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ATTAINING THE SPREAD AT CARDINALS WHICH ARE NOT STRONG LIMITS

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It is shown to be consistent with set theory that there is a cardinal κ and a Hausdorff space X such that $\mathrm{cf}(\kappa) > \omega$ and $\mathrm{sp}(X) = \kappa$ and X contains no discrete subspace of cardinality κ; also, if X is a Hausdorff space such that $\mathrm{cf}(\mathrm{sp}(X) = \omega$ and X does not attain its spread, then X contains a subspace of a certain canonical form with the same spread.

1. Preliminaries. The spread of a topological space X, abbreviated as $\mathrm{sp}(X)$ is defined as a supremum of cardinalities of certain subspaces:

$$\mathrm{sp}(X) = \sup\{|Y|: Y \text{ is a discrete subspace of } X\}$$

where $|Y|$ is the cardinality of Y. For brevity, we say that spread must be attained at κ iff every Hausdorff space X for which $\mathrm{sp}(X) = \kappa$ has a discrete subspace of cardinality κ. A natural question to ask, then, is whether spread must be attained at every cardinal κ. The answer is clearly yes if κ is a successor cardinal. If κ is a limit cardinal, it is trivial to construct spaces which are, say, T_1 but not T_2 which have a spread of κ but no discrete subspace of cardinality κ, thus necessitating the word "Hausdorff" in our definition of attaining the spread. Juhasz and Hajnal have found classes of limit cardinals at which spread is attained; they also have a class of spaces for which, if the spread has cofinality ω, then the space has a discrete subspace of the cardinality of the spread (we say that the space attains its spread). Here we look for counterexamples: it is found consistent with the axioms of set theory to have Hausdorff spaces of uncountable cofinality which do not attain their spread; in the case of countable cofinality, it is shown that a Hausdorff space which does not attain its spread contains a space of a certain canonical form which has the same spread.

Notation and conventions. Lower case Greek letters are reserved for ordinals, which may or may not be cardinals; κ is reserved for cardinals, which are assumed to be initial ordinals.

We remind the reader of some basic concepts about ordinals.

DEFINITION 1. $\mathrm{cf}(\alpha) = \beta$ iff β is the least ordinal such that for some function $f\colon \beta \to \alpha$, f is increasing and $\sup(\text{range } f) = \alpha$. $\mathrm{cf}(\alpha)$ is the *cofinality* of α.

DEFINITION 2. α is *regular* iff $\alpha = \mathrm{cf}(\alpha)$. α is *singular* otherwise. We note that regular ordinals are always cardinals.

DEFINITION 3. κ is a *limit cardinal* iff for every cardinal τ, $\kappa \neq \tau^+$. κ is a *strong limit cardinal* iff for every cardinal $\tau < \kappa$, $2^\tau < \kappa$.

DEFINITION 4. κ is *weakly inaccessible* iff κ is a regular limit cardinal. κ is *strongly inaccessible* iff κ is a regular strong limit cardinal.

Since our purpose is to find counterexamples or describe what they must look like if we could find them, it would be useful to know where not to look. The following result of Hajnal and Juhasz tells us, and also insures that a counterexample must be a consistency result, i.e. it cannot exist in all models of set theory.

THEOREM 5. (Juhasz, Hajnal [2 and 3]) *If κ is a weakly compact or a singular strong limit cardinal, and X is a Hausdorff space of cardinality κ, then X has a discrete subspace of cardinality κ.*

Thus in model of GCH spread is attained at singular cardinals. In fact in the constructible universe L all our questions about spread are settled, since a cardinal κ of L which is a regular limit cardinal and not weakly compact has a κ-Suslin line, and this line has spread κ which is not attained (see Juhasz [6]). Weakly compact cardinals play no further role in this paper, and the curious reader is referred to Juhasz [6] for a definition.

The results in this paper were originally proved in longer proofs using combinatorics in an inelegant fashion. The author is grateful to Ken Kunen and Istvan Juhasz for pointing out how they could be shortened; she also thanks the referee for helpful comments on organization.

2. The case $\mathrm{cf}(\kappa) > \omega$. Theorem 5 tells us that in order to not attain the spread at κ, GCH must be violated below κ in a strong fashion. Theorem 6 will set up machinery for constructing spaces which do not attain their spread from spaces of small spread and large cardinality. Corollary 7 will show that there is a class of cardinals for which this construction works which is large in the sense that Easton forcing makes it easy for us to find models of

set theory in which this class is cofinal in the class of all ordinals. Corollary 8 will point to a subclass of the class of Corollary 7 whose consistency follows from any large cardinal axiom. Corollary 9 will connect the machinery to the existence of large spaces with small width, which has been shown to be consistent by Hajnal and Juhasz [5].

