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INTEGRABILITY THEOREMS FOR POWER SERIES EXPANSIONS OF TWO VARIABLES

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Let $f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}x^m y^n$ in the triangle $x + y \leqq 1$, $x, y \geqq 0$, or in the quarter-disk $x^2 + y^2 < 1$, $x, y \geqq 0$. This paper show some relations between L-integrability of $f(x,y)$, with certain multipliers, and the coefficients $a_{m,n}$.

1. DEFINITION. A real-valued function $f(x, y)$ is said to be harmonic in a domain D in R^2 if it is 2-times continuously differentiable in D and satisfies Laplace's equation

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{for any } (x, y) \in D\,.$$

Throughout the paper, the letter C, with or without a suffix, denotes a positive constant, not necessarily the same at each appearance.

Heywood [3] proved a result as follows:

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $0 \leqq x < 1$, that $\gamma < 1$, and that there are positive numbers ε, C such that $a_n \geqq -Cn^{-(\gamma+\varepsilon)}$ for all sufficiently large n. Then $(1 - x)^{-\gamma} f(x) \in L(0, 1)$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges absolutely.

We shall show two analogues of his result for power series expansions of two variables.

Kiselman [4] proved the following theorem.

THEOREM A. *If $f(x, y)$ is harmonic in the disk $x^2 + y^2 < r_0^2$ $(r_0 > 0)$, but not in any open disk of larger radius centred on the origin, then the power series expansion*

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n \tag{1}$$

converges absolutely in the square K: $|x| + |y| < r_0$, uniformly on every compact subset of K. It diverges at all points exterior to K for which $x \neq 0$, and $y \neq 0$.

Further, the following theorem is known (see [2, p. 189 and 200] and [4]).

THEOREM B. *Suppose that $f(x, y)$ is harmonic in the disk*

$$x^2 + y^2 < r_0^2\,,$$

and that $f(x, y)$ *has the power series expansion* (1) *in the square* K, *where* K *is defined as in Theorem A. Let* $P_N(x, y)$ *be defined by*

$$P_N(x, y) = \sum_{m+n=N} a_{m,n} x^m y^n \qquad (N = 0, 1, 2, \cdots) .$$

Then the polynomial expansion

$$f(x, y) = \sum_{N=0}^{\infty} P_N(x, y)$$

of $f(x, y)$ *converges uniformly and absolutely in* $x^2 + y^2 \leqq r^2$ *for any* $0 < r < r_0$, *where* $P_N(x, y)$ *are harmonic.*

We give the following four theorems.

THEOREM 1. *Suppose that a double power series* (1) *converges absolutely in the triangle*

$$(2) \qquad T:\ x + y < 1, \qquad x,\ y \geqq 0 ,$$

that $\gamma < 1$, *and that there are positive numbers* ε, C *such that*

$$(3) \qquad a_{m,n} \geqq -C(m + n + 1)^{m+n-\gamma-\varepsilon+1/2}(m + 1)^{-(m+1/2)}(n + 1)^{-(n+1/2)}$$

for all sufficiently large $m + n$. *Then* $(1 - x - y)^{-\gamma} f(x, y)$ *is Lebesgue-integrable on* T *if and only if*

$$(4) \qquad \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} a_{m,n}$$

converges absolutely.

THEOREM 2. *Suppose that* $f(x, y)$ *is harmonic in the quarter-disk*

$$(5) \qquad Q:\ x^2 + y^2 < 1 , \quad x,\ y \geqq 0 ,$$

and that $f(x, y)$ *has the power series expansion* (1) *in the triangle* T, *where* T *is defined by* (2). *Then, under the assumption* (3), *the function* $(1 - x - y)^{-\gamma} f(x, y)$, $\gamma < 1$, *is Lebesgue-integrable on* T *if and only if the series* (4) *converges absolutely.*

Theorem 2 is an obvious consequence of Theorem A ($r_0 = 1$) and Theorem 1, and so we omit the proof.

