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ON A SPACE BETWEEN BH AND B_∞

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Let $F \subseteq E_\infty \xrightarrow{p} B_\infty$ be the classifying space of fibre homotopy equivalence classes of fibrations with fiber F. An obstruction to the existence of a retraction $B_\infty \xrightarrow{r} E_\infty$ is developed. This obstruction is shown to vanish when F is a stable space. Consequences of the existence of a retraction are indicated.

One asks the following question: Let $F \subseteq E_\infty \xrightarrow{p_\infty} B_\infty$ be a universal fibration for $\mathscr{H}(\_, F)$, for which F do there exist retractions $r: B_\infty \to E_\infty$?

An obstruction theoretic answer to this question is given below. The method is first to observe that F must be a loop space. Then construct a fibration $F \subseteq E_+ \xrightarrow{p_+} B_+$, admitting a retraction $r: B_+ \to E_+$, that is "universal" with respect to this property. Finally, compute the fibre of the classifying map $B_+ \to B_\infty$. As an application of this computation it is shown that a retraction exists whenever F is a stable space.

Our interest in this question grew out of an observation in [7]. In particular, let $L(\pi, n)$ be the classifying space for $\mathscr{H}(\_, K(\pi, n))$, (fibre homotopy equivalence classes of fibrations with fibre $K(\pi, n)$). Let $K(\pi, n) \subseteq E_\infty \xrightarrow{p} L(\pi, n)$ be the universal fibration [1], in [7] it was observed that there is a retraction of $L(\pi, n)$ to E_∞. This observation was used in [6] and implicitly in [7] to facilitate the computation of the cohomology of $L(\pi, n)$. 2.13, below, generalizes that part of the main result of [7] and seems to give the appropriate setting for that theorem.

We would like to thank the referee for his helpful comments and especially for allerting us to the difficulties in the work of Allaud and Dold.

1. Preliminaries.

We follow Allaud [1], (see Dold [2] for an alternate treatment)[1]. In addition, we make the blanket assumption that all spaces are in $\mathscr{C}$ the category of spaces of the homotopy type of countable connected C.W complexes (resp. $\mathscr{C}_0$ base pointed) according to [1], [5] the constructions below do not carry us out of these categories.

[1] Peter May has noted a difficulty in both works. In particular, neither work supplies well defined set valued functors for the theory of fibrations. These, of course, are required for application of Browns theorem. May will discuss various remedies in a forthcoming monograph.

DEFINITION 1.1. Let F be in $\mathscr{C}$ and let (X, x_0) be in $\mathscr{C}_0$. A *fibre space over X with fibre F* (written $(\mathscr{E}, g)$) consists of a sequence of spaces and maps. $F \xrightarrow{g} E \xrightarrow{p} X$ such that:

(a) The triple $\mathscr{E} = (E, p, X)$ is a fibre space.
(b) $g: F \to p^{-1}(x_0)$ is a homotopy equivalence.

DEFINITION 1.1a. If we further assume:
(1) $(F, b_0) \in \mathscr{C}_0$
(2) $\mathscr{E}^0 = (E, p, X, s)$ is a fibre space with cross-section.
(3) $g: (F, b_0) \to (p^{-1}(x_0), s(x_0))$ is a base point preserving homotopy equivalence.
We call $(\mathscr{E}^0, g)$ a *fibre space with cross-section.*

DEFINITION 1.2. Let $(\mathscr{E}_1, g_1)$ and $(\mathscr{E}_2, g_2)$ be fibre spaces. A *fibre map* $(\mathscr{E}_1, g_1) \xrightarrow{f} (\mathscr{E}_2, g_2)$ is a triple of maps $(g, \bar{f}, f')$

$$\begin{array}{ccccc}
F_1 & \xrightarrow{g_1} & E_1 & \xrightarrow{p_1} & X_1 \\
g \downarrow & I & \bar{f} \downarrow & II & f' \downarrow \\
F_2 & \xrightarrow{g_2} & E_2 & \xrightarrow{p_2} & X_2
\end{array}$$

such that I commutes up to homotopy and II commutes. We denote the category of fibre spaces and maps by $\mathscr{F}$.

