress momentarily in order to strengthen the results of Proposition 3.5. It follows from [4, 16.10] that if P is a prime ideal of D, then each prime ideal of $D[X]$ contained in $P[X]$ is the extension of a prime ideal of D. We show that the following analogue holds in $D[[X]]$.

COROLLARY 3.6. *If D is an SFT-ring and if P is a prime ideal of D, then the only prime ideals of $D[[X]]$ contained in $P[[X]]$ have* *the form* $P_1[[X]]$ *for some prime ideal* P_1 *of* D.

Proof. Suppose that D is an SFT-ring, let P be a prime ideal of D, and let Q be a nonzero prime ideal of $D[[X]]$ such that $Q \subset P[[X]]$. If $Q \cap D = P' \neq (0)$, then $Q \supseteqq \sqrt{P'D[[X]]} = P'[[X]]$ [1, Thm. 1], so by considering $D[[X]]/P'[[X]] \cong (D/P')[[X]]$, we may assume that $Q \cap D = (0)$. In view of Proposition 3.5, we may also assume that P is not minimal in D; hence, there exists a prime ideal P_1 of D such that $P_1 \subset P$ and there are no prime ideals properly contained between P and P_1 [4, 19.3]. We further assume that $P_1[[X]] \not\supseteqq Q$. Let $p \in P - P^2$ and $p_1 \in P_1 - P_1^2$.

$$S = \{p^k s(X) \mid k \in \omega_0,\ s(X) \in D[[X]] - P[[X]]\}$$

is a multiplicative system in $D[[X]]$ and $Q \cap S = \phi$. Let

$$f(X) \in Q - P_1[[X]]$$

and set $A = (f(X), p_1)D[[X]]$. Suppose that $r(X), t(X) \in D[[X]]$ are such that $r(X)f(X) + p_1 t(X) = p^k s(X) \in S$. Since $P_1 D_P = \bigcap_{n=1}^{\infty} P^n D_P$ [4, 14.1], there exist $y, z \in D$, $y \notin P$, $z \in P_1$, such that $yp_1 = zp^k$. Therefore, $yr(X)f(X) = p^k(ys(X) - zt(X)) \in S$, contrary to the fact that $Q \cap S = \phi$. Thus, $A \cap S = \phi$, and there exists a prime ideal Q_1 of $D[[X]]$ such that $A \subseteqq Q_1$ and $Q_1 \cap S = \phi$. Clearly, $Q_1 \subset P[[X]]$, and since $p_1 \in Q_1 \cap D \subset P[[X]] \cap D$, it follows that $Q_1 \cap D = P_1$. But then we have $P_1[[X]] \subset Q_1 \subset P[[X]]$ which yields, on reducing to $(D/P_1)[[X]]$, a contradiction to Proposition 3.5. We conclude that no such Q exists.

PROPOSITION 3.7. *If* D *is an SFT-ring, then* J *is a Dedekind domain.*

Proof. Since $J = J' \cap L$, J is a Krull domain [4, 36.10]. Therefore, it suffices to show that $\dim J = 1$ [4, 35.16]. Let QJ be a nonzero prime ideal of J, where Q is a prime ideal in $D[[X]]$ such that $Q \cap D = (0)$. We first suppose that $QD'[[X]]$ is contained in some minimal prime ideal Q' of $D'[[X]]$. We cannot have $Q' = P_1[[X]]$ for any prime ideal P_1 of D', for if we set $P_\alpha = P_1 \cap D$, then

$$P_1[[X]] \cap D[[X[] = P_\alpha[[X]]\ .$$

But $P_\alpha \in \Pi$, so by Proposition 3.5, $P_\alpha[[X]]$ is minimal in $D[[X]]$. Since we must have $Q \subseteqq QD'[[X]] \cap D[[X]] \subseteqq Q' \cap D[[X]]$, it follows that $Q' \cap D' = (0)$. Consequently, $Q'J'$ is a minimal prime ideal of J'. Since $Q'J' \cap J \supseteqq QJ \neq (0)$, QJ is a minimal prime ideal in J.

We now consider the possibility that $QD'[[X]] = (QU)D'[[X]]$ is contained in no minimal prime ideal of $D'[[X]]$. But if this is the case, then by Lemma 3.4, $X^n \in QU$ for some $n \in \omega$. Since

$$Q = QJ \cap D[[X]] = (QU)J \cap D[[X]] ,$$

it follows that $X \in Q$. But $Q \neq (X)D[[X]]$, for $(X)D'[[X]]$ is a minimal prime ideal of $D'[[X]]$ and clearly, $(X)D'[[X]] \supsetneqq (X)D[[X]]$. Therefore, $Q \supset (X)D[[X]]$, from which it is immediate that $Q \cap D \neq (0)$. Since this contradicts our assumption on Q, we conclude that QJ is minimal in J.

