PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

A REPRESENTATION THEOREM FOR REAL CONVEX FUNCTIONS

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The Krein-Milman theorem is used to prove the following result. A nonnegative function f on [0, 1] is convex if, and only if, there exist nonnegative Borel measures μ_1 and μ_2 on [0, 1] such that

$$f(x)=\int_0^x (1-\xi)^{-1}(x-\xi)d\mu_1(\xi)+\int_x^1 [1-(x/\xi)]d\mu_2(\xi),$$

for every $x \in [0,1]$. An example is given for which the representation is not unique.

1. Extremal elements. Let C be the set of all nonnegative convex real-valued functions on [0, 1]. Since the sum of two nonnegative convex functions is in C and since a nonnegative real multiple of a convex function is a convex function, the set C is a convex cone. It is the purpose of this paper to determine the extremal elements of this cone and to show that for the convex functions an integral representation in terms of extremal elements is possible (see **[1]** for terminology). We prove the following theorem which characterizes the extremal elements of C.

THEOREM 1. *The set of extremal elements of C consists of the following functions, where $m > 0$:*

$e_+(m,\xi;x)=0$, $x \in [0,\xi]$ *and* $m(x-\xi)$ *for* $x \in [\xi,1]$, *where* $0 \leqq \xi < 1$;
$e_+(m,1;x)=0$, $x \in [0,1)$ *and* m *for* $x=1$;
$e_-(m,0;x)=0$, $x \in (0,1]$ *and* m *for* $x=0$;
$e_-(m,\xi;x)=m(\xi-x)$, $x \in [0,\xi]$ *and* 0 *for* $x \in [\xi,1]$ *where* $0<\xi \leqq 1$.

Proof. Let f be a function in C which assumes exactly one positive value in [0, 1]. If $f=c>0$, then $f(x)=cx+c(1-x)$ for $x \in [0,1]$ and hence, f is not an extremal element of C. If f is not constant, then f must be positive at one end point of [0, 1], since f is continuous on (0, 1) **[5, p. 109]**. It is evident that the two functions which are positive only at 0 and 1, respectively, are extremal elements of C. If $f \neq 0$ on [0, 1) and

$$f(1) \neq f_-(1)=\lim_{x\to 1-} f(x),$$

then $f = f_1 + f_2$, where $f_1 = 0$ on $[0,1)$, $f_1(1) = f(1) - f_-(1) > 0$ and $f_2 = f - f_1$, and hence f is not an extremal element. Similarly, if f is an extremal element and $f \neq 0$ on $(0,1]$, then f must be continuous at 0. Thus, all of the remaining extremal elements of C must be continuous on $[0,1]$.

Let $f \in C$ such that f is not constant and is continuous on $[0,1]$, and let $f(x_0) = \inf\{f(x)\colon 0 \leqq x \leqq 1\}$. If $f(x_0) > 0$, then $f = f(x_0) + [f - f(x_0)]$ and so f is not an extremal element of C. If f is not monotonic, then a nonproportional decomposition of f can be given by $f = f_1 + f_2$, where $f_1(x) = f(x)$ for $x \in [0, x_0]$, $f_1(x) = f(x_0)$ for $x \in [x_0, 1]$ and $f_2 = f - f_1$. Hence, all continuous extremal elements of C must assume the value 0 and must be monotonic.

Let $f \in C$ such that f is continuous and monotonic and $\inf\{f(x)\colon x \in [0,1]\} = 0$. Suppose, without loss of generality, that f'_- assumes at least two positive values in $(0,1)$; we know that f'_- exists and is left-continuous and nondecreasing on $(0,1)$, since f is convex [**5**, p. 109]. Let $0 < x_1 < x_2 < 1$ be such that $0 < f'_-(x_1) < f'_-(x_2)$ and define $f_1(x) = f(x)$, $x \in [0, x_1]$, and $f_1(x) = f(x_1) + f'_-(x_1)(x - x_1)$, for $x \in [x_1, 1]$. Then $f_1 \in C$ and $f_2 = f - f_1 \in C$; that is, $f = f_1 + f_2$ and hence, f is not an extremal element of C. Thus, if f is a continuous extremal element of C, then f'_- (and f'_+) assumes exactly one nonzero value in $(0,1)$.

For $m > 0$, define the functions $e_+(m, \xi; \cdot)$ and $e_-(m, \xi; \cdot)$ as in the statement of the theorem. It is easily seen that $e_+(m, \xi; \cdot)$, where $0 \leqq \xi < 1$, and $e_-(m, \xi; \cdot)$, where $0 < \xi \leqq 1$, are extremal elements of C and moreover, they are the only continuous extremal elements of C. Thus, for $m > 0$ and $0 \leqq \xi \leqq 1$,

$$\operatorname{extr} C = \{e_+(m, \xi; \cdot)\} \cup \{e_-(m, \xi; \cdot)\},$$

where extr C denotes the set of extremal elements of C. This completes the proof of Theorem 1.

2. Integral representations.

The set of functions $C - C = C + (-C)$ is the smallest linear space containing the convex cone C. With the topology of simple convergence, $C - C$ is a Hausdorff locally convex space such that for each $x \in [0,1]$, the linear functional L_x defined by $L_x(f) = f(x)$ is continuous.

THEOREM 2. *In $C - C$, the cone C has a compact base C_0. Moreover, the extreme points of C_0 form a compact set.*

Proof. The linear functional F defined on $C - C$ by $F(f) = f(x + 2h) - 2f(x + h) + f(x)$, for $[x, x + 2h] \subset [0,1]$, is continuous in the topology of simple convergence. By definition, C is the intersection of a collection of closed half-spaces corresponding to such functional. Hence, C is closed in $C-C$.