THEOREM 3. *Suppose that a double power series* (1) *converges absolutely in the quarter-disk* Q, *where* Q *is defined by* (5), *that* $\gamma < 1$, *and that there are positive numbers* ε, C *such that*

$$(6)\qquad a_{m,n} \geqq \begin{cases} -C(m+n+1)^{(m+n+1)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \qquad\qquad \times (n+1)^{-(n+1)/2} & (\text{even } m, n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \qquad\qquad \times (n+1)^{-(n+1)/2} & (\text{odd } m \text{ and even } n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \qquad\qquad \times (n+1)^{-n/2} & (\text{even } m \text{ and odd } n) \\ -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \qquad\qquad \times (n+1)^{-n/2} & (\text{odd } m, n) \end{cases}$$

for all sufficiently large $m + n$. *Then the function*

$$\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$$

is Lebesgue-integrable on Q *if and only if the series*

$$(7)\qquad \sum_{m,n=0}^{\infty} (m+n+1)^{-(m+n+3)/2+\gamma}(m+1)^{m/2}(n+1)^{n/2} a_{m,n}$$

converges absolutely.

REMARK 1. In Theorem 3, it is easily seen that (6) may be replaced by a stronger condition

$$a_{m,n} \geqq -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2}(n+1)^{-n/2} \qquad (m, n = 0, 1, 2, \cdots)$$

for all sufficiently large $m + n$.

THEOREM 4. *Suppose that* $f(x, y)$ *is harmonic in the quarter-disk* Q, *where* Q *is defined by* (5), *and that* $f(x, y)$ *has the power series expansion* (1) *in the triangle* T, *where* T *is defined by* (2). *Then, under the assumption* (6), *the function* $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$, $\gamma < 1$, *is Lebesgue-integrable on* Q *if and only if the series* (7) *converges absolutely.*

Theorem 4 is a consequence of Theorem B ($r_0 = 1$) and Theorem 3. In § 2, we shall prove Theorem 1 and give an example for Theorem 2. Further, in § 3, we shall prove Theorems 3 and 4.

2. *Proof of Theorem* 1. First, suppose that $(1 - x - y)^{-\gamma} f(x, y)$ is Lebesgue-integrable on T. Without loss of generality, we suppose that $\gamma + \varepsilon$ is a noninteger value < 1. For, we get

$$a_{m,n} \geqq -C(m+n+1)^{m+n-\gamma-\varepsilon'+1/2}(m+1)^{-(m+1/2)}(n+1)^{-(n+1/2)}$$

for $0 < \varepsilon' < \varepsilon$. We have, for any $(x, y) \in T$,

$$
\begin{aligned}
(1-x-y)^{\gamma+\varepsilon-1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\gamma-\varepsilon)}{\Gamma(N+1)\,\Gamma(1-\gamma-\varepsilon)}(x+y)^N \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\gamma-\varepsilon)}{\Gamma(N+1)} \\
&\quad \times \sum_{\substack{m+n=N \\ m,n\geqq 0}} \binom{m+n}{n} x^m y^n \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{m,n=0}^{\infty} \frac{\Gamma(m+n-\gamma-\varepsilon+1)}{\Gamma(m+1)\Gamma(n+1)} x^m y^n \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{m,n=0}^{\infty} b_{m,n} x^m y^n ,
\end{aligned} \tag{8}
$$

say, where $\Gamma(u)$ is the Gamma function. By Stirling's formula (see e.g. [1, p. 24])

$$
\Gamma(u) = \sqrt{2\pi}\, u^{u-1/2} e^{-u+\eta/12u} \qquad \text{for any } u > 0 ,
$$

where η is a number independent of u between 0 and 1, we obtain

$$
C_1 u^{u-1/2} e^{-u} \leqq \Gamma(u) \leqq C_2 u^{u-1/2} e^{-u} \qquad \text{for any } u \geqq u_0 \tag{9}
$$

if u_0 is a fixed positive number. Hence we get easily

$$
C_3 \lambda_{m,n} \leqq b_{m,n} \leqq C_4 \lambda_{m,n} \qquad \text{for all } m, n \geqq 0 , \tag{10}
$$

where

$$
\lambda_{m,n} = (m+n+1)^{m+n-\gamma-\varepsilon+1/2}(m+1)^{-(m+1/2)}(n+1)^{-(n+1/2)}
$$

(notice $u_0 \geqq \min(1-\gamma-\varepsilon, 1)$). Let

$$
g(x, y) = C_5 \Gamma(1-\gamma-\varepsilon)(1-x-y)^{\gamma+\varepsilon-1} , \qquad C_5 \geqq C/C_3 .
$$

