PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

UNCONDITIONAL SCHAUDER DECOMPOSITIONS OF NORMED IDEALS OF OPERATORS BETWEEN SOME l_p-SPACES

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Given a Banach space E, let

$$l(E) = \sup_{F \in \mathscr{F}(E)} \inf_{\{P_i\}} \sup_{N, \pm} \left\| \sum_{i=1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\|$$

where $\mathscr{F}(E)$ denotes the collection of all finite-dimensional subspaces of E, the infimum ranges over all possible sequences of finite-rank operators $P_i : F \to E$ which satisfy the equality $\Sigma P_i(f) = f$ for all $f \in F$, and $r(P)$ denotes the rank of an operator P.

It is shown that there are finite-dimensional spaces with arbitrarily large $l(E)$ values, and infinite-dimensional spaces E with $l(E) = \infty$. The specific examples with $l(E) = \infty$ yield also information on the rapidity of growth of unconditional Schauder decompositions of E into finite-dimensional spaces.

Clearly if E is finite-dimensional

$$l(E) = \inf_{\{P_i\}} \sup_{N, \pm} \left\| \sum_{i=1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\|$$

where the infimum ranges over all sequences $P_i : E \to E$ satisfying $\Sigma_{i \geq 1} P_i(x) = x$ for all $x \in E$.

It is also obvious from the definition that the value $l(E)$ is not greater than the local unconditional constant $\chi_u(E)$ introduced in [3] which is defined similarly, the only difference being that for $\chi_u(E)$ only sequences $\{P_i\}$ with $r(P_i) = 1$ for all i are considered. Spaces E with finite $\chi_u(E)$ were called in [3] spaces with local unconditional structure. If E is complemented in a space with an unconditional basis then clearly $\chi_u(E) < \infty$.

Besides this generalization the result stated above answers a question of Professor H. P. Rosenthal by providing examples of spaces which do not have unconditional Schauder decompositions into finite-dimensional spaces all of the same dimension p, for any $p = 1, 2, 3, \cdots$; spaces E with $l(E) = \infty$ clearly cannot have such decompositions.

Specifically it is shown in section 2 that if E is the space of operators on l_2 equipped with any ideal norm α, then $l(E) = \infty$ unless α is equivalent to the Hilbert-Schmidt norm for operators on l_2. This implies Lewis' ([6]) characterization of the ideals of operators on l_2 which have local unconditional structure. In addition, it is proved in section 3 that the space E of operators mapping l_1 to c_0 normed with any perfect ideal norm α which is not equivalent to the operator norm $\|\circ\|$, also has $l(E)=\infty$. Additional results on spaces with $l(E)=\infty$ will appear in a forthcoming paper by Professor P. Saphar and this author.

If $l(E)=\infty$, then by Proposition 1 E does not have property P_k for any integer $k=1,2,\cdots$; according to Lindenstrauss and Zippin ([7]) a Banach space E has property P_k if there is a $\lambda>0$ such that for every finite-dimensional subspace F of E there is a Boolean algebra of projections $\mathcal{B}$ on E with $\sup\{\|P\|;\, P\in\mathcal{B}\}\leqq\lambda$ and k vectors $\{x_i\}_1^k$ in E such that F is contained in the closed linear span of $\{P(x_i);\, i=1,\cdots,k,\ P\in\mathcal{B}\}$.

The terminology will generally follow that of [4]. $\mathcal{R}^n$ will denote the n-dimensional linear space, $\{e_i\}_1^n$ the usual unit basis. Given any vector $\epsilon=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$ with $\epsilon_i=\pm 1$, h_ϵ will denote the linear operator on $\mathcal{R}^n$ defined by $h_\epsilon(e_i)=\epsilon_i e_i$ for all i. For any permutation σ of $\{1,2,\cdots,n\}$, g_σ will denote the operator defined by $g_\sigma(e_i)=e_{\sigma(i)}$ for all i.

G will be the compact group of all isometries on l_2^n and dg its unique normalized Haar measure. S will be the unit sphere in l_2^n $\{x\in l_2^n;\, \|x\|_2=1\}$ and dx will stand for the probability measure on S defined by

$$\int_S f(x)dx=\int_G f(g(e))dg, \qquad f\in C(S)$$

where $e\in S$ is any fixed point.

Given any ideal norm α ([4]) and a Banach space E, $\alpha(E)$ will stand for the value $\alpha(1_E)$ where 1_E is the identity operator on E. α^* will denote the adjoint ideal norm of α. α is perfect if $\alpha^{**}=\alpha$. $[L(E,F),\alpha]$ will be the space of all operators $T: E\to F$ with $\alpha(T)<\infty$, and E' denotes the conjugate of E.