If $f \in C$, then f is bounded by $f(0)+f(1)$, and it follows from the Tychonoff theorem that the normalized convex functions, namely

$$C_0 = \{f \in C: f(0)+f(1) = 1\},$$

form a compact base for C. If we let

$$E_+ = \{e_+((1-\xi)^{-1}, \xi; \cdot): 0 \leqq \xi < 1\} \cup \{e_+(1,1;\cdot)\} \text{ and}$$
$$E_- = \{e_-(\xi^{-1}, \xi; \cdot): 0 < \xi \leqq 1\} \cup \{e_-(1,0;\cdot)\},$$

then $\operatorname{ext} C_0 = E_+ \cup E_-$, where $\operatorname{ext} C_0$ denotes the set of extreme points of C_0.

Let $U_0 = \{f \in C-C: |f(0)| < 1/2\}$ and $U_1 = \{f \in C-C: |f(1)| < 1/2\}$. Then U_0 and U_1 are open sets, $E_+ \subset U_0$, $E_- \subset U_1$ and $U_0 \cap E_- = U_1 \cap E_+ = \emptyset$. Hence, $E_+ \cup E_-$ is a separation of $\operatorname{ext} C_0$. If we define $\alpha_+: [0,1] \to E_+$ by $\alpha_+(\xi) = e_+((1-\xi)^{-1}, \xi; \cdot)$, for $0 \leqq \xi < 1$, and $\alpha_+(1) = e_+(1,1;\cdot)$, then α_+ is a continuous bijection. Since $[0,1]$ is a compact space and E_+ is a Hausdorff space, then α_+ is a homeomorphism. Likewise, $\alpha_-: [0,1] \to E_-$, defined by $\alpha_-(\xi) = e_-(\xi^{-1}, \xi; \cdot)$, for $0 < \xi \leqq 1$, and $\alpha_-(0) = e_-(1,0;\cdot)$, is a homeomorphism. Hence, $\operatorname{ext} C_0 = E_+ \cup E_-$ is a compact set, and the proof is complete.

The mappings α_+ and α_- introduced in the proof of Theorem 2 will now be used to prove the representation theorem.

THEOREM 3. *For each $f \in C$, there exist nonnegative Borel measures μ_1 and μ_2 on $[0,1]$ such that*

$$f(x) = \int_0^x (1-\xi)^{-1}(x-\xi)\,d\mu_1(\xi) + \int_x^1 [1-(x/\xi)]\,d\mu_2(\xi)$$

for every $x \in [0,1]$.

Proof. Let $f \in C_0$. (Since each nonzero function in C is a positive scalar multiple of some function in C_0, we need only consider those functions in C_0.) Then, since C_0 and $\operatorname{ext} C_0$ are compact subsets of the locally convex space $C-C$, by the Krein-Milman representation theorem there exists a probability measure μ on $\operatorname{ext} C_0$ such that

$$L(f) = \int_{\operatorname{ext} C_0} L\,d\mu,$$

for every continuous linear functional L on $C - C$ [3, p. 6]. Thus,

$$f(x) = L_x(f) = \int_{\mathrm{ext}C_0} L_x d\mu = \int_{E_+} L_x d\mu + \int_{E_-} L_x d\mu,$$

for all $x \in [0,1]$. Define μ_1 on each Borel subset B of $[0,1]$ by $\mu_1(B) = \mu[\alpha_+(B)]$; that is, $\mu_1 = \mu\alpha_+$. Since $L_x[\alpha_+(\xi)] = 0$, for $x \in [0,\xi]$ and $(1-\xi)^{-1}(x-\xi)$, for $x \in [\xi, 1]$, then

$$\int_{E_+} L_x d\mu = \int_{(\alpha_+)^{-1}(E_+)} L_x \alpha_+ d(\mu\alpha_+) = \int_0^1 L_x[\alpha_+(\xi)] d\mu_1(\xi)$$
$$= \int_0^x (1-\xi)^{-1}(x-\xi) d\mu_1(\xi)$$

[2, p. 163]. Similarly,

$$\int_{E_-} L_x d\mu = \int_x^1 \xi^{-1}(\xi - x) d\mu_2(\xi),$$

where $\mu_2 = \mu\alpha_-$, and the theorem is proved.

3. Remarks. If μ_1 and μ_2 are nonnegative Borel measures on $[0,1]$ and

$$f(x) = \int_0^x (1-\xi)^{-1}(x-\xi) d\mu_1(\xi) + \int_x^1 \xi^{-1}(\xi - x) d\mu_2(\xi),$$

for every $x \in [0,1]$, then it is easily seen that f is in C. The measures μ_1 and μ_2 which appear in the statement of Theorem 3 are not necessarily unique because the probability measure μ in the proof of Theorem 3 will not always be unique. This follows from the fact that

$$(1/4)(f_1 + f_2 + f_3 + f_4) = (1/8)(f_3 + f_4 + 3f_5 + 3f_6),$$

where $f_1 = e_+(1,0;\cdot)$, $f_2 = e_-(1,1;\cdot)$, $f_3 = e_+(4,(3/4);\cdot)$, $f_4 = e_-(4,(1/4);\cdot)$, $f_5 = e_+((4/3),(1/4);\cdot)$ and $f_6 = e_-((4/3),(3/4);\cdot)$. We also note that $C - C$ properly contains the functions of bounded convexity on $[0,1]$ [4].

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Received January 31, 1974.

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