Then, it is clear that $(1-x-y)^{-\gamma} g(x, y)$ is Lebesgue-integrable on T. Thus, by assumption,

$$
(1-x-y)^{-\gamma}\{f(x, y) + g(x, y)\} = (1-x-y)^{-\gamma} \times \sum_{m,n=0}^{\infty} (a_{m,n} + C_5 b_{m,n}) x^m y^n
$$

is Lebesgue-integrable on T. By (3) and (10), we heve

$$
a_{m,n} + C_5 b_{m,n} \geqq a_{m,n} + C\lambda_{m,n} \geqq 0
$$

for all sufficiently large $m+n$. Hence we get

$$
\begin{aligned}
&\iint_T (1-x-y)^{-\gamma} \left\{ \sum_{m,n=0}^{\infty} (a_{m,n} + C_5 b_{m,n}) x^m y^n \right\} dxdy \\
&\qquad = \sum_{m,n=0}^{\infty} (a_{m,n} + C_5 b_{m,n}) \iint_T (1-x-y)^{-\gamma} x^m y^n dxdy ,
\end{aligned} \tag{11}
$$

where the right-side series converges absolutely. Using the change of variable $x = (1 - y)u$, we have, for all $m, n \geqq 0$,

$$\begin{aligned}\iint_T &(1 - x - y)^{-\gamma} x^m y^n dxdy \\ &= \int_0^1 dy \int_0^{1-y} (1 - x - y)^{-\gamma} x^m y^n dx \\ &= \int_0^1 (1 - y)^{m+1-\gamma} y^n dy \int_0^1 (1 - u)^{-\gamma} u^m du \\ &= \frac{\Gamma(n + 1)\Gamma(m + 2 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{\Gamma(m + 1)\Gamma(1 - \gamma)}{\Gamma(m + 2 - \gamma)} \\ &= \Gamma(1 - \gamma) \cdot \frac{\Gamma(m + 1)\Gamma(n + 1)}{\Gamma(m + n + 3 - \gamma)} .\end{aligned}$$

Hence, from (9), we get

$$\begin{aligned}&C_6(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} \\ &\qquad \leqq \iint_T (1 - x - y)^{-\gamma} x^m y^n dxdy \\ &\qquad \leqq C_7(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2}\end{aligned} \tag{12}$$

for all $m, n \geqq 0$. Thus, by (11) and (12),

$$\sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2}(a_{m,n} + C_5 b_{m,n}) \tag{13}$$

converges absolutely. Further, from (10)

$$\begin{aligned}&\sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} b_{m,n} \\ &\qquad \leqq C_4 \sum_{m,n=0}^{\infty} (m + n + 1)^{-2-\varepsilon} < \infty .\end{aligned} \tag{14}$$

By (3) and (10), we get

$$| a_{m,n} | \leqq a_{m,n} + 2C\lambda_{m,n} \leqq a_{m,n} + 2C_5 b_{m,n} \qquad (C_5 \geqq C/C_3)$$

for all sufficiently large $m + n$. Hence, from (13) and (14), the series (4) converges absolutely.

Conversely we suppose that the series (4) converges absolutely, and will deduce that $(1 - x - y)^{-\gamma} f(x, y)$ is Lebesgue-integrable on T. For this part of the argument we do not assume (3). We have in fact

$$\begin{aligned}&\iint_T (1-x-y)^{-\gamma}\,|f(x,y)|\,dxdy\\&\quad\leqq \iint_T (1-x-y)^{-\gamma}\left\{\sum_{m,n=0}^{\infty}|a_{m,n}|\,x^m y^n\right\}dxdy\\&\quad= \sum_{m,n=0}^{\infty}|a_{m,n}|\iint_T (1-x-y)^{-\gamma}x^m y^n dxdy\\&\quad\leqq C_7\sum_{m,n=0}^{\infty}(m+n+1)^{-m-n+\gamma-5/2}(m+1)^{m+1/2}(n+1)^{n+1/2}\,|a_{m,n}|<\infty\end{aligned}$$

by (12). Thus Theorem 1 is proved.