Recall that if E and F are finite-dimensional $[L(E,F),\alpha]'$ is the space $[L(F,E,),\alpha^*]$ where the correspondence is given by

$$\langle S,T\rangle=\operatorname{trace}(ST), \qquad S\in L(E,F), \qquad T\in L(F,E).$$

π_p and i_p $(1\leqq p\leqq\infty)$ will denote the p-absolutely summing and p-integral norms respectively. All Banach spaces are taken over the reals as the results can be easily carried over to the complex case with some changes in the constants.

LEMMA 1. *If* $A\in L(X,E)$, $B\in L(E,X)$ *and* BA *is the identity on* X, *then* $l(X)\leqq\|A\|\,\|B\|\,l(E)$.

Proof. Let $F \in \mathcal{F}(X)$ and $\epsilon > 0$ be given. $A(F) \in \mathcal{F}(E)$, so there are $P_i: A(F) \to E$ with $\Sigma P_i A(f) = A(f)$ for all $f \in F$ and

$$\sup_{\pm, N} \left\| \sum_1^N \pm \sqrt{r(P_i)}\, P_i \right\| \leqq l(E) + \epsilon.$$

Set $Q_i = BP_iA$, then $r(Q_i) \leqq r(P_i)$ and $\Sigma Q_i(f) = f$ for all $f \in F$, and

$$\sup_{\pm, N} \left\| \sum_1^N \pm \sqrt{r(Q_i)}\, Q_i \right\| \leqq \|A\| \|B\| (l(E) + \epsilon).$$

As ϵ and F are arbitrary, the result follows.

PROPOSITION 1. *If E has property P_k for some integer k, then $l(E) < \infty$.*

Proof. There is a $\lambda > 0$ such that if $F \subset E$ is any finite-dimensional subspace there is a subset $\{x_i\}_1^k \subset E$ and a Boolean algebra of projections $\mathcal{B}$ on E, with $\sup\{\|P\|; P \in \mathcal{B}\} \leqq \lambda$ and $F \subset \mathrm{span}\{P(x_j);\ j = 1, \cdots, k, P \in \mathcal{B}\}$.

Using elementary arguments similar to Proposition 1 of [7], let $\{y_1, \cdots, y_p\}$ be a basis of F. Given ϵ, there exists a subset of n-disjoint elements $\{P_i\}_1^n \subset \mathcal{B}$ with $\Sigma_1^n P_i = I$, a subset $\{z_r\}_1^p \subset \mathrm{span}\{P_i(x_j);\ i = 1, \cdots, n,\ j = 1, \cdots, k\}$ with $\|z_r - y_r\| < \epsilon$ for every $r = 1, \cdots, p$. It is easy to see that if $\epsilon > 0$ is sufficiently small there is a $1-1$ operator T on E satisfying $T(y_r) = z_r$ for all r and $\|T\|, \|T^{-1}\| < 2$.

Let R_i be the restriction to F of $T^{-1}P_iT$, $i = 1, \cdots, n$. Then $r(R_i) \leqq k$, ΣR_i is the identity on F and

$$\begin{aligned}\sup_{\pm} \left\| \sum_1^n \pm \sqrt{r(R_i)}\, R_i \right\| &\leqq \sqrt{k} \sup_{\pm} \left\| \sum_1^n \pm T^{-1} P_i T \right\| \\ &\leqq 4\sqrt{k} \sup_{\pm} \left\| \sum_1^n \pm P_i \right\| \leqq 8\sqrt{k} \sup_J \left\| \sum_{i \in J} P_i \right\| \\ &\leqq 8\sqrt{k}\, \lambda,\end{aligned}$$

this proves $l(E) \leqq 8\sqrt{k}\,\lambda$.

The following elementary generalization of Hölder's inequality will be used.

LEMMA 2. *Let x_k, y_k $k = 1, 2, \cdots, n$ be vectors in $\mathcal{R}^m$. Then*

$$\left(\sum_{k=1}^{n} \langle x_k, y_k\rangle\right)^2 \leqq m \sum_{j=1}^{n} \sum_{k=1}^{n} \langle x_k, y_j\rangle^2.$$

Proof. Assume without loss of generality that $\operatorname{span}\{y_k\} = \mathcal{R}^m$. Fix the sequence $\{y_k\}$ and consider the problem of minimizing the function

$$f(\{x_k\}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle x_k, y_j\rangle^2 \text{ under the restriction } \sum_{k=1}^{n} \langle x_k, y_k\rangle = 1.$$

