PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

MONOTONE BASES IN L_p

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We prove that every monotone basis (decomposition) for $L_p(\mu)$, $1 < p < \infty$, is unconditional. The structure of such bases is closely related to that of the usual Haar basis. This structure is described here, and it is shown that there is an uncountable number of mutually non-equivalent monotone bases for L_p. The structure of monotone bases in L_1 is also considered, and the equivalence question there is characterized in analytic terms.

Introduction. The Theorem (2.1), that every monotone decomposition, and in particular every monotone basis for $L_p(\mu)$, $1 < p < \infty$, is unconditional was discovered also by A. Pełczyński and H. P. Rosenthal **[10]**. The remainder of §2 deals with the structure of monotone bases in $L_p(\mu)$ $(1 < p < \infty)$. In Theorem 2.2 we obtain a representation of a monotone basis for $L_p(0,1)$ as a direct l_p-sum of what we call generalized Haar bases (which are in turn a natural generalization of the classical Haar system). Finally we show that there is a continuum of non-equivalent generalized Haar bases in L_p.

In §3 we study monotone bases on $L_1(0,1)$. First we show how a general monotone basis in $L_1(0,1)$ is obtained from generalized Haar bases, and then we characterize analytically the equivalence of two generalized Haar bases in $L_1(0,1)$.

Section 1 contains notation and preliminaries. Several open questions are stated throughout the paper.

We wish to thank Professors T. Figiel, W. B. Johnson and H. P. Rosenthal for many helpful discussions regarding the material presented here.

1. Notation and preliminaries. We use standard Banach space notation. A sequence of closed subspaces X_n of a Banach space X is said to be a (Schauder) decomposition if every $f \in X$ can be uniquely expressed as $f = \sum_{i=1}^{\infty} f_i$, where $f_i \in X_i$ for all i. The decomposition is called unconditional if $\sum_{i=1}^{\infty} f_i$ converges unconditionally for all f. This is equivalent to the condition $K = \sup\{\|P_E\|; E \subseteq N \text{ finite}\} < \infty$ where P_E is defined by: $P_E f = \sum_{i \in E} f_i$. K is called the unconditional constant of the decomposition. A decomposition is called monotone if $P_n = P_{\{1,2\cdots n\}}$ is a contractive (i.e. norm 1) projection for all n. Thus a monotone decomposition corresponds to a sequence (P_i) of contractive projections satisfying $P_i P_j = P_{\min(i,j)}$.

If $(\Omega, \mathcal{S}, \mu)$ is a measure space (μ is assumed to be finite unless otherwise stated), we shall refer to its L_p-space as $L_p(\mu)$, $L_p(\mathcal{S})$, or $L_p(\Omega)$ according to convenience. If $\Omega_0 \subseteq \Omega$ we shall identify $L_p(\Omega_0)$ with functions in $L_p(\Omega)$ vanishing off Ω_0. If $\mathcal{J}$ is a sub σ-ring of $\mathcal{S}$, $S(\mathcal{J})$ will denote the support of $\mathcal{J}$; i.e. its greatest element, and the conditional expectation $\mathcal{E}_{\mathcal{J}} f = \mathcal{E}_{\mathcal{J},\mu} f$ of $f \in L_1(\Omega, \mathcal{S}, \mu)$ with respect to $\mathcal{J}$ and μ is defined as the unique $g \in L_1(\Omega_0, \mathcal{J}, \mu)$ satisfying $\int_E g d\mu = \int_E f d\mu$ for all $E \in \mathcal{J}$. $\mathcal{E}_{\mathcal{J}}$ is a contractive projection of $L_p(\mathcal{S})$ onto $L_p(\mathcal{J})$, for any $p \geqq 1$. For a function f, $S(f)$ will denote the support of f; for a set A, $\sim A$ will denote the complement of A. m is Lebesgue measure on $[0, 1]$.