DEFINITION 1.2a. If $(\mathscr{E}_1^0, g_1)$ and $(\mathscr{E}_2^0, g_2)$ are fibre spaces with cross-section and $(\mathscr{E}_1^0, g_1) \xrightarrow{f} (\mathscr{E}_2^0, g_2)$ is a fibre map such that $\bar{f}s_1 = s_2 f'$ then f is called a *section preserving fibre map*. We denote the resulting category by $\mathscr{F}^0$.

DEFINITION 1.3. Let $(\mathscr{E}, g)$ be a fibre space over X with fibre F. Let $f: Y \to X$ be a base pointed map. We have the usual induced fibration $(f^{-1}(\mathscr{E}), g)$ with fibre F. f induces $(f^{-1}(\mathscr{E}), g) \xrightarrow{f} (\mathscr{E}, g)$ in $\mathscr{F}$.

DEFINITION 1.3a. If $(\mathscr{E}^0, g)$ carries a cross-section then so does $f^{-1}(\mathscr{E}^0, g)$ and f is in $\mathscr{F}^0$.

DEFINITION 1.4. Restricting to a fixed fibre F there are contravariant functors $\mathscr{H}(\_, F)$ and $\mathscr{H}_0(\_, F)$ from $\mathscr{C}_0$ to the category of sets. $\mathscr{H}(X, F)$ (resp. $\mathscr{H}_0(X, F)$), is the set of fibre homotopy equivalence classes of fibrations over X (resp. base section preserving) with fibre F.

We now state two theorems, the first of which appears in [1]. The second can be proved by mimicking that proof. We offer an alternative proof below.

THEOREM 1.5. *Given F, there exists a space $B_\infty \in \mathscr{C}_0$ and a fibre space $(\mathscr{E}_\infty, g_\infty)$ over B_∞ with fibre F such that the natural transformation $[X, B_\infty]_0 \xrightarrow{T} \mathscr{H}(X, F)$ given by $T[f] = (f^{-1}(\mathscr{E}_\infty), g_\infty)$ is a 1–1 onto set map.*

THEOREM 1.5a. *There exists a space $B_\dagger \in \mathscr{C}_0$ and a fibre space $(\mathscr{E}_\dagger, g_\dagger)$ over $B_\dagger$ with fibre F such that the natural transformation $[X, B_\dagger]_0 \xrightarrow{T_0} \mathscr{H}_0(X, F)$ given by $T_0[f] = (f^{-1}(\mathscr{E}_\dagger), g_\dagger)$ is a 1–1 onto set map.*

We prove 1.5a by observing that, in fact, $B_\dagger$ is the total space of the universal fibration over B_∞.

DEFINITION 1.6. If $\mathscr{E} = (E, p, X)$ is a fibre space with fibre F. Let E^F consist of all maps from F into E which are a homotopy equivalence into some $p^{-1}(x)$, $x \in X$. Fixing $b \in F$, let $\tilde{p}f = pf(b)$. Set $\mathscr{E}^F = (E^F, \tilde{p}, X)$. $\mathscr{E}^F$ is a principal $H(F)$-fibration where $H(F)$ is the space of homotopy equivalences of F to itself.

DEFINITION 1.6a. If $\mathscr{E}_0 = (E, p, X, s)$ is a fibre space with base pointed fibre (F, b) and cross-section s. Let $E_0^F \subseteq E^F$ consist of all maps f such that f is a base point preserving homotopy equivalence into $(p^{-1}(x), s(x))$ for some x. $\mathscr{E}_0^F$ is a principal $H_0(F)$-fibration. Where $H_0(F)$ is the space of base point preserving homotopy equivalences.

We also need the following construction.