Using the "Lagrange multipliers" method, set

$$\phi(\{x_k\}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle x_k, y_j\rangle^2 - \lambda\left(\sum_{k=1}^{n} \langle x_k, y_k\rangle\right).$$

At the minimum value for f, which must exist, $\partial\phi/\partial x_{ki} = 0$ for all $k = 1, 2, \cdots, n$, $i = 1, 2, \cdots, m$, where $x_k = \sum_{i=1}^{m} x_{ki} e_i$. The equations in vector form are then

$$\sum_{j=1}^{n} \langle x_k, y_j\rangle y_j = \lambda y_k \text{ for all } k = 1, 2, \cdots, n.$$

Clearly $\lambda \neq 0$, so the operator $A = \sum_{j=1}^{n} y_j \otimes y_j$ satisfies the equations $A(x_k) = \lambda y_k$ for all k, hence has an inverse A^{-1}. Then

$$m = \operatorname{trace}(A^{-1}A) = \sum_{j=1}^{n} \langle y_j, A^{-1}(y_j)\rangle = \sum_{j=1}^{n} \lambda^{-1} \langle y_j, x_j\rangle$$
$$= \lambda^{-1}$$

finally,

$$f(\{x_k\}) = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x_k, y_j\rangle^2 = \sum_{k=1}^{n} \langle \lambda y_k, x_k\rangle = \lambda = m^{-1}.$$

2. Unconditional decomposition of ideals of operators between Hilbert spaces. The main result proved here is the following:

THEOREM 1. *Let α be an ideal norm, $E = [L(l_2^n, l_2^n), \alpha]$, and let $\alpha(n) = \max\{\max\{\alpha(A)/\pi_2(A), \pi_2(A)/\alpha(A)\}; A \in L(l_2^n, l_2^n)\}$. Then,*

$$e^{1/2}(\pi/2)^2 l(E) \geqq \alpha(n).$$

Proof. Let $u = \sum_{i=1}^{m} A_i \otimes B_i$ be any rank-m operator mapping E to E, where $A_i \in E' = [L(l_2^n, l_2^n), \alpha^*]$ and $B_i \in E$. We shall write

$$A_i(e_j) = \sum_{k=1}^{n} a_{ijk} e_k \quad \text{and} \quad B_i(e_j) = \sum_{k=1}^{n} b_{ijk} e_k$$

for all $i = 1, \cdots, m$, $j = 1, \cdots, n$, where $\{e_k\}_1^n$ is the unit basis of l_2^n. Denote by K_F the unit ball of a given Banach space F, and let $K = K_\epsilon \times K_{E'}$ be the product of the unit balls. Define on K the probability measure μ by

$$\mu(f) = \int_G \int_G 2^{-n} \sum_\epsilon \int_S \int_S f(([\alpha(A)^{-1} h_\epsilon g A h) \times (y \otimes x)) dy dx dg dh$$

where $f \in C(K)$, $A \in L(l_2^n, l_2^n)$ is a fixed non-zero operator, and Σ_ϵ denotes the sum over all 2^n possible choices of $\epsilon = (\pm 1, \pm 1, \cdots, \pm 1)$.

The operator u defines a function of $C(K)$ which is denoted by $\langle u, \circ \rangle$ and defined as $\langle u, a \times b \rangle = \langle u(a), b \rangle = \operatorname{trace}(b(u(a)))$, $a \in E$, $b \in E'$.

Then,

$$\begin{aligned} &\alpha(A)\mu(|\langle u, \circ \rangle|) \\ &= \int_G \int_G 2^{-n} \sum_\epsilon \int_S \int_S |\langle y \otimes x, u(h_\epsilon g A h) \rangle| \, dy dx dg dh \\ &= \int_G \int_G 2^{-n} \sum_\epsilon \int_S \int_S |\langle (u(h_\epsilon g A h))(x), y \rangle| \, dy dx dg dh. \end{aligned}$$

It is well known ([1]) that for any $v \in L(l_2^n, l_2^n)$ with $v_{jk} = \langle v(e_j), e_k \rangle$

$$(\pi_1(l_2^n))^2 \int_S \int_S |\langle v(x), y \rangle| \, dy dx = \pi_1(v) \geqq \pi_2(v) = \left(\sum_{j,k=1}^{n} v_{jk}^2 \right)^{1/2},$$

therefore

$$\begin{aligned} &(\pi_1(l_2^n))^2 2^{-n} \sum_\epsilon \int_S \int_S |\langle (u(h_\epsilon g A h))(x), y \rangle| \, dy dx \\ &\geqq 2^{-n} \sum_\epsilon \pi_2 \left(\sum_{i=1}^{m} (\operatorname{trace}(h_\epsilon g A h A_i)) B_i \right) \\ &= 2^{-n} \sum_\epsilon \left[\sum_{j,k=1}^{n} \left(\operatorname{trace} \left\{ h_\epsilon g A h \left(\sum_{i=1}^{m} b_{ijk} A_i \right) \right\} \right)^2 \right]^{1/2}. \end{aligned}$$