The contractive projections in $L_p(\mu)$ were characterized by Douglas **[4]** (for $p = 1$) and Ando **[1]** (for $1 < p < \infty, p \neq 2$) as follows (cf. also **[9]**):

THEOREM A. (i) *Let* $1 < p < \infty$, $p \neq 2$. *If P is a contractive projection in $L_p(\mu)$, then there is a measure ν on $\mathcal{S}$, an isometry T of $L_p(\mu)$ onto $L_p(\nu)$, and a sub σ-ring $\mathcal{J}$ of $\mathcal{S}$, so that*

$$TPT^{-1} = \mathcal{E}_{\mathcal{J},\nu}.$$

(ii) *Let* $p = 1$. *If P is a contractive projection in $L_1(\mu)$, there are ν, T and $\mathcal{J}$ as in* (i) *and a norm* 1(nilpotent) *operator* $N: L_1(\sim S(\mathcal{J})) \to L_1(\mathcal{J})$ *so that*

$$TPT^{-1}(f) = \mathcal{E}_{\mathcal{J},\nu} f + N(f_{|\sim S(\mathcal{J})}).$$

We outline the proof, since a similar construction will be used later. The main part of the proof is to show the following special case:

Fact 1. If P is a projection in $L_p(\mu)$ $(1 \leqq p < \infty, p \neq 2)$ and χ_Ω is in the range $R(P)$ of P then there is a sub σ-algebra $\mathcal{J}$ of $\mathcal{S}$ so that $P = \mathcal{E}_{\mathcal{J}}$.

Also needed is:

Fact 2. Every closed subspace X of $L_p(\mu)$ $(1 \leqq p < \infty)$ contains a function k with greatest support $S(k)$ (i.e. for all $f \in X$, $S(f) \subseteq S(k)$ μ—a.e.).

Now, let k_0 be an element with greatest support in $R(P)$, (we shall then write, $S(P) = S(k_0)$) and let $k = k_0 + \chi_{\sim S(P)}$. Define ν by $d\nu = |k|^p d\mu$ and T: $L_p(\mu) \to L_p(\nu)$ by $Tf = f/k$, if $f \in L_p(\mu)$. $Q = TPT^{-1}$ is a contractive projection in $L_p(\nu)$, and $\chi_{S(Q)} = \chi_{S(P)} \in R(Q)$. Therefore by Fact 1, $Q_{|L_p(S(P))} = \mathcal{E}_{\mathcal{J},\nu}$ for some subσ-ring $\mathcal{J}$ of $\mathcal{S}$ with $S(\mathcal{J}) = S(P)$. Denoting $Q_{|L_p(\sim S(P))} = N$ we have:

$$Qf = \mathcal{E}_{\mathcal{J}}(f_{|S(P)}) + N(f_{|\sim S(P)}).$$

Now, if $1 < p < \infty$, then $L_p(\nu)$ is smooth and hence contractive projections in $L_p(\nu)$ are uniquely determined by their range, (cf. [3]), implying that $N = 0$. For $p = 1, N$ can be any contraction.

The proof of our first result essentially extends Theorem A to sequences of contractive projections (P_i) satisfying $P_iP_j = P_{\min(i,j)}$. We then apply the following result of Burkholder and Gundy (cf. [2], Theorem 9).

THEOREM B. *Let $1 < p < \infty$. If $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ is an increasing sequence of sub σ-algebras of $\mathcal{S}$ which generate the σ-algebra $\mathcal{J}$, then the monotone Schauder decomposition $(R(\mathcal{E}_{\mathcal{J}_i} - \mathcal{E}_{\mathcal{J}_{i-1}}), i = 1, 2, \cdots)$ for $L_p(\mathcal{J})$ is unconditional. Moreover, there is a constant K_p, depending only on p so that the unconditional constant of this decomposition is smaller then K_p.*

2. Monotone bases in L_p $(1 < p < \infty)$.

THEOREM 2.1. *Let (P_i) be a sequence of contractive projections in $L_p(\Omega, \mathcal{S}, \mu)$, $(1 < p < \infty, p \neq 2)$, with $P_iP_j = P_{\min(i,j)}$. Then there is a measure ν on Ω, an isometry T of $L_p(\mu)$ onto $L_p(\nu)$ and a sequence of sub σ-rings $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \subseteq \mathcal{S}$ so that $P_i = T^{-1}\mathcal{E}_{\mathcal{J}_i}T$.*

Proof. We first note that Theorem A (and the definition of T in its proof) implies:

(*) If $h, f \in R(P)$, then $h \cdot \chi_{S(f)} \in R(P)$.

