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SIMPLE EXTENSIONS OF MEASURES AND THE PRESERVATION OF REGULARITY OF CONDITIONAL PROBABILITIES

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Throughout this paper, the following notation will be adopted. $(\Omega, \mathfrak{A}, P)$ will be a probability space with $\mathfrak{B}$ a sub σ-field of $\mathfrak{A}$. H will denote a subset of Ω not in $\mathfrak{A}$ and $\mathfrak{A}'$ will be the σ-field generated by $\mathfrak{A}$ and H. P_e will be a simple extension of P to $\mathfrak{A}'$ if P_e is a probability measure on $\mathfrak{A}'$ with $P_e|_{\mathfrak{A}} = P$.

The ability to extend the regularity of the conditional probability $P^{\mathfrak{B}}$ to regularity of $P_e^{\mathfrak{B}}$ has been explored earlier for canonical extensions of measures. The main results of this paper are:

(a) If $P_c^{\mathfrak{B}}$ is regular for some canonical extension P_c of P to $\mathfrak{A}'$, then $P_e^{\mathfrak{B}}$ is regular for any simple extension P_e of P to $\mathfrak{A}'$.

(b) For some choice of $(\Omega, \mathfrak{A}, P)$, $\mathfrak{B}$ and H, $P^{\mathfrak{B}}$ is regular but for *no* P_e is $P_e^{\mathfrak{B}}$ regular. This will essentially extend the Dicudonné example,

Our notation regarding (regular) conditional probabilities will be consistent with [1].

For extendability see [4]. The example for (b) occurs in [2].

PROPOSITION 1. *Any simple extension P_e can be expressed as the sum of a canonical extension of P plus a finite signed measure on $\mathfrak{A}$. (Since the construction is carried out in a unique manner, this decomposition of P_e will be called the canonical decomposition of P_e.)*

Proof. As in [1], let K be a set which extends P canonically to $\mathfrak{A}'$. For any $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$ write

$$P_e(A') = P(A'K^c) + P_e(A_1HK) + P_e(A_2H^cK) .$$

It may be supposed that $P(K) \neq 0$. Thus, let $\alpha_\Omega \equiv P_e(HK)/P(K)$ and define a set function ε on $\mathfrak{A}$ such that for every $A \in \mathfrak{A}$

$$\varepsilon(A) = P_e(AHK) - \alpha_\Omega P(AK) .$$

It is immediate that ε is a finite signed measure. It also follows that for any $A \in \mathfrak{A}$

$$P_e(AH^cK) = \beta_\Omega P(AK) - \varepsilon(A)$$

where $\beta_\Omega \equiv 1 - \alpha_\Omega$ inasmuch as it can be written that

$$P(A) = P_e(A) = P_e(AH + AH^c) = \\ P(AK^c) + \alpha_\Omega P(AK) + \varepsilon(A) + P_e(AH^cK) .$$

Thus, for $A' \in \mathfrak{A}'$

$$P_e(A') = \underline{P(A'K^c)} + \underline{\alpha_\Omega P(A_1K)} + \varepsilon(A_1) \\ + \underline{\beta_\Omega P(A_2K)} - \varepsilon(A_2) .$$

(Let the sum of the underlined measures be called the *canonical part* of P_e.)

It is clear that the extension, P_e, of Proposition 1 is canonical if and only if the signed measure ε is identically zero.

LEMMA 2. *The signed measure ε is absolutely continuous with respect to P.*

Proof. Let $B \in \mathfrak{A}$ be a positive set for ε according to its Jordan decomposition and let $A \in \mathfrak{A}$ with $P(A) = 0$. Then,

$$P_e(ABHK) \leqq P(ABK) \leqq P(A) = 0 \tag{2.1}$$

and so $\varepsilon(AB) = 0$. If $C(=B^c)$ is a negative set for ε then it follows that $\varepsilon(AC) = 0$ where one merely inserts C for B in (2.1). Hence $\varepsilon \ll P$.

LEMMA 3. *If $\Omega_0 \in \mathfrak{A}$ with $P(\Omega_0) = 1$ then $\varepsilon(\Omega_0) = 0$.*

Proof. Immediate.

The following lemma is needed before the main result can be presented.