It is well known and easy to show that if (Ω, Σ, μ) is a probability space and $f_i \in C(\Omega)$ $i = 1, \cdots, k$, then

$$\mu \left(\left[\sum_1^n |f_i|^2 \right]^{1/2} \right) \geqq \left[\sum_1^n (\mu(|f_i|))^2 \right]^{1/2},$$

therefore

$$(\pi_1(l_2^n))^2 2^{-n} \sum_{\epsilon} \int_S \int_S |\langle (u(h_\epsilon gAh))(x), y\rangle|\, dydx$$
$$\geqq \left[\sum_{j,k=1}^{n} \left(2^{-n} \sum_{\epsilon} \left| \operatorname{trace}\left\{ h_\epsilon gAh \left(\sum_{i=1}^{m} b_{ijk} A_i \right)\right\} \right| \right)^2 \right]^{1/2}$$
$$\geqq e^{-1/2} \left[\sum_{s,j,k=1}^{n} \left\langle gAh \left(\sum_{i=1}^{m} b_{ijk} A_i \right)(e_s), e_s \right\rangle^2 \right]^{1/2}$$

the last is Khinchin's inequality (the constant $e^{-1/2}$ is due to [9]). Thus

$$\alpha(A)(\pi_1(l_2^n))^2 e^{1/2} \mu(|\langle u, \circ\rangle|)$$
$$\geqq \int_G \int_G \left[\sum_{s,j,k=1}^{n} \left\langle gAh \left(\sum_{i=1}^{m} b_{ijk} A_i \right)(e_s), e_s \right\rangle^2 \right]^{1/2}$$
$$\geqq \left[\sum_{s,j,k=1}^{n} \left(\int_G \int_G \left| \left\langle gAh \left(\sum_{i=1}^{m} b_{ijk} A_i \right)(e_s), e_s \right\rangle \right| dgdh \right)^2 \right]^{1/2}$$
$$= \left[\sum_{s,j,k=1}^{n} \left(\int_G \left\| Ah \left(\sum_{i=1}^{m} b_{ijk} A_i \right)(e_s) \right\|_2 (\pi_1(l_2^n))^{-1} dh \right)^2 \right]^{1/2}.$$

Set $w = \sum_{i=1}^{m} b_{ijk} A_i$, then

$$\int_G \|Ahw(e_s)\|_2\, dh$$
$$= \int_G \left(\sum_{t=1}^{n} \langle Ahw(e_s), e_t\rangle^2 \right)^{1/2} dh$$
$$\geqq \left[\sum_{t=1}^{n} \left(\int_G |\langle Ahw(e_s), e_t\rangle|\, dh \right)^2 \right]^{1/2}$$
$$= \left[\sum_{t=1}^{n} \|A'(e_t)\|_2^2 \|w(e_s)\|_2^2 \right]^{1/2} (\pi_1(l_2^n))^{-1},$$

this implies

$$\alpha(A) e^{1/2} (\pi_1(l_2^n))^4 \mu(|\langle u, \circ\rangle|)$$
$$\geqq \left[\sum_{j,k,s,t=1}^{n} \|A'(e_t)\|_2^2 \|w(e_s)\|_2^2 \right]^{1/2}$$
$$= \pi_2(A) \left(\sum_{j,k,s,r=1}^{n} \left(\sum_{i=1}^{m} b_{ijk} a_{isr} \right)^2 \right)^{1/2}$$
$$\geqq \pi_2(A) \left| \sum_{j,k=1}^{n} \sum_{i=1}^{m} b_{ijk} a_{ikj} \right| m^{-1/2} \qquad \text{(Lemma 2)}$$
$$= \pi_2(A) m^{-1/2} |\operatorname{trace}(u)|.$$

Let now $P_i \in L(E, E)$, $i = 1, 2, \cdots, N$. Then

$$\begin{aligned} e^{1/2}(\pi_1(l_2^n))^4 \max_{\pm} \left\| \sum_{i=1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\| &\geqq e^{1/2}(\pi_1(l_2^n))^4 \mu\left(\sum_{i=1}^{N} \sqrt{r(P_i)}\, |\langle P_i, \circ \rangle| \right) \\ &\geqq (\pi_2(A)/\alpha(A)) \left| \sum_{i=1}^{N} \operatorname{trace}(P_i) \right| . \end{aligned}$$