Let $k_1 \in R(P_1)$ with $S(k_1) = S(P_1)$. If $S(P_2) \supsetneqq S(P_1)$ use (*) to choose $k_2 \in R(P_2)$ with $S(k_1) \cap S(k_2) = \varnothing$ and $S(k_1) \cup S(k_2) = S(P_2)$. If $S(P_2) = S(P_1)$ we proceed to P_3 and continue in this manner. We obtain a (possibly finite) sequence (k_i) of disjointly supported functions and integers $n(1) \leqq n(2) \leqq \cdots$ with the property that for each each i,

$$S(P_i) = \bigcup_{j=1}^{n(i)} S(k_j)$$

and $k_j \in R(P_i)$ for $j \leqq n(i)$. We may assume $k = \Sigma_1^\infty k_i \in L_p(\mu)$ and proceed to define ν and T as in the proof of Theorem A, i.e. $d\nu = |k|^p d\mu$ and $Tf = f/k$. Clearly $Q_i = TP_iT^{-1}$ satisfies $Q_i(\chi_{S(Q_i)}) = \chi_{S(Q_i)}$ and $Q_iQ_j = Q_{\min(i,j)}$. By Theorem A there are sub σ-rings $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \subseteq \mathcal{S}$ with $Q_i = \mathcal{E}_{\mathcal{J}_i}$ for i.

COROLLARY 1. *A monotone decomposition in $L_p(\mu)$, $(1 < p < \infty)$ is unconditional with constant $\leqq K_p$, where K_p depends only on p.*

Proof. For $p = 2$ this is well known. If $p \neq 2$, we apply 2.1, and observe, that in the notation of its proof, we have:

(a) $$Q_i f = \sum_j \chi_{S(k_j)} \mathscr{E}_{\mathscr{F}_i}(\chi_{S(k_j)} \cdot f), \qquad f \in L_p(\nu),$$

(b) for fixed j the non zero projections $f \to \chi_{S(k_j)} \mathscr{E}_{\mathscr{F}} f$ in $L_p(S(k_j))$ are conditional expectations with respect to *σ-algebras* on $S(k_j)$, and
(c) the direct l_p-sum of projections of norm smaller then K_p has norm smaller than K_p. The rest follows from Theorem B.

REMARK. Corollary 1 holds for arbitrary measures μ. In fact, for any given $f \in L_p(\mu)$ we can find a sub σ-ring $\Sigma_0 \subseteq \Sigma$ so that $L_p(\Sigma_0)$ is separable, contains f and is an invariant subspace for each projection P_n (cf. **[11]**, Lemma 1 and its proof). Then by Corollary 1,

$$\left\| \sum_n \epsilon_n (P_n - P_{n-1}) f \right\| \leqq K_p \, \| f \|$$

for all $\epsilon_n = \pm 1$.

COROLLARY 2. *If* (X_i) *is a monotone decomposition for* $L_p(\mu)$, $(1 < p < \infty)$ *with each* X_i *finite-dimensional, then there is a monotone basis* (x_i) *and integers* $1 = n(0) < n(1) < \cdots$ *so that* $X_i = [x_j;\, n(i-1) \leqq j < n(i)]$.

We proceed to describe more precisely the monotone bases in $L_p(0,1)$, $1 < p < \infty, p \neq 2$. For clarity of exposition we shall state the results for separable $L_p(\mu)$ where μ is a purely nonatomic probability measure.

A system of sets $(A_{n,i};\, i \leqq 2^n, n = 0,1,2,\cdots)$ is called a *dyadic tree* if for all n and $i \leqq 2^n$

$$A_{n+1,2i-1} \cap A_{n+1,2i} = \varnothing$$

and

$$A_{n+1,2i-1} \cup A_{n+1,2i} = A_{n,i}.$$

DEFINITION. Let $1 < p < \infty$, and let $(A_{n,i}, i \leqq 2^n, n = 0,1,2,\cdots)$ be a dyadic tree in $\mathscr{S}$. The *generalized Haar system* $(h_k, k = 1,2,\cdots)$ with respect to $(A_{n,i})$ is defined as follows:

$$h_1 = h_{0,1} = \chi_{A_{0,1}} / \| \chi_{A_{0,1}} \|_p$$

and:

$$h_{2^{n-1}+i} = h_{n,i} = H_{n,i} / \| H_{n,i} \|_p,$$

where

$$H_{n,i} = \{\chi_{A_{n,2i-1}}/\mu_{(A_{n,2i-1})} - \chi_{A_{n,2i}}/\mu_{(A_{n,2i})}\},$$
$$\text{for } i \leqq 2^{n-1},\ n \geqq 1.$$