EXAMPLE FOR THEOREM 2. Let

$$f(x,y)=\Re(1-z)^{-2}=\frac{(1-x)^2-y^2}{\{(1-x)^2+y^2\}^2}\quad (z=x+iy,\ i=\sqrt{-1}).$$

Then $f(x, y)$ is harmonic in the disk $x^2+y^2<1$. Since

$$f(x,y)=\Re\sum_{N=0}^{\infty}(N+1)z^N=\sum_{N=0}^{\infty}(N+1)\sum_{m+2n=N}(-1)^n\binom{m+2n}{2n}x^m y^{2n}$$

in the disk $x^2+y^2<1$, we get

$$f(x,y)=\sum_{m,n=0}^{\infty}(-1)^n\frac{\Gamma(m+2n+2)}{\Gamma(m+1)\Gamma(2n+1)}x^m y^{2n}$$

in the square $|x|+|y|<1$, by Theorem A. When $a_{m,n}$ denote the (m, n)th coefficients of this power series expansion, we have, from (9),

$$\begin{aligned}&C_1(m+2n+1)^{m+2n+3/2}(m+1)^{-(m+1/2)}(2n+1)^{-(2n+1/2)}\\&\qquad\leqq |a_{m,2n}|\leqq C_2(m+2n+1)^{m+2n+3/2}(m+1)^{-(m+1/2)}(2n+1)^{-(2n+1/2)}\end{aligned}$$

and $a_{m,2n+1}=0$. First we put $\gamma<-1$. Then the sequence $\{a_{m,n}\}$ satisfies (3) for $\varepsilon=-(\gamma+1)/2$. Now we have

$$\begin{aligned}&\iint_T (1-x-y)^{-\gamma}\,|f(x,y)|\,dxdy\\&\quad=\int_0^1(1-x)^{-\gamma-1}dx\int_0^1\frac{(1-u)^{-\gamma+1}(1+u)}{(1+u^2)^2}du<\infty\end{aligned}$$

by the change of variable $y=(1-x)u$. Further we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty}(m+n+1)^{-m-n+\gamma-5/2}(m+1)^{m+1/2}(n+1)^{n+1/2}\,|a_{m,n}|\\&\qquad\leqq C_2\sum_{m,n=0}^{\infty}(m+n+1)^{\gamma-1}<\infty\,.\end{aligned}$$

Next we set $\gamma=-1$. Then $\{a_{m,n}\}$ does not satisfy (3), but we notice $\varepsilon=0$. It is clear that

$$\iint_T (1-x-y)\,|f(x,y)|\,dxdy = \int_0^1 \frac{(1-u)^2(1+u)}{(1+u^2)^2}\,du < \infty .$$

But we get

$$\sum_{m,n=0}^{\infty} (m+n+1)^{-m-n-7/2}(m+1)^{m+1/2}(n+1)^{n+1/2}\,|a_{m,n}|$$
$$\geqq C_1 \sum_{m,n=0}^{\infty} (m+2n+1)^{-2} > \frac{C_1}{4} \sum_{m,n=0}^{\infty} (m+n+1)^{-2} = \infty .$$

Thus this example ($\gamma = -1$) show that we cannot set $\varepsilon = 0$ in (3) without destroying the validity of Theorem 2.

3. In order to prove Theorem 3, we need the following lemma.

LEMMA. *Suppose that $\mu < 1$, and that $A(x, y)$ is defined by*

$$A(x,y) = (1+x+y+xy)(1-x^2-y^2)^{\mu-1}$$

in the quarter-disk Q, where Q is defined by (5). *Then $A(x, y)$ has the power series expansion*

$$A(x,y) = \sum_{m,n=0}^{\infty} d_{m,n}x^m y^n , \quad C_1\delta_{m,n} \leqq d_{m,n} \leqq C_2\delta_{m,n} \quad (C_1, C_2 > 0) \tag{15}$$

in Q, where

$$\delta_{m,n} = \begin{cases} (m+n+1)^{(m+n+1)/2-\mu}(m+1)^{-(m+1)/2} \\ \qquad \times (n+1)^{-(n+1)/2} & (\textit{even } m, n) \\ (m+n+1)^{(m+n)/2-\mu}(m+1)^{-m/2} \\ \qquad \times (n+1)^{-(n+1)/2} & (\textit{odd } m \textit{ and even } n) \\ (m+n+1)^{(m+n)/2-\mu}(m+1)^{-(m+1)/2} \\ \qquad \times (n+1)^{-n/2} & (\textit{even } m \textit{ and odd } n) \\ (m+n+1)^{(m+n-1)/2-\mu}(m+1)^{-m/2} \\ \qquad \times (n+1)^{-n/2} & (\textit{odd } m, n) . \end{cases}$$