As α and P_i are arbitrary, the inequality is true for α^* and P'_i too, noting that P'_i maps $[L(l_2^n, l_2^n), \alpha^*]$ to itself, and as

$$\sum_1^N \operatorname{trace}(P'_i) = \sum_1^N \operatorname{trace}(P_i) \quad \text{and} \quad \left\| \sum_1^N \pm \sqrt{r(P'_i)}\, P'_i \right\| = \left\| \sum_1^N \pm \sqrt{r(P_i)}\, P_i \right\|,$$

it follows for arbitrary non-zero operators A, B on l_2^n that

$$\begin{aligned} &e^{1/2}(\pi_1(l_2^n))^4 \max_{\pm} \left\| \sum_1^N \pm \sqrt{r(P_i)}\, P_i \right\| \\ &\geqq \max\{\pi_2(A)/\alpha(A),\ \pi_2(B)/\alpha^*(B)\} \left| \operatorname{trace}\left(\sum_1^N P_i \right) \right| . \end{aligned}$$

Finally, if $\Sigma_{i \geqq 1} P_i(x) = x$ for all $x \in E$, then $\underline{\operatorname{trace}}(\Sigma_{i \geqq 1} P_i) = n^2$ and the result follows from the inequality $\pi_1(l_2^n) \leqq \sqrt{\pi n/2}$ ([2]).

COROLLARY 1. *If α is not equivalent to the Hilbert-Schmidt norm for operators on l_2, then*

$$l([L(l_2^n, l_2^n), \alpha]) \xrightarrow[n \to \infty]{} \infty \quad \textit{and} \quad l([L(l_2, l_2), \alpha]) = \infty.$$

Proof. Let $J_n : [L(l_2^n, l_2^n), \alpha] \to [L(l_2, l_2), \alpha]$ be the natural inclusion and $P_n : [L(l_2, l_2), \alpha] \to [L(l_2^n, l_2^n), \alpha]$ be the natural projection. By Lemma 1, since $\|J_n\|, \|P_n\| \leqq 1$ and $P_n J_n$ is the identity on $L(l_2^n, l_2^n)$ then

$$\begin{aligned} l([L(l_2, l_2), \alpha]) &\geqq l([L(l_2^n, l_2^n), \alpha]) \\ &\geqq (2/\pi)^2 e^{-1/2} \alpha(n) \xrightarrow[n \to \infty]{} \infty. \end{aligned}$$

Let H be a Hilbert space, $c_p(H)$ be the closure of all finite-rank operators $A: H \to H$ in the c_p norm σ_p defined by: $\sigma_p(A) = [\mathrm{trace}(A^*A)^{p/2}]^{1/p}$ if $1 \leqq p < \infty$, and $\sigma_\infty(A) = \|A\|$ if $p = \infty$ (**[8]**).

COROLLARY 2. *$l(c_p(l_2^n)) \geqq n^{|1/p-1/2|} e^{-1/2}(2/\pi)^2$ and $l(c_p(l_2)) = \infty$ if $p \neq 2$.*

Proof. Taking $\alpha = \sigma_p$, the result follows from the fact that $\sigma_p(n) \geqq n^{|1/p-1/2|}$, Theorem 1 and Lemma 1.

Let $[L_0(l_2, l_2), \alpha]$ be the closure of the finite-rank operators on l_2 normed by the ideal norm α.

If α is not equivalent to the Hilbert-Schmidt norm, then the following result shows that $[L_0(l_2, l_2), \alpha]$ does not have an unconditional Schauder decomposition into finite-diemnsional spaces if their dimensions are not sufficiently rapidly increasing.

THEOREM 2. *If p_n $n = 1, 2, \cdots$, is a sequence of integers for which $\alpha(n)p_n^{-1/2} \to \infty$, then $[L_0(l_2, l_2), \alpha]$ does not have an unconditional Schauder decomposition into finite-dimensional spaces E_i having the following property: For any n, there is a subset I_n of integers for which $[L(l_2^n, l_2^n), \alpha]$ is contained in $\Sigma_{i \in I_n} \otimes E_i$ where* $\dim(E_i) \leqq p_n$ *for all $i \in I_n$.*