The system $(h_{n,i})$ is determined by the conditions: $h_{n,i}$ is a linear combination of $\chi_{A_{n,2i-1}}$ and $\chi_{A_{n,2i}}$ which is positive on $A_{n,2i-1}$ and satisfies:

$$\| h_{n,i} \|_p = 1 \quad \text{and} \quad \int h_{n,i} d\mu = 0 \qquad (n \geqq 1).$$

If $(A_{n,i})$ are the dyadic intervals in $[0,1]$ and μ is the Lebesgue measure on $[0,1]$, this gives the usual Haar system in L_p. It is easily seen, that a generalized Haar system is a monotone basic sequence, which spans the space $L_p(\mathcal{J})$, where $\mathcal{J}$ is the σ-algebra generated by the $A_{n,i}$. If $\mathcal{J} = \mathcal{S}$ we must have $\mu(A_{n,i}) \to 0$; on the other hand if the $A_{n,i}$ are intervals in $[0,1]$ and $m(A_{n,i}) \to 0$, then $\mathcal{J} = \mathcal{S}$.

THEOREM 2.2. *Let (x_k) be a normalized monotone basis for $L_p(\mu)$, μ purely nonatomic, $1 < p < \infty, p \neq 2$. Then there is a measure ν, an isometry T of $L_p(\mu)$ onto $L_p(\nu)$ which sends (x_k) to a basis (y_k), and a sequence* (possibly finite) *of disjoint sets (E_n) in $\mathcal{S}$, covering Ω, so that (y_k) is the union of disjoint subsequences $(y_i^n, i = 1, 2, \cdots)$, $n = 1, 2, \cdots$ where for each n, $(y_i^n, i = 1, 2, \cdots)$ is a permutation of a generalized Haar basis for $L_p(E_n)$.*

Proof. By Theorem 2.1 we may assume that $P_i = \mathcal{E}_{\mathcal{J}_i}$ for each i, where $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ are sub σ-rings of $\mathcal{S}$, and $P_i\colon L_p(\mu) \to [x_1, \cdots, x_i]$ are the projections associated with the basis (x_i). For each i, we have: $L_p(\mathcal{J}_i) = R(P_i) = [x_1, \cdots, x_n]$ and so $\mathcal{J}_i$ is generated by i atoms. For each i there are two cases:

1°. $S(P_i) = S(P_{i-1})$ 2°. $S(P_i) \supsetneqq S(P_{i-1})$.

In case 1°, $\mathcal{J}_i$ is obtained from $\mathcal{J}_{i-1}$ by splitting some set A in $\mathcal{J}_{i-1}$ into two sets. Clearly $S(x_i) = A$ and $\int_A x_i = 0$. In case 2° $\mathcal{J}_i$ is obtained from $\mathcal{J}_{i-1}$ by adding an atom D disjoint from the $i-1$ atoms of $\mathcal{J}_{i-1}$. Then $P_{i-1}\chi_D = 0$ so that $x_i = \pm \chi_D / \nu(D)^{1/p}$ (being norm 1). We enumerate all the x_i obtained in 2° as $\{x_1^n\colon n = 1, 2, \cdots\}$ and for each n enumerate the functions $\{x_i\colon S(x_i) \subseteq S(x_1^n)\}$ as $(x_j^n)_{j=2}^{\infty}$. This is clearly the required partition.

REMARK. In the above Theorem we could have let ν = Lebesgue measure m, on $[0,1]$. Indeed there exist disjoint intervals $E_n \subseteq [0,1]$ with

$m(E_n) = \mu(S(x_1^n))$ and a map ϕ from $\cup \mathcal{J}_i$ into the intervals contained in $[0,1]$ which preserves inclusion, disjointness and measure, such that for any x_i of type 2°, $\phi(\{t: x_i > 0\})$ is to the left of $\phi(\{t: x_i < 0\})$. This map extends to an isomorphism of the measure space $(\Omega, \mathcal{J}, \mu)$ onto the Lebesgue measure space on $[0,1]$. Thus to study monotone bases in $L_p(\mu)$, one need only study generalized Haar systems with respect to dyadic trees and one can assume that the interval where x_i is positive is to the left of the interval where it is negative.