THEOREM 6. *Let κ be a limit cardinal of uncountable cofinality, and suppose there exists a Hausdorff space $\langle X, \mathscr{T}\rangle$, $|X| = \kappa$, such that the spread of $\langle X, \mathscr{T}\rangle$ is less than the cofinality of κ. Then there is a finer topology $\mathscr{T}'$ on X such that $\langle X, \mathscr{T}'\rangle$ has a spread of κ which is not attained.*

Proof. Let $\xi = \mathrm{cf}(\kappa)$. We may write the set X then as the disjoint sum $X = \sum_{\alpha<\xi} X_\alpha$, where $|X_\alpha| = \kappa_\alpha$ for every $\alpha < \xi$, $\kappa = \sup\{\kappa_\alpha: \alpha < \xi\}$, and if $\alpha < \beta < \xi$ then $\kappa_\alpha < \kappa_\beta$. Let $\mathscr{T}'$ be the topology on X derived from sub-basic sets of the following form:
Suppose $x \in X$. Then for some unique α, $x \in X_\alpha$.

Let $u \in \mathscr{T}$, $x \in u$. Then $\{x\} \cup (u - X_\alpha) \in \mathscr{T}'$. Since each $\langle W_\alpha, \mathscr{T}'\rangle$ is discrete, $\langle X, \mathscr{T}'\rangle$ has spread κ. $\langle X, \mathscr{T}'\rangle$ is Hausdorff because $\langle X, \mathscr{T}\rangle$ is. We need to show that $\langle X, \mathscr{T}'\rangle$ does not attain its spread.

Proceeding by contradiction, suppose Y is a discrete subspace of $\langle X, \mathscr{T}'\rangle$ of cardinality κ. Then since $\mathrm{cf}(\kappa) = \xi$, there is some $Y' \subset Y$ with $|Y'| = \xi$ and for every $\alpha < \xi$, $|Y' \cap X_\alpha| \leqq 1$. If Y is discrete, so is Y'. Let $U = \{u_x: x \in Y'\}$ be a subset of $\mathscr{T}'$ where $x \in u_x$ and if $y \neq x$ then $u_x \cap u_y \cap Y' = \varnothing$. We may assume each u_x is a basic open set, hence of the form

$$u_x = Z_x \cup \left(u_x^* - \bigcup_{\alpha \in A_x} X_\alpha\right)$$

where A_x is a finite subset of ξ, Z_x a finite subset of $\bigcup_{\alpha \in A_x} X_\alpha$, $x \in Z_x \cap u_x^*$, and $u_x^* \in \mathscr{T}$. Hence $Y' \subset u_x^* \cap Z_x$ for every $x \in Y'$

But then each $u_x^* \cap Y'$ is finite, and since Y' has a cover by sets in $\mathscr{T}$ which are finite when relativized to Y', it is easily seen that Y' is a discrete subspace of $\langle X, \mathscr{T}\rangle$ of cardinality, ξ. But this contradicts the hypothesis that $\xi > \mathrm{sp}\,\langle X, \mathscr{T}\rangle$ and the proof of Theorem 6 is complete.

COROLLARY 7. *Let κ be a limit cardinal of uncountable cofinality, and suppose there is some $\tau < \mathrm{cf}(\kappa)$ such that $2^\tau > \kappa$. Then spread* *is not attained at* κ.

Proof. The set of functions 2^τ under the product topology has a basis of cardinality τ, since the set of functions into 2 whose domains are finite subsets of τ is isomorphic to a basis. But the spread of a space cannot be larger than the cardinality of some basis, so any subspace of 2^τ has spread $\leqq \tau$. In particular, any $X \subset 2^\tau$ where $|X| = \kappa$ can be used in the hypothesis of Theorem 6.

Now Easton forcing (1) gives us a technique for the following: let M be a transitive model of ZFC + GCH and let F be a nondecreasing function whose domain is the cardinals of M such that if κ is a regular cardinal of M then $\mathrm{cf}\,(F(\kappa)) > \kappa$. Then there is a model of set theory, N, which has the same cardinals as M, where cardinals have the same cofinalities they had in M, and in which if κ is regular they $2^\kappa = F(\kappa)$. Using this technique it is easy to get models of set theory in which the class of cardinals satisfying Corollary 7 is cofinal in the class of cardinals of the model.

COROLLARY 8. *Let* κ *be weakly inaccessible but not strongly inaccessible. Then the spread need not be attained at* κ.

Proof. Then the hypothesis of Corollary 7 is satisfied.

For a last example of cardinals to which Theorem 6 applies, we look at a model of set theory due to Juhasz and Hajnal and found in [5]. Here a forcing argument is used over a model M in which $2^\kappa = \kappa^+$ to make a model with the same cardinals and cofinalities in which 2^κ is still κ^+ but $2^{(\kappa^+)}$ is now "as large as you want" and there is a hereditarily κ-separable Hausdorff space (equivalently a space of width κ) of cardinality $2^{(\kappa^+)}$. In particular, we may make $2^{(\kappa^+)} > \omega_{\kappa^+}$. Since a width of κ implies a spread which is $\leqq \kappa$, this model justifies the conclusion of

COROLLARY 9. *It is consistent with the axioms of set theory to have a cardinal* κ *such that* $2^\kappa = \kappa^+$ *and spread is not attained at* ω_{κ^+}.

An explicit examination, not performed here, of each of the spaces of Corollaries 7 and 9 shows that none of them is regular. It will also be noted that in these corollaries a cardinal bounded by the cofinality of κ has the large power set necessary to avoid Theorem 5. So the following open questions remain:

Must spread be attained in the class of regular spaces?