Proof. Assume to the contrary $[L_0(l_2, l_2), \alpha]$ has such an unconditional decomposition. Fix n and consider the factorization

$$[L(l_2^n, l_2^n), \alpha] \xrightarrow{J_n} [L_0(l_2, l_2), \alpha] \xrightarrow{P_i} E_i \xrightarrow{T_i} [(L_0(l_2, l_2), \alpha] \xrightarrow{Q_n} [L(l_2^n, l_2^n), \alpha]$$

where $i \in I_n$, J_n and T_i are the natural inclusion operators, P_i and Q_n are the natural projections. Let $R_i = Q_n T_i P_i J_n$, then $r(R_i) \leqq \dim(E_i) \leqq p_n$ for all $i \in I_n$, and $\Sigma_{i \in I_n} R_i(x) = x$ for all $x \in L(l_2^n, l_2^n)$. Then

$$\begin{aligned}
\sup_{\pm, N} \left\| \sum_{i=1}^{N} \pm P_i \right\| &\geqq \sup_{\pm} \left\| \sum_{i \in I_n} \pm P_i \right\| \\
&\geqq \sup_{\pm} \left\| \sum_{i \in I_n} \pm R_i \right\| \\
&\geqq \sup_{\pm} \left\| \sum_{i \in I_n} \pm \sqrt{r(R_i)}\, R_i \right\| p_n^{-1/2} \\
&\geqq p_n^{-1/2}\, l([L(l_2^n, l_2^n), \alpha]) \\
&\geqq (2/\pi)^2 e^{-1/2} p_n^{-1/2} \alpha(n) \xrightarrow[n \to \infty]{} \infty,
\end{aligned}$$

which is a contradiction.

REMARKS. If $l(E)=\infty$, this does not necessarily imply that E does not have an unconditional decomposition into finite-dimensional spaces. In fact, by **[5]**, the space $c_p(l_2)$ for all $1<p<\infty$ has such a decomposition. Theorem 2 therefore informs us on the rapidity of growth of the dimensions of many unconditional decompositions of $c_p(l_2)$ $(p\neq 2)$ and is an answer to the question posed to this author by Professor A. Pełczyński at the June 1973 international conference on Banach spaces at Wabash, Indiana. The author learned from Professor J. Lindenstrauss that he has proved $c_p(l_2)$ imbeds in a Banach space with an unconditional basis for any $1<p<\infty$.

Finally, it should be mentioned that the condition imposed on I_n in Theorem 2 is a very natural one, since l_p has an unconditional basis and is isomorphically complemented in $c_p(l_2)$ hence $c_p(l_2)$ has an unconditional Schauder decomposition such that an infinite number of spaces have dimensions equal to 1.

3. Unconditional decompositions in $[L(l_1, c_0), \alpha]$.

THEOREM 3. *Let α be any ideal norm, $E=[L(l_1^n, l_\infty^n), \alpha]$. Then for any operator $B \in L(l_1^n, l_\infty^n)$*

$$e^2 l(E)\,\|B\| \geqq \alpha(B).$$

Proof. Let $u=\Sigma_{i=1}^m A_i \otimes B_i$ be any rank-m operator mapping $E'=[L(l_\infty^n, l_1^n), \alpha^*]$ to E', where $A_i \in E$ and $B_i \in E'$. Set

$$A_i(e_j)=\sum_{k=1}^n a_{ijk} f_k, \qquad B_i(f_j)=\sum_{k=1}^n b_{ijk} e_k$$

where $\{e_k, f_k\}_{k=1}^n$ is the usual biorthonormal set for l_1^n.

Let $A \in L(l_\infty^n, l_1^n)$ be an arbitrary non-zero operator. Define on $K=K_{E'}\times K_E$ the probability measure μ by

$$\mu(f)=\frac{2^{-4n}}{(n!)^2}\sum_{\epsilon,\theta,\phi,\lambda}\ \sum_{\pi,\sigma} f(([\alpha^*(A)]^{-1}h_\theta g_\pi A g_\sigma h_\epsilon)\times(\phi\otimes\lambda))$$

$(f \in C(K))$, where the first Σ sums over all possible vectors $\epsilon, \theta, \phi, \lambda$ of the form $(\pm 1, \pm 1, \cdots, \pm 1)$, and the second Σ sums over all possible permutations π, σ of the set $\{1, 2, \cdots, n\}$.