We turn now to the question of equivalence of Haar bases for L_p, $1 < p \neq 2$. A basis (x_n) is said to be *K-equivalent* to a basis (y_n), $(x_n) \overset{K}{\sim} (y_n)$, if for all n and all scalars $\alpha_1, \cdots, \alpha_n$,

$$K^{-1}\left\|\sum_1^n \alpha_i x_i\right\| \leqq \left\|\sum_1^n \alpha_i y_i\right\| \leqq K\left\|\sum_1^n \alpha_i x_i\right\|.$$

If $(h_{n,i})$ is a generalized Haar basis for L_p we define its generalized Rademacher functions r_n by:

$$r_n = 2^{(1-n)/p}(h_{n,1} + h_{n,2} + \cdots + h_{n,2^{n-1}}).$$

THEOREM 2.3. *There exist two nonequivalent generalized Haar bases for $L_p(0,1)$, $(1 < p < \infty, p \neq 2)$.*

Proof. Let $(h_{n,i})$, (r_i) denote the classical Haar and Rademacher systems. By Khintchine's inequality (cf. **[12]**), (r_n) is equivalent to the usual basis of l_2. We shall construct a generalized system $(h'_{n,i})$, (r'_n) so that (r'_{2^n}) is equivalent to the usual basis for l_p, and hence $(h_{n,i}) \not\sim (h'_{n,i})$.

It is easy to check, that if $h = a\chi_{E_1} - b\chi_{E_2}$ is a generalized Haar function, then $\|h_{|E_2}\|$ approaches 1 as $m(E_2)/m(E_1) \to 0$. (This does not happen of course for $p = 1$).

We shall have $(r'_{2^n}) \sim$ usual basis of l_p if there are disjoint sets E_n so that:

$$\int_{E_k} |r'_{2^k}|^p > 1 - 4^{-(k+1)p}, \qquad k = 1, 2, \cdots \tag{1}$$

and

$$\int_{E_k} \sum_{j=1}^{k-1} |r'_{2^j}|^p < 4^{-(k+1)p}, \qquad k = 1, 2, \cdots \tag{2}$$

(see **[7]**, proof of Theorem 2).

Let $h'_{0,1} = 1$, $h'_{1,1} = h_{1,1}$, and assume that $(h'_{k,i})$ and E_j are chosen for $1 \leqq k \leqq 2n-1$, $i \leqq 2^{k-1}$, $1 \leqq j \leqq n-1$, so that (1) and (2) hold for

$k=1,\cdots,n-1$. Let $(A_{k,i},\ k \leqq 2n-1,\ i \leqq 2^k)$ be the underlying intervals. For each $i \leqq 2^{2n-1}$, divide $A_{2n-1,i}$ into two disjoint intervals $A_{2n,2i-1}$ and $A_{2n,2i}$ with $m(A_{2n,2i})$ so small that $\| h'_{2n,i}|_{A_{2n,2i}} \| > 1-4^{-(n+1)p}$, and $\sum_{i \leqq 2^{2n-1}} m(A_{2n,2i}) \leqq \epsilon_n$, $\epsilon_n > 0$ being chosen so that $M(E) \leqq \epsilon_n$ implies

$$\int_E \sum_{j \leqq n-1} |r'_{2j}|^p < 4^{-(n+1)p}.$$

Let $E = \bigcup_{i \leqq 2^{2n-1}} A_{2n,2i}$. Then we have:

$$\int_{E_n} |r'_{2n}|^p = \frac{1}{2^{2n-1}} \sum_{i=1}^{2^{2n-1}} \| h'_{2n,i|A_{2n,2i}} \|^p > 1 - 4^{-(n+1)p}.$$

Thus (1) and (2) hold for $k=n$. Define now the functions $h'_{2n+1,i}$, $i \leqq 2^{2n}$ by splitting each $A_{2n,i}$ into two intervals of equal measure. (This ensures that $m(A_{n,i}) \to 0$ and so $(h_{n,i})$ is a basis for all of L_p).