What happens when κ is not a strong limit but $2^\tau < \kappa$ for all $\tau < \mathrm{cf}(\kappa)$?

3. The case $\mathrm{cf}(\kappa) = \omega$. Here we do not have a counterexample, but if one exists we know what it must look like.

THEOREM 10. *Suppose X is a Hausdorff space whose spread has cofinality ω. Then X contains a downward subspace with the same spread.*

Theorem 10 tells us that if we want a counterexample to attaining a spread of countable cofinality we need only look at the class of downward spaces, which we define forthwith.

DEFINITION 11. Let X be a topological space. Then X is *downward* iff X is set-theoretically the disjoint sum $\sum_{n\in\omega} X_n$ where each X_n is a discrete subspace and for every $m \in \omega$, $\bigcup_{n<m} X_n$ is open.

The proof of Theorem 10 relies on a combinatorial theorem of Hajnal, which we state as

THEOREM 12. (Hajnal) *Let f be a function mapping a set X into its power set and such that for some cardinal $\tau < |X|$, $|f(x)| < \tau$ for every $x \in X$. Then there is a set $Y \subseteq X$, $|Y| = |X|$, such that if x, y are elements of Y then $x \notin f(y)$. Then set Y is called a free set for f.*

It is clear that if f is a function taking each element of X into an open neighborhood, then the set Y which is free for f is also discrete in the topology for X. The proof of Theorem 12 can be found in Juhasz [6].

Proof of Theorem 10. Let X be a Hausdorff space whose spread has cofinality ω, $\mathrm{sp}(X) = \kappa$, and suppose κ is the limit of the strictly increasing sequence κ_n, $n \in \omega$. Then X contains disjoint discrete subspaces X_n where each X_n has cardinality κ_n, so without loss of generality we assume X is the union of these X_n's. There are three cases to consider.

Case 1. Every open set in X has cardinality κ. Then since X is Hausdorff it has a countable infinite family of disjoint open sets, call them u_n, $n \in \omega$. Then each u_n, having cardinality κ, contains at least κ_n elements from some X_{m_n}, say $u_n \cap X_{m_n} = Y_n$. Then

$Y = \bigcup_{n \in \omega} Y_n$ is discrete, and hence trivially downward.

Case 2. For some $\tau < \kappa$, every point in X has a neighborhood of cardinality strictly less than τ. Then we let f be the map taking each point in X into such a small neighborhood. By Theorem 12 we then have a free set Y for f of cardinality κ, which as we have noted is a discrete subspace and hence downward.

Case 3. For every $n < \omega \mid \{x \in X: x$ has a neighborhood of cardinality $\geqq \kappa_n\} \mid = \kappa$. By Case 1 we may without loss of generality assume that no point in X has a neighborhood of cardinality κ. We now may proceed to construct a downward space by induction.

Assume for $i < n$ we have discrete spaces $Y_i \subset X$ such that if $i < j < n$ then each point of Y_i has a neighborhood whose intersection with Y_j is empty, that each Y_i has cardinality κ_i, and for each $i < n$ there is some m_i such that each point of Y_i has a neighborhood in X of cardinality $\leqq \kappa_{m_i}$, say to the point y we have assigned the small neighborhood u_y. Let $Z_n = \bigcup_{i<n} \bigcup_{y \in Y_i} u_y$. Since $|Z_n| < \kappa$ there is some m_n such that $\{x \in X - Z_n: x$ has a neighborhood of cardinality $\leqq \kappa_{m_n}\} = B_n$ has cardinality $\geqq \kappa_n$. But then for some k_n, $B_n \cap X_{k_n} \geqq \kappa_n$. Let $Y_n \subset B_n \cap X_{k_n}$ of the required cardinality, and to each $y \in Y_n$ associate a neighborhood u_y of cardinality $\leqq \kappa_{m_n}$. By construction, the space $Y = \bigcup_{n \in \omega} Y_n$ is a downward subspace of X.

Theorem 10 is proved. But in the proof we actually learn more, since if X has a subspace X' of cardinality κ in which either Case 1 or 2 holds, spread is attained. Thus a counterexample must contain a space which is not only downward, but in which every subspace of cardinality κ satisfies Case 3 of the proof of the theorem.

In fact a theorem of Juhasz and Hajnal's tells us more.

THEOREM 13. *If X is a strongly Hausdorff space whose spread has countable cofinality, X attains its spread. (A strongly Hausdorff space is a Hausdorff space in which every infinite subset has an infinite subset which can be separated by a disjoint open family in the original space.)*

The proof of Theorem 13 is given in Juhasz [6] and in its light the question of attaining a spread of countable cofinality reduces to the question: does every space whose spread has countable cofinality have a strongly Hausdorff subspace with the same spread? Actually, since every known example of a Hausdorff space which is not strongly Hausdorff is essentially countable (i.e. a countable space tacked on to any other space which is strongly Hausdorff) an open question of considerable interest is the following: for any cardinal κ is there a Hausdorff space of cardinality κ such that no subspace of cardinality κ is strongly Hausdorff?

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