Let $E \xrightarrow{p} X$ be a fibration with fibre F. Let $E^2 \xrightarrow{p} E$ be the "pull back" over E of $E \xrightarrow{p} X$

$$\begin{array}{ccc} E^2 & \xrightarrow{t} & E \\ s \upharpoonleft\!\downharpoonright p^2 & \nearrow & \downarrow p \\ E & \xrightarrow{p} & X \end{array}$$

The cross-section s is induced by the identity map. Fixing $b \in F$ we have a map $\alpha: (E^2)_0^F \to E^F$ given by $\alpha(f) = t \cdot f$.

LEMMA 1.7. *α is a homeomorphism of $(E^2)_0^F$ to E^F.*

Proof. The inverse of α is $\beta\colon E^F \to (E^2)_0^F$ given by

$$\beta(f) = f(b) \times f\colon F \to E \times E.$$

THEOREM 1.8. *Let $E_\infty \xrightarrow{p_\infty} B_\infty$ as in 1.5 then $(E_\infty^2)_0^F \xrightarrow{p_\infty^2} E_\infty$ is a universal principal $H_0(F)$-fibration.*

Proof. By 1.7 $\alpha_*\colon \pi_i((E_\infty^2)_0^F) \cong \pi_i(E_\infty^F)$. By Theorem 4.1 of **[1]** $\pi_i(E_\infty^F) = 0$.

COROLLARY 1.9. *$E_\infty^2 \xrightarrow{p_\infty^2} E_\infty$ is universal for $\mathscr{H}_0(\_, F)$.*

Proof. Given $\mathscr{E}^0 = E \underset{s}{\overset{p}{\rightleftarrows}} X$ we have, using the universality of B_∞

$$\begin{array}{ccc} E & \longrightarrow & E_\infty \\ p \downarrow\uparrow s & & \downarrow p_\infty \\ X & \longrightarrow & B_\infty \end{array}$$

hence $\mathscr{E}^0$ is induced by a map into E_∞. But using 1.8 and trivial modifications of the arguments in **[4]** §7, we have $\mathscr{E}^0$ and $\mathscr{E}^{0\prime}$ are base section fibre homotopy equivalent iff the induced maps into E_∞ are homotopic.

This explicit model will facilitate later computations. We will also need the following observation about fibrations with cross-sections.

DEFINITION 1.10. Let $(\mathscr{E}^0, g)$ be a fibration over X with fibre (F, b) and cross-section s. There is the usual associated fibration $(\Omega\,\mathscr{E}^0, \Omega\, g)$ over X with fibre $\Omega\,(F)$ where $(\Omega\, p)^{-1}(x) = \Omega\,(p^{-1}(x))$ based at $s(x)$.

LEMMA 1.11. *Let $(\mathscr{E}^0, g) = F \xrightarrow{g} E \overset{p}{\rightleftarrows} X$ be a fibration with cross-section. Let $F_s \overset{i}{\subseteq} X \xrightarrow{s} E$ be the fibration of the map s, then the pull back*

$$\begin{array}{ccc} F_s \subseteq s^{-1}(X) & \dashrightarrow & X \\ p' \downarrow & & \downarrow s \\ X & \xrightarrow{s} & E \end{array}$$

is fibre homotopy equivalent with cross-section to $(\Omega\,\mathscr{E}^0, \Omega\, g)$.

Proof. First note $s^{-1}(X) \subseteq E^I$ and $s^{-1}(X) = \{f: I \to E \mid f(0)$ and $f(1) \in s(X)\}$ hence there is the natural inclusion $\Omega(\mathscr{E}^0) \subseteq s^{-1}(X)$ covering X, where $\Omega(\mathscr{E}^0)$ here denoted the total space of $(\Omega \mathscr{E}^0, \Omega(g))$. There is map $\gamma: s^{-1}(x) \to \Omega E^{\text{free}}$ given by $\gamma(f) = (spf)^{-1} * f$, where $*$ denotes path composition. Since $(\Omega p)\gamma = (pf)^{-1} * pf$, γ can be deformed to a fibre map $\gamma': s^{-1}(x) \to \Omega(\mathscr{E}^0)$ such that $\gamma' i = id$. By a standard argument one shows $i\gamma$ is cross-section preserving fibre homotopic to the identity.