The operator u defines a function denoted by $\langle u, \circ\rangle$ in $C(K)$ by

$$\langle u, a\times b\rangle=\langle b, u(a)\rangle=\operatorname{trace}(b(u(a))),\ a\in K_{E'},\ b\in K_E.$$

Then,

$$\begin{aligned}
&\alpha^*(A)\mu(|\langle u, \circ\rangle|) \\
&\quad = \frac{2^{-4n}}{(n!)^2} \sum_{\epsilon,\theta,\phi,\lambda} \sum_{\pi,\sigma} |\langle u(h_\theta g_\pi A g_\sigma h_\epsilon), \phi \otimes \lambda\rangle| \\
&\quad = \frac{2^{-4n}}{(n!)^2} \sum \sum |\langle (u(h_\theta g_\pi A g_\sigma h_\epsilon))(\lambda), \phi\rangle|
\end{aligned}$$

Observe that if $v \in L(l_\infty^n, l_1^n)$, then by applying Khinchin's inequality twice it follows that

$$2^{-2n} \sum_{\lambda,\phi} |\langle v(\lambda), \phi\rangle| \geqq e^{-1}\left(\sum_{i,j=1}^{n} \langle v(f_i), f_j\rangle^2\right)^{1/2},$$

and so

$$\begin{aligned}
&e\alpha^*(A)\mu(|\langle u, \circ\rangle|) \\
&\quad \geqq \frac{2^{-2n}}{(n!)^2} \sum_{\epsilon,\theta} \sum_{\pi,\sigma} \left(\sum_{i,j=1}^{n} \langle (u(h_\theta g_\pi A g_\sigma h_\epsilon))(f_i), f_j\rangle^2\right)^{1/2} \\
&\quad \geqq (n!)^{-2} \sum_{\pi,\sigma} \left[\sum_{i,j=1}^{n} \left(\sum_{\epsilon,\theta} 2^{-2n} |\langle (u(h_\theta g_\pi A g_\sigma h_\epsilon))(f_i), f_j\rangle|\right)^2\right]^{1/2} \\
&\quad = (n!)^{-2} \sum_{\pi,\sigma} \left[\sum_{i,j=1}^{n} \left(\sum_{\epsilon,\theta} 2^{-2n} \left|\sum_{k=1}^{m} b_{kij} \operatorname{trace}(A_k h_\theta g_\pi A g_\sigma h_\epsilon)\right|\right)^2\right]^{1/2}.
\end{aligned}$$

Again, by Khinchin's inequality for any $v: l_\infty^n \to l_\infty^n$

$$2^{-n} \left|\sum_{\epsilon} \operatorname{trace}(vh_\epsilon)\right| \geqq e^{-1/2}\left(\sum_{s=1}^{n} \langle v(f_s), e_s\rangle^2\right)^{1/2},$$

therefore

$$\begin{aligned}
&e^{1/2} \sum_{\epsilon,\theta} 2^{-2n} \left|\sum_{k=1}^{m} b_{kij} \operatorname{trace}(A_k h_\theta g_\pi A g_\sigma h_\epsilon)\right| \\
&\quad \geqq 2^{-n} \sum_{\theta} \left[\sum_{s=1}^{n} \left\langle \left(\sum_{k=1}^{m} b_{kij} A_k\right) h_\theta g_\pi A g_\sigma(f_s), e_s\right\rangle^2\right]^{1/2} \\
&\quad \geqq \left[\sum_{s=1}^{n} \left(2^{-n} \sum_{\theta} \left|\left\langle h_\theta g_\pi A g_\sigma(f_s), \left(\sum_{k=1}^{m} b_{kij} A'_k\right)(e_s)\right\rangle\right|\right)^2\right]^{1/2},
\end{aligned}$$

and another application of Khinchin's inequality shows that for any $x \in l_1^n$, $y \in l_\infty^n$

$$\sum_{\theta} 2^{-n} |\langle h_\theta(x), y\rangle| \geqq e^{-1/2}\left(\sum_{r=1}^{n} x_r^2 y_r^2\right)^{1/2}.$$

So

$$e \sum_{\epsilon,\theta} 2^{-2n} \left| \sum_{k=1}^{m} b_{kij} \operatorname{trace}(A_k h_\theta g_\pi A g_\sigma h_\epsilon) \right|$$
$$\geqq \left[\sum_{r,s=1}^{n} \langle g_\pi A g_\sigma(f_s), f_r \rangle^2 \left\langle \sum_{k=1}^{m} b_{kij} A'_k(e_s), e_r \right\rangle^2 \right]^{1/2},$$