We now state the principal result of this paper.

THEOREM 3.8. *Let D be Prüfer domain with* $\dim D = n$. *The following statements are equivalent*:

(1) *D is an SFT-ring.*

(2) $\operatorname{Dim} D[[X]] = n + 1$.

(3) $\operatorname{Dim} D[[X]] < \infty$.

Proof. It is clear that (2) implies (3) and it is shown in [1] that (3) implies (1). We show that (1) implies (2) by induction of n. But if $n = 1$, then D is a Dedekind domain, so the theorem holds. Now suppose that $\dim D = n > 1$ and let $(0) \subset Q_1 \subset \cdots \subset Q_k$, $k > 1$, be a chain of prime ideals of $D[[X]]$. Since Q_2 is not minimal, it follows from Proposition 3.7 that $Q_2 \cap D \neq (0)$. In particular, $Q_2 \cap D \supseteqq P_\alpha$ for some $P_\alpha \in \Pi$. But then $Q_2 \supseteqq \sqrt{P_\alpha D[[X]]} = P_\alpha[[X]]$ [1, Thm. 1], and the containment is proper since $P_\alpha[[X]]$ is minimal in $D[[X]]$. This yields a chain $(0) \subset Q_2/P_\alpha[[X]] \subset \cdots \subset Q_k/P_\alpha[[X]]$ of $k-1$ prime ideals in $D[[X]]/P_\alpha[[X]] \cong (D/P_\alpha)[[X]]$. Since D/P_α is a Prüfer domain [4, 18.5] which is, by Proposition 2.3, an SFT-ring, our induction hypothesis implies that $k - 1 \leqq n$. Consequently, $k \leqq n + 1$. But we already know that $\dim D[[X]] \geqq n + 1$, so equality must hold.

4. EXAMPLE. Suppose that $\dim R = n$. We have seen that

$$\dim R[X] = n + 1 \nRightarrow \dim R[[X]] = n + 1 ;$$

for if D is any n-dimensional Prüfer domain which is not an SFT-ring, then $\dim D[X] = n + 1$ while $\dim D[[X]] = \infty$. We now give an example which shows that

$$\dim R[[X]] = n + 1 \nRightarrow \dim R[X] = n + 1 .$$

Thus, let F be a field and $K = F(Y)$ a simple transcendental extension of F. Let $V = K + M$ be a rank one discrete valuation ring with maximal ideal M (e.g., take $V = K[[Z]]$) and set $D = F + M$.

Then D is integrally closed and M is the unique nonzero prime ideal of D [4, App. 2]. In particular, $\dim D = 1$. But D is not a Prüfer domain, so $\dim D[X] = 3$ [4, 25.13]. For $m \in M$, we have

$$mV[[X]] \subseteqq M[[X]] \subseteqq D[[X]] ,$$

whence it follows that $(V[[X]])_{V-(0)} = (D[[X]])_{D-(0)}$. But

$$\dim (V[[X]])_{V-(0)} = 1 ,$$

so if Q is a prime ideal of $D[[X]]$ such that $Q \cap D = (0)$, then Q is minimal in $D[[X]]$. Moreover, it is clear that $Q = Q' \cap D[[X]]$ for some minimal prime ideal Q' of $V[[X]]$, $Q' \neq M[[X]]$. Thus, in order to see that $M[[X]]$ is minimal in $D[[X]]$, it suffices to see that $Q' \cap D[[X]] \not\subseteq M[[X]]$ for each such Q'. Therefore, let

$$\xi(X) = \sum_{i=0}^{\infty} \xi_i X^i \in Q' - M[[X]]$$

and let r be the smallest integer for which $\xi_r \notin M$. Since ξ_r is a unit in V, we assume that $\xi_r = 1$. If $u(X) = \sum_{i=0}^{\infty} \xi_{r+i} X^i$, then $u(X)$ is a unit in $V[[X]]$ and

$$\xi(X)u(X)^{-1} = \left(\sum_{i=0}^{r-1} \xi_i X^i\right) u(X)^{-1} + X^r \in Q' \cap D[[X]] .$$

Hence, $Q' \cap D[[X]] \not\subseteq M[[X]]$.

Now let $(0) \subset Q_1 \subset Q_2$ be a chain of prime ideals of $D[[X]]$. Then $Q_2 \cap D \neq (0)$; hence $Q_2 \supseteqq M[[X]]$. But $M[[X]]$ is minimal, so the containment is proper. It follows that Q_2 is maximal in $D[[X]$ and that $\dim D[[X]] = 2$.

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