We close this section with one final technical observation. The proof is straightforward and we leave it undone.

The following notation is useful. Given a map $X \xrightarrow{f} Y$ we let $X_f = X \times_{f \times 0} Y^I$, and let $X_f \xrightarrow{f'} Y$ be the standard replacement of $X \xrightarrow{f} Y$ by a fibration.

Let $F \xrightarrow{g} E \underset{}{\overset{p}{\rightleftarrows}} B$ be a fibration with retraction. We have the following diagram.

$$\begin{array}{ccccc} E & = & E & \longrightarrow & E_p \\ r \upharpoonleft\!\downharpoonright p & & r' \upharpoonleft\!\downharpoonright p & & \upharpoonleft\!\downharpoonright \\ B & \xrightarrow{i} & B_r & = & B_r \end{array}$$

THEOREM 1.12.

$$\begin{array}{ccc} E & \longrightarrow & i^{-1}(E_p) \\ \downarrow p & & \downarrow p' \\ B & = & B \end{array}$$

is a fibre homotopy equivalence.

2. The Space B_+.

We will be concerned with the question of when there is a retraction of B_∞ to E_∞. We begin with certain simple technical facts we will need in the following.

THEOREM 2.1. *Let $F \to E \underset{r}{\overset{p}{\rightleftarrows}} B$ be a fibration with r a retraction to the total space. Then the composition $\Omega F_r \to \Omega B \to F$ is a homotopy equivalence.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_i(E) & \xrightarrow{p_*} & \pi_i(B) & \xrightarrow{\partial} & \pi_{i-1}(F) & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longleftarrow & \pi_i(E) & \xleftarrow{r_*} & \pi_i(B) & \xleftarrow{i_*} & \pi_i(F_r) & \longleftarrow & 0 \end{array}$$

We show ∂i_* is an isomorphism by a simple diagram chase. f is then the geometric realization of this map.

∂i_* is a monomorphism. $\partial i_* x = 0 \Rightarrow \exists\, y \cdot \ni \cdot p_* y = i_* x$ but $r_* p_* y = y$ so $y = r_* p_* y = r_* i_* x = 0$ so $i_* x = 0 \Rightarrow x = 0$.

∂i_* is an epimorphism. Let $z \in \pi_{i-1}(F)$, $z = \partial y = \partial(y - p_* r_* y)$ but $r_*(y - p_* r_* y) = 0$ so $\exists x \cdot \ni .\ \ i_* x = y - p_* r_* y$.

LEMMA 2.2. *Let H be an H-space. Let $(\mathscr{E}_\infty, g_\infty)$ be the universal fibration for H then the sequence $0 \to \pi_i(E_\infty) \to \pi_i(B_\infty) \to \pi_{i-1}(H) \to 0$ is exact and splits.*

Proof. Let $H \to * \to BH$ be the universal principal H-fibration.

Consider the diagram.

$$\begin{array}{ccccc} H & \longrightarrow & * & \longrightarrow & BH \\ \| & & \downarrow & & \downarrow \\ H & \longrightarrow & E_\infty & \longrightarrow & B_\infty \end{array}$$

This leads to the following diagram of homotopy group.

$$\begin{array}{ccc} \pi_i(BH) & \xrightarrow[\cong]{\partial} & \pi_{i-1}(H) \\ \tau_* \downarrow & & \| \\ \pi_i(B_\infty) & \xrightarrow{\partial_\infty} & \pi_{i-1}(H) \end{array}$$

∂_∞ is therefore split by $\tau_* \partial^{-1}$.