and writing $A(f_s) = \Sigma_{t=1}^{n} a_{s,t} e_t$ $(s = 1, \cdots, n)$,

$$e^2 \alpha^*(A) \mu(|\langle \mu, \circ \rangle|)$$
$$\geqq (n!)^{-2} \sum_{\pi,\sigma} \left[\sum_{i,j,r,s=1}^{n} \left(\sum_{k=1}^{m} b_{kij} a_{krs} \right)^2 a^2_{\sigma(s),\pi^{-1}(r)} \right]^{1/2}$$
$$\geqq \left[\sum_{i,j,r,s=1}^{n} \left(\sum_{k=1}^{m} b_{kij} a_{krs} \right)^2 \left(\sum_{\pi,\sigma} (n!)^{-2} |a_{\sigma(s),\pi^{-1}(r)}| \right)^2 \right]^{1/2}$$
$$= n^{-2} \left[\sum_{i,j,r,s=1}^{n} \left(\sum_{k=1}^{m} b_{kij} a_{krs} \right)^2 \right]^{1/2} \left(\sum_{p,q=1}^{n} |a_{p,q}| \right)$$
$$\geqq n^{-2} m^{-1/2} \left(\sum_{p,q=1}^{n} |a_{p,q}| \right) |\operatorname{trace}(u)|$$

where the last inequality is due to Lemma 2.

By duality it follows that for all $B \in L(l_1^n, l_\infty^n)$

$$n^2 e^2 m^{1/2} \| B \| \mu(|\langle u, \circ \rangle|) \geqq \alpha(B) |\operatorname{trace}(u)|,$$

hence for any sequence of operators $P_i \colon E \to E$ satisfying $\Sigma P_i(x) = x$ for all $x \in E$, and for any integer N,

$$\| B \| n^2 e^2 \sup_{\pm} \left\| \sum_{1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\|$$
$$= \| B \| n^2 e^2 \sup_{\pm} \left\| \sum_{1}^{N} \pm \sqrt{r(P'_i)}\, P'_i \right\|$$
$$= \| B \| n^2 e^2 \sup_{\|x\|=\|y'\|=1} \sum_{1}^{N} |\langle P'_i(x), y' \rangle| \sqrt{r(P'_i)}$$
$$\geqq \| B \| n^2 e^2 \sum_{1}^{N} \sqrt{r(P'_i)}\, \mu(|\langle P'_i, \ \rangle|)$$
$$\geqq \alpha(B) \sum_{1}^{N} \operatorname{trace}(P'_i) \xrightarrow[N \to \infty]{} \alpha(B) n^2$$

and the Theorem is established.

COROLLARY 4. *If α is a perfect ideal norm not equivalent to the operator norm $\|\circ\|$ for operators from l_1 to c_0, then $l([L(l_1^n, l_\infty^n), \alpha]) \underset{n\to\infty}{\longrightarrow} \infty$ and $l([L(l_1, c_0), \alpha]) = \infty$.*

Proof. Suppose $l([L(l_1^n, l_\infty^n), \alpha]) \leqq \lambda < \infty$ for all n. Then $\|B\| \leqq \alpha(B) \leqq e^2\lambda \|B\|$ for all compact operators B from l_1 to c_0. Therefore for every operator $B \in L(l_1, c_0)$, $\|B\| \leqq \alpha^{**}(B) \leqq e^2\lambda \|B\|$.

But as α is perfect, $\alpha = \alpha^{**}$, so α is equivalent to the operator norm $\|\circ\|$, which is a contradiction.

REMARKS. Observe that if $\alpha = \|\circ\|$ for operators from l_1 to c_0, then $L(l_1^n, l_\infty^n)$ has an unconditional basis with basis constant equal to 1, the usual basis $f_k \otimes f_j$ $(k, j = 1, \cdots, n)$.

By duality if β is a perfect ideal norm not equivalent to the integral norm $i_1(=\|\circ\|^*)$ for operators from c_0 to l_1, then again

$$l([L(l_\infty^n, l_1^n), \beta]) \underset{n\to\infty}{\longrightarrow} \infty \quad \text{and} \quad l([L(c_0, l_1), \beta]) = \infty.$$

As in Theorem 2, if $\beta(n) = \sup\{\alpha(B)/\|B\|; B \in L(l_1^n, l_\infty^n)\}$ and p_n is a sequence of integers satisfying $\beta(n)p_n^{-1/2} \underset{n\to\infty}{\longrightarrow} \infty$, then the space of compact operators from l_1 to c_0 normed by α does not have an unconditional Schauder decomposition into finite-dimensional spaces E_i with the following property: For any integer n there is a subset I_n of integers such that $L(l_1^n, l_\infty^n)$ is a subspace of $\Sigma_{i\in I_n} \otimes E_i$ where $\dim(E_i) \leqq p_n$ for each $i \in I_n$.

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Received April 29, 1975. Supported in part by N.S.F. Grant GP-34193.

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