LEMMA 4. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B} \subset \mathfrak{A}$. Let P_0 be another measure on $\mathfrak{A}$ with $P = P_0$ on $\mathfrak{B}$ and $P \ll P_0$. Suppose $P_0^{\mathfrak{B}}$ is regular. Then, $P^{\mathfrak{B}}$ is regular.*

Proof. Let $p_0(\cdot, \cdot \mid \mathfrak{B})$ be a version of $P_0^{\mathfrak{B}}$ such that $p_0(\omega, \cdot \mid \mathfrak{B})$ is a measure ($P_0\mid_{\mathfrak{B}}$ a.e.). Also, let $X = dP/dP_0$ where for all $A \in \mathfrak{A}$

$$P(A) = \int_A X\, dP_0 .$$

Hence, define

$$h(\omega, A) = \int_A X(\omega') p_0(\omega, d\omega' \mid \mathfrak{B}) . \tag{4.1}$$

From (4.1) it is immediate that $h(\cdot, A)$ is $\mathfrak{B}$-measurable for every $A \in \mathfrak{A}$ and for fixed $\omega \in \Omega$, $h(\omega, \cdot)$ is a measure on $\mathfrak{A}$. It remains to show that for any $B \in \mathfrak{B}$

$$\int_B h(\omega, A)P(d\omega) = P(AB) .$$

To show this, begin by establishing that

$$X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot \mid \mathfrak{B}))\ P_0|_{\mathfrak{B}} \text{ a.e.}$$

This follows at once by observing that

$$\int_\Omega X(\omega')p(\omega, d\omega' \mid \mathfrak{B}) = (E^{\mathfrak{B}}X)(\omega)$$

and

$$\int_\Omega (E^{\mathfrak{B}}X)(\omega)P_0(d\omega) = \int_\Omega X(\omega)P_0(d\omega) = 1 .$$

Next, write

$$X = \lim_{n\to\infty} X_n \quad \text{where} \quad X_n = \sum_{k=1}^{m_n} \zeta_{k,n}(\Psi A_{k,n}) \quad \text{where}$$

$\zeta_{k,n}$ is a real constant, $(\Psi A_{k,n})$ is the characteristic function of $A_{k,n} \in \mathfrak{A}$ and $\{X_n\}_{n\geq 1}$ is an increasing sequence.

Finally, since $X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot \mid \mathfrak{B}))\ P_0|_{\mathfrak{B}}$ a.e., the monotone convergence theorem can be used on the following chain of equalities to give the desired result:

$$\begin{aligned}
\int_B h(\omega', A)P(d\omega') &= \int_B \left\{\int_A X(\omega)p_0(\omega', d\omega \mid \mathfrak{B})\right\} P(d\omega') \\
&= \int_B \left\{\lim_{n\to\infty} \sum_{k=1}^{m_n} \zeta_{k,n}p_0(\omega', A_{k,n}A \mid \mathfrak{B})\right\} P(d\omega') \\
&= \lim_{n\to\infty} \left\{\int_B \sum_{k=1}^{m_n} \zeta_{k,n}\, p_0(\omega', AA_{k,n} \mid \mathfrak{B})\right\} P(d\omega') \\
&\qquad\qquad\qquad\qquad (\text{since } P = P_0 \text{ on } \mathfrak{B}) \\
&= \lim_{n\to\infty} \left\{\int_B \sum_{k=1}^{m_n} \zeta_{k,n}p_0(\omega', AA_{k,n} \mid \mathfrak{B})\right\} P_0(d\omega') \\
&= \lim_{n\to\infty} \left\{\sum_{k=1}^{m_n} \zeta_{k,n}P_0(AA_{k,n}B)\right\} \\
&= \lim_{n\to\infty} \left\{\int_{AB} \sum_{k=1}^{m_n} \zeta_{k,n}(\Psi A_{k,n})(\omega)P_0(d\omega)\right\} \\
&= \int_{AB} X(\omega)P_0(d\omega) = P(AB) .
\end{aligned}$$

Lemma 4 gives immediately

THEOREM 5. *Let* $(\Omega, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, *and* $\mathfrak{A}'$ *be given. Let* P_e *be any simple extension of* P *to* $\mathfrak{A}'$. *Let* $P^{\mathfrak{B}}$ *be regular. A sufficient condition that* $P_e^{\mathfrak{B}}$ *be regular is that* $P_c^{\mathfrak{B}}$ *be regular where* P_c *is the canonical part of* P_e. (*Let* K *be the set which extends* P *canonically to* $\mathfrak{A}'$ *as in* [1].)