2.3. Let $BH \to E_\dagger \underset{s}{\overset{p_\dagger}{\rightleftarrows}} B_\dagger$ be the universal fibration for $\mathscr{H}_0(-, BH)$. By 2.1 H is the fibre of s. Setting $B_+ = E_\dagger$, $E_+ = B_\dagger$, $s = p_+$ and $p_\dagger = r_+$ we have the following diagram.

$$
\begin{array}{ccccc}
 & & H & = & H \\
 & & \downarrow & & \downarrow \\
T' & \xrightarrow{i'} & E_+ & \xrightarrow{\sigma} & E_\infty \\
g\downarrow & & r_+ \upharpoonleft\!\downharpoonright p_+ & & \downarrow p_\infty \\
T & \xrightarrow{i} & B_+ & \xrightarrow{\rho} & B_\infty
\end{array}
$$

where T is the fibre of ρ and T' is the fibre of σ.

It is the space T that we are interested in since, as we now know, it measures the obstruction to retracting B_∞ to E_∞.

THEOREM 2.4. *Let $H \to E \underset{r}{\overset{p}{\rightleftarrows}} X$ be a fibration such that r is a retraction of X to E then the classifying map $f\colon X \to B_\infty$ factors through B_+. That is, there exists $f'\colon X \to B_+$ and a diagram of fibre maps.*

$$
\begin{array}{ccccc}
E & \xrightarrow{\bar{f}'} & E_+ & \longrightarrow & E_\infty \\
r \upharpoonleft\!\downharpoonright p & & r_+ \upharpoonleft\!\downharpoonright p_+ & & \downarrow p_\infty \\
X & \xrightarrow{f'} & B_+ & \xrightarrow{\rho} & B_\infty
\end{array}
$$

with $\rho f' \sim f$ and $\bar{f}' r \sim r_+ f'$.

Moreover f' induces a unique map on homotopy. That is if $\rho f'_1 \sim \rho f'_2$ then $(f'_1)_* = (f'_2)_*\colon \pi_i(X) \to \pi_i(B_+)$.

Proof. Consider the fibration

$$
\begin{array}{c}
X_r \\
\tilde{p} \upharpoonleft\!\downharpoonright r' \\
E
\end{array}
$$

where $\tilde{p} \sim p$ is a cross-section. Since $B_+ = E_+$, we have the fibre map

$$
\begin{array}{ccc}
X_r & \overset{f'}{\text{——}} & B_+ \\
\tilde{p} \upharpoonleft\!\downharpoonright r' & & p_+ \upharpoonleft\!\downharpoonright r_+ \\
E & \overset{f}{\text{——}} & E_+
\end{array}
$$

and the associated diagram of fibrations

$$\begin{array}{ccc} E_{\tilde{p}} & \xrightarrow{f} & E_+ \\ \tilde{p}' \downarrow & & \downarrow p_+ \\ X_r & \xrightarrow{f'} & B_+ \end{array}$$

but $\tilde{p} \sim p$ implies that $E_p \xrightarrow{p'} X_r$ is fibre homotopy equivalent to $E_{\tilde{p}} \xrightarrow{p'} X_r$ and by 1.12 we have

$$\begin{array}{ccc} E & \longrightarrow & E_p \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{i} & X_r \end{array}$$

a fibre map. Let $f' = i\bar{f}'$.

In order to verify uniqueness consider the diagram

$$\begin{array}{ccccccc} 0 \to \pi_i(E) & \underset{r_*}{\overset{p_*}{\rightleftarrows}} & \pi_i(X) & \xrightarrow{\partial} & \pi_{i-1}(H) \to 0 \\ \Big\| \bar{f}'_* & & \Big\| f'_* & & \Big\| \\ 0 \to \pi_i(E_+) & \underset{p_{+*}}{\overset{r_{+*}}{\rightleftarrows}} & \pi_i(B_+) & \xrightarrow{\partial} & \pi_{i-1(H)} \quad \to 0 \end{array}$$

Since $\bar{f}'r \sim r_+f'$ and E_+ is universal for $H_0(\_, BH)$ we have $\bar{f}'_*$ is unique (indeed $\bar{f}'$ is unique up to homotopy). The uniqueness of f'_* follows from the splitting of the diagram.