Proof. We have, for any $(x, y) \in Q$,

$$\begin{aligned}(1-x^2-y^2)^{\mu-1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\mu)}{\Gamma(N+1)\Gamma(1-\mu)}(x^2+y^2)^N \\ &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\mu)}{\Gamma(N+1)\Gamma(1-\mu)} \sum_{\substack{m+n=N \\ m,n\geqq 0}} \binom{m+n}{m} x^{2m}y^{2n} \\ &= \sum_{m,n=0}^{\infty} \frac{1}{\Gamma(1-\mu)} \cdot \frac{\Gamma(m+n+1-\mu)}{\Gamma(m+1)\Gamma(n+1)} x^{2m}y^{2n} \\ &= \sum_{m,n=0}^{\infty} p_{m,n}x^{2m}y^{2n} ,\end{aligned}$$

say. Then we get

$$A(x, y) = \sum_{m,n=0}^{\infty} p_{m,n}(x^{2m}y^{2n} + x^{2m+1}y^{2n} + x^{2m}y^{2n+1} + x^{2m+1}y^{2n+1}) . \tag{16}$$

We put

$$d_{m,n} = \begin{cases} p_{m/2,n/2} & \text{(even } m,\ n) \\ p_{(m-1)/2,n/2} & \text{(odd } m \text{ and even } n) \\ p_{m/2,(n-1)/2} & \text{(even } m \text{ and odd } n) \\ p_{(m-1)/2,(n-1)/2} & \text{(odd } m,\ n) . \end{cases}$$

Now, from (16) and (9), we get easily (15). Thus the Lemma is proved.

Proof of Theorem 3. First, suppose that $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$ is Lebesgue-integrable on Q. Without loss of generality, we may suppose that $\gamma + \varepsilon$ is a noninteger < 1. Let

$$h(x, y) = (1 + x + y + xy)(1 - x^2 - y^2)^{\gamma+\varepsilon-1} \tag{17}$$

in Q. Then, by the Lemma ($\mu = \gamma + \varepsilon$), we have

$$h(x, y) = \sum_{m,n=0}^{\infty} k_{m,n}x^m y^n , \qquad C_1\theta_{m,n} \leqq k_{m,n} \leqq C_2\theta_{m,n} \tag{18}$$

in Q, where $k_{m,n}$ and $\theta_{m,n}$ are defined respectively like $d_{m,n}$ and $\delta_{m,n}$ in the Lemma with $\mu = \gamma + \varepsilon$. Clearly, the function

$$\begin{aligned} &\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} h(x, y) \\ &= (1 + x + y + xy)\,\{1 + (x^2 + y^2)^{1/2}\}^{\gamma+\varepsilon-1}\,\{1 - (x^2 + y^2)^{1/2}\}^{\varepsilon-1} \end{aligned}$$

is Lebesgue-integrable on Q. Hence, by assumption, the function

$$\begin{aligned} &\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma}\,\{f(x, y) + C_3 h(x, y)\} \\ &\qquad = \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n})x^m y^n \end{aligned}$$

is Lebesgue-integrable on Q, where $C_3 \geqq C/C_1$. Further, by (6) and (18), we have

$$a_{m,n} + C_3 k_{m,n} \geqq a_{m,n} + C\theta_{m,n} \geqq 0 \tag{19}$$

for all sufficiently large $m + n$. Thus we get

$$\begin{aligned} &\iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \left\{\sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n})x^m y^n\right\} dxdy \\ &\qquad = \sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n}) \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dxdy , \end{aligned} \tag{20}$$

where the right-side series converges absolutely. By the change of variables

$$x = r \cos v\,, \qquad y = r \sin v \qquad (0 \leqq r < 1,\ 0 \leqq v \leqq \pi/2)\,,$$

we get

$$\begin{aligned}\iint_Q &\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dxdy \\ &= \int_0^1 (1 - r)^{-\gamma} r^{m+n+1} dr \int_0^{\pi/2} \sin^m v \ \cos^n v \ dv \\ &= \frac{\Gamma(m + n + 2)\Gamma(1 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{1}{2} \cdot \frac{\Gamma((m + 1)/2)\Gamma((n + 1)/2)}{\Gamma((m + n)/2 + 1)}\,.\end{aligned}$$