Proof. It is immediate that $P_c|_{\mathfrak{B}} = P = P_e|_{\mathfrak{B}}$. Thus the proof will be complete by Lemma 4 if it can be shown that $P_e \ll P_c$. To do so, suppose $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. If $P_c(A') = 0$, it follows that $P(A_1K) = P(A_2K) = 0$. Thus

$$\varepsilon(A_1K) = \varepsilon(A_2K) = 0$$

by Lemma 2. But, by Proposition 1 it follows that $\varepsilon(A) = \varepsilon(AK)$ for all $A \in \mathfrak{A}$; hence $\varepsilon(A_1) = \varepsilon(A_2) = 0$ and thus $P_e(A') = 0$.

COROLLARY 6. *With the notation of Theorem* 5, *assume* $P_c^{\mathfrak{B}}$ *is regular with* $0 < \alpha_\Omega < 1$. *Let* $P_{c'}$ *be any other canonical extension of* P *to* $\mathfrak{A}'$, *then* $P_{c'}^{\mathfrak{B}}$ *is regular.*

Proof. $P_{c'} \ll P_c$ and the proof is complete by Lemma 4.

The representation of an arbitrary simple extension as constructed in Proposition 1 helps establish the following interesting

PROPOSITION 7. *Let* $(\Omega, \mathfrak{A}, P)$ *be given with* $\mathfrak{A}$ *countably generated and* $\{\omega\} \in \mathfrak{A}$ *for all* $\omega \in \Omega$. *Suppose* $H \notin \mathfrak{A}$ *with* $P_*(H) = 0$ *and* $P^*(H) = 1$. *Then there exists no simple extension* P_e *of* P *to* $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$ *such that* $P_e^{\mathfrak{A}}$ *is regular.*

Proof. With H so chosen, it follows that the set K associated with the canonical part of P_e has P-measure one.

By Proposition 1 write

$$P_e(A') = \alpha_\Omega P(A_1K) + \varepsilon(A_1) + \beta_\Omega P(A_2K) - \varepsilon(A_2)$$

for any $A' \in \mathfrak{A}'$ with $A' \in A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. It may be assumed that $0 < \alpha_\Omega < 1$; otherwise, P_e would be canonical (see [1]) and the result would follow directly as in [3], p. 210.

Suppose there exists a version of $P_e^{\mathfrak{A}}$, $p_e(\cdot\,, \cdot\,|\,\mathfrak{A})$, such that $p_e(\omega, \cdot\,|\,\mathfrak{A})$ is a measure on $\mathfrak{A}'$. Define

$$B \equiv \{\omega \mid p_e(\omega, H \mid \mathfrak{A}) = 0\}\ .$$

It follows that $P(B) < 1$, otherwise write

$$0 = \int_B p_e(\omega, H \mid \mathfrak{A}) P_e(d\omega) = P_e(BH) = \alpha_\Omega P(BK) + \varepsilon(B)$$
$$= \alpha_\Omega P(B) + \varepsilon(B) = \alpha_\Omega ,$$

where $P(B) = 1$ and $\varepsilon(B) = 0$ by Lemma 3, and get $\alpha_\Omega = 0$, a contradiction.

Define a set E where E is the set of points ω for which it is not true that $p_e(\omega, D \mid \mathfrak{A}) = (\psi D)(\omega)$ identically for all $D \in \mathfrak{A}$ (where ψD is the characteristic function of D). Since $\mathfrak{A}$ is countably generated, $P(E) = 0$ (see [3, p. 210]).

It then follows that $(E \cup B)^c \subset H$. Suppose otherwise; that is, $\omega \in (E \cup B)^c$ and $\omega \in H^c$ and get

$$\begin{aligned} p_e(\omega, \{\omega\} \cup H \mid \mathfrak{A}) &= p_e(\omega, \{\omega\} \mid \mathfrak{A}) + p_e(\omega, H \mid \mathfrak{A}) \\ &= (\psi\{\omega\})(\omega) + p_e(\omega, H \mid \mathfrak{A}) > 1 , \end{aligned}$$

a contradiction.

But $P((E \cup B)^c) > 0$ and $(E \cup B)^c \subset H$. This contradicts construction of H and so $P_e^{\mathfrak{A}}$ cannot be regular for any simple extension of P to $\mathfrak{A}'$.

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