It is reasonable to expect that f' is unique up to homotopy, however B_+ is not classifying space in sense of Brown [2]. Fibre spaces admitting retractions do not form a suitable category.

To infer that f' is unique up to homotopy from the above requires knowledge the two fibre maps that are homotopic along the base are homotopic in the total space. In general of course this is not the case.

THEOREM 2.5. *The fibration $H \to E_\infty \xrightarrow{p_\infty} B_\infty$ admits a retraction r, iff the fibration $T \to B_+ \xrightarrow{p} B_\infty$ admits a cross-section, s.*

Proof.

$\Rightarrow$: Let $E_\infty \xrightarrow{p_\infty} B_\infty$ admit a retraction. Therefore, by 2.4 we can factor the identity map id: $B_\infty \to B_\infty$ through ρ. That is find $s' B_\infty \to B_+$ with $\rho s' \sim id$. Let s be a cross-section homotopic to s.

$\Leftarrow$: Consider the diagram.

$$\begin{array}{ccccc} E_\infty & \xrightarrow{t} & E_+ & \xrightarrow{\sigma} & E_\infty \\ p_\infty \downarrow & & p_+ \uparrow\!\downarrow r_+ & & p_\infty \downarrow \phi \\ B_\infty & \xrightarrow{s} & B_+ & \xrightarrow{\rho} & B_\infty \end{array}$$

Note. Here we use the fact that B_∞ is universal to conclude that $s^{-1}(E_+) \to B_\infty$ is fibre-homotopy equivalent to $E_\infty \xrightarrow{p_\infty} B_\infty$.

Define

$$\bar{r} = \sigma r_+ s.$$

$$\begin{aligned} \bar{r} p_\infty &= \sigma r_+ s p_\infty \\ &= \sigma r_+ p_+ t \\ &= \sigma t \end{aligned}$$

but σt is a homotopy equivalence so set $r = (\sigma t)^{-1} \bar{r}$

We now study the problem of sectioning the fibration $T \to B_+ \xrightarrow{\rho} B_\infty$.

Consider again the diagram.

$$(2.6) \qquad \begin{array}{ccccc} & & H & = & H \\ & & \downarrow & & \downarrow \\ T' & \overset{i'}{—} & E_+ & \overset{\sigma}{—} & E_\infty \\ g \downarrow & & r_+ \uparrow\!\downarrow p_+ & & \downarrow p_\infty \\ T & \xrightarrow{i} & B_+ & \xrightarrow{\rho} & B_\infty \end{array}$$

THEOREM 2.7. $g: T' = T$

Proof. By symmetry of fibre products the fibre of ζ and the fibre of σ are the same.

Description of T'. Given a fibration $(\mathscr{E}_0, g)$ with cross-section. There is the "loop" fibre map.

$$\begin{array}{ccccc}
(F^F)_0 & \xrightarrow{g} & E_o^F & \xrightarrow{p} & B \\
\downarrow \Omega(F) & & \downarrow \Omega(F) & & \| \\
\Omega(F) & \xrightarrow[\Omega g]{} & \Omega(\mathscr{E}^0) & \xrightarrow[\Omega(p)]{} & B
\end{array}$$

Given by fibre-wise looping of the mapping space. We apply this construction to $BH \to B_+ \underset{p_+}{\overset{r}{\rightleftarrows}} E_+$ (see 2.3) in conjuction with 1.11 to obtain the following diagram.

$$\begin{array}{ccccc}
(BH)_0^{BH} & \xrightarrow{\Omega} & (H^H)_0 & = & (H^H)_0 \\
\downarrow & & \downarrow & & \downarrow \\
B_+^{BH} & \longrightarrow & (\Omega(B_+))^H & \longrightarrow & (E_\infty^2)^H \\
\downarrow r_+ & & \downarrow & & \downarrow \\
E_+ & = & E_+ & \xrightarrow{p} & E_\infty
\end{array}$$

Using the outer edges of 2.4 combined with 1.8 yield the following communtative diagram of homotopy group.

$$\begin{array}{ccc}
\pi_n(E_+) & \xrightarrow{\sigma_*} & \pi_n(E_\infty) \\
\partial \| & & \| \partial \\
\pi_{n-1}((BH)^{BH}) & \xrightarrow{\Omega} & \pi_{n-1}(H_0^H)
\end{array}$$

and this plus the usual elementary considerations about mapping spaces of fibre spaces shows.