Thus, from (9), we get

$$\begin{aligned}&C_4(m + n + 1)^{-(m+n+3)/2+\gamma}(m + 1)^{m/2}(n + 1)^{n/2} \\ &\qquad \leqq \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dxdy \\ &\qquad \leqq C_5(m + n + 1)^{-(m+n+3)/2+\gamma}(m + 1)^{m/2}(n + 1)^{n/2}\end{aligned} \tag{21}$$

for all $m, n \geqq 0$. Hence, by (20),

$$\sum_{m,n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma}(m + 1)^{m/2}(n + 1)^{n/2}(a_{m,n} + C_3 k_{m,n}) \tag{22}$$

converges absolutely. Further, by (18), we have

$$\begin{aligned}&\sum_{m,n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma}(m + 1)^{m/2}(n + 1)^{n/2} k_{m,n} \\ &\qquad \leqq C_2 \sum_{m,n=0}^{\infty} \{(m + n + 1)^{-1-\varepsilon}(m + 1)^{-1/2}(n + 1)^{-1/2} \\ &\qquad + (m + n + 1)^{-3/2-\varepsilon}(n + 1)^{-1/2} + (m + n + 1)^{-3/2-\varepsilon}(m + 1)^{-1/2} \\ &\qquad + (m + n + 1)^{-2-\varepsilon}\} < \infty\,.\end{aligned} \tag{23}$$

By (6) and (18), we get

$$|a_{m,n}| \leqq a_{m,n} + 2C\theta_{m,n} \leqq a_{m,n} + 2C_3 k_{m,n} \qquad (C_3 \geqq C/C_1)$$

for all sufficiently large $m + n$. Hence, from (22) and (23), the series (7) converges absolutely.

Conversely we suppose that series (7) converges absolutely, and will deduce that $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$ is Lebesgue-integrable on Q. For this part of the argument we do not assume (6). We have in fact

$$\iint_Q \{1-(x^2+y^2)^{1/2}\}^{-\gamma} \mid f(x, y) \mid dxdy$$
$$\leqq \iint_Q \{1-(x^2+y^2)^{1/2}\}^{-\gamma} \left\{\sum_{m,n=0}^{\infty} \mid a_{m,n} \mid x^m y^n\right\} dxdy$$
$$= \sum_{m,n=0}^{\infty} \mid a_{m,n} \mid \iint_Q \{1-(x^2+y^2)^{1/2}\}^{-\gamma} x^m y^n dxdy$$
$$\leqq C_5 \sum_{m,n=0}^{\infty} (m+n+1)^{-(m+n+3)/2+\gamma}(m+1)^{m/2}(n+1)^{n/2} \mid a_{m,n} \mid < \infty$$

by (21). Thus Theorem 3 is proved.

REMARK 2. From (17), it is easily seen that

$$C_1 h(x, y) \leqq \{1-(x^2+y^2)^{1/2}\}^{\gamma+\varepsilon-1} \leqq C_2 h\,(x, y)$$

in Q.

Proof of Theorem 4. By Theorem B ($r_0 = 1$), we get

$$f(x, y) = \sum_{N=0}^{\infty} \sum_{m+n=N} a_{m,n} x^m y^n$$

in Q. We define $h(x, y)$ by (17). Then it is sufficient for us to notice that

$$f(x, y) + C_3 h(x, y) = \sum_{N=0}^{\infty} \sum_{m+n=R} a_{m,n} x^m y^n + C_3 \sum_{m,n=0}^{\infty} k_{m,n} x^m y^n$$
$$= \sum_{N=0}^{\infty} \sum_{m+n=N} (a_{m,n} + C_3 k_{m,n}) x^m y^n$$
$$= \sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n}) x^m y^n$$

in Q, in view of (18) and (19), where the last right-side series converges absolutely. Thus Theorem 4 is a consequence of Theorem 3.

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