THEOREM 2.9. *$\Omega(T')$ is homotopy equivalent to the fibre of $BH_0^{BH} \xrightarrow{\Omega} H_0^H$ which we denote by L.*

Description of L. We begin by noting that the map Ω is equivalent to the map $(BH)_{ID}^{BH} \xrightarrow{\tau^*} (BH)_\tau^{S\,\Omega BH}$ induced by the canonical map. $S\,\Omega\, BH \xrightarrow{\tau} BH$. The subscripts refer to the appropriate component of mapping spaces.

Considering τ as an inclusion we have the well known lemma:

LEMMA 2.10. *$(BH)^{BH}_{ID} \xrightarrow{\tau^*} (BH)^{S\Omega BH}_{\tau}$ is a fibration with fibre $L = \{f \in (BH)^{BH} \mid f/S\,\Omega\, BH = \tau\}$.*

If we further assume BH is a loop space then for any X all components of BH^X have the same homotopy type. Indeed

THEOREM 2.11. *Let BH be a loop space then L is homotopy equivalent to $(BH^{BH/S\Omega BH})_0$ where $(BH/S\,\Omega\, BH)_0$ is the cofibre of τ.*

Proof. This is a standard argument using associativity and the existence of a homotopy inverse.

EXAMPLE 1.12. Suppose we define an $n-1$ connected space H to be stable if $\pi_i(H)=0$, $i>2n-3$ and BH is a loop space. $S\,\Omega\, BH \to BH$ is an isomorphism in homology through dimension $2n-2$. But $\pi_i(BH)=0$, $i>2n-2$ so

$$\pi_j(L) = \pi_j((BH^{BH/S\Omega})_0) = [S^j \wedge (BH/S\,\Omega\, BH), BH]_0 = 0, \quad j \geqq 0.$$

We therefore have

THEOREM 2.13. *Let H be stable then $\pi_j(B_+) \overset{\rho}{\cong} \pi_j(B_\infty)$, all j. Hence, ρ is a homotopy equivalence.*

This generalizes 2.2 of **[7]**.

Final Remarks 2.14. The importance of the existence of a retraction to the total space was discussed in **[6]** and **[7]**. In particular, the existence of a retraction implies $E_\infty^{p,q}=0$, $q>0$ in the spectral sequence of the fibration $H \subseteq E_+ \xrightarrow{p} B_+$. Indeed, $H^*(E_+) = E_\infty^{*,0}$ splits off.

Since $H \subseteq * \xrightarrow{p} BH$ of course admits a retraction to the total space. The classifying map $BH \to B_\infty$ factors through B_+.

2.15.
$$\begin{array}{ccccc}
H \subseteq * & \longrightarrow & E_+ & \longrightarrow & E_\infty \\
\downarrow & & \downarrow p_+ & & \downarrow p_\infty \\
BH & \longrightarrow & B_+ & \xrightarrow{\rho} & B_\infty
\end{array}$$

Using 2.15, the observations about spectral sequences above and the comparison theorem, one would like to conclude

$$H^*(B_+) \cong H^*(E_+) \oplus H^*(BH).$$

In general this is not true because the coefficient system $H^*(F)$ over B_+ is not trivial. However, the machinery of **[6]** and **[7]** carries over to this more general situation and allows specific computations to be carried out. Hopefully, such information can be used to infer information about B_∞ and /or the classifying map $BH \to B_\infty$ in such specific situations.

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