Using an idea of J. Hennefeld **[5]**, we can now prove:

COROLLARY. *There is an uncountable family of mutually nonequivalent generalized Haar bases for L_p.*

Proof. Let (E_n) be a partition of $[0,1]$ into infinitely many disjoint adjacent intervals, ordered from left to right. Define part of the tree $(A_{n,i})$ as follows: for any $n \geqq 1$ let $A_{n,2^n-1} = E_n$, and $A_{n,2^n} = \bigcup_{j>n} E_j$. Now, given a sequence (ϵ_n), $\epsilon_n = \pm 1$, complete the tree $(A_{n,i})$ so that the system $(h_{n,i}) = \mathscr{H}$ satisfies the condition that for $\epsilon_m = 1$ the sequence $\{h \in \mathscr{H};\ S(h) \subseteq E_m\}$ in its natural ordering is equivalent to the usual Haar basis (without constant term), while for $\epsilon_m = -1$ it is equivalent to the basis $h'_{n,i}$ of 2.3. Different sequences (ϵ_n) yield non-equivalent systems $(h_{n,i})$.

Questions. (1) Does every generalized Rademacher system span a complemented subspace of L_p? If so could this be used to construct an $\mathscr{L}_p$ space not isomorphic to any of those already known?

(2) Do there exist two non-permutatively equivalent generalized Haar bases? (We can show that (for $p>2$) some permutation of the generalized Haar basis constructed in 2.3 has its generalized Rademacher system equivalent to the unit vector basis of l_2.)

3. Monotone bases in L_1. Monotone bases in L_1 are also built from generalized Haar bases, however the "interlace" is somewhat more involved, due to the larger variety of contractive projections in L_1 (cf. Theorem A(ii).):

THEOREM 3.1. *Let (x_k) be a normalized monotone basis for $L_1(\mu)$, μ purely non atomic. Then there is an isometry T of $L_1(\mu)$ onto some $L_1(\nu)$, which sends (x_k) to a basis (y_k), and a sequence* (possibly finite) *of disjoint sets E_n in $\mathcal{S}$, covering Ω, so that (y_k) is the union of disjoint subsequences $(y_i^n, i = 1, 2, \cdots)$, $n = 1, 2, \cdots$, where for each n, the sequence: $\chi_{E_n}/\|\chi_{E_n}\|$, $y_2^n, y_3^n, \cdots$ is a generalized Haar basis for $L_1(E_n)$. Moreover, $y_1^n = c_n\chi_{E_n} + f_n$ where $\|f_n\| \leqq \|c_n\chi_{E_n}\|$ and f_n is a combination of the elements (y_k) preceding y_1^n in the original sequence (y_k).*

Proof. Let (P_n) be the projections associated with the basis (x_n). Using Theorem A(ii) and the proof of Theorem 2.1, we get an isometry T of $L_1(\mu)$ onto some $L_1(\nu)$ and a sequence of sub σ-rings $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \subseteq \mathcal{S}$ so that the projections $Q_i = TP_iT^{-1}$ have the form $Q_if = \mathcal{E}_{\mathcal{J}_i}f + N_i(f_{\sim S(\mathcal{J}_i)})$, N_i being some norm 1 operator from $L_1(\sim S(\mathcal{J}_i), \mathcal{S}, \nu)$ to $L_1(\Omega, \mathcal{J}_i, \nu)$. Let $y_i = Tx_i$. We have two cases: 1° $S(Q_i) = S(Q_{i-1})$ and 2° $S(Q_i) \supsetneqq S(Q_{i-1})$. In case 1° $\mathcal{J}_i$ is obtained by splitting an atom A in $\mathcal{J}_{i-1}$ and y_i is a Haar function supported on A, while in case 2° $\mathcal{J}_i$ is obtained by adding an atom D disjoint from $S(\mathcal{J}_{i-1})$. In the latter case $Q_{i-1}(\chi_D - N_{i-1}\chi_D) = 0$, so $y_i = c(\chi_D - N_{i-1}\chi_D)$, where $\|N_{i-1}\chi_D\| \leqq \|\chi_D\|$ and $N_{i-1}\chi_D$ is $\mathcal{J}_{i-1}$-measurable.

In the rest of this section we examine the question when two generalized Haar bases in $L_1[0,1]$ are equivalent. If $(h_{n,i})$ is such a basis, then a sequence $(h_{n,i(n)}, n = 0, 1, 2, \cdots)$ will be called a chain if $S(h_{n,i(n)}) \subseteq S(h_{n-1,i(n-1)})$ for all n. Now, *two generalized Haar bases $(h_{n,i})$ and $(g_{n,i})$ are equivalent if (and only if) there is K so that every chain of $(h_{n,i})$ is K-equivalent to the corresponding chain in $(g_{n,i})$.* In fact, suppose that $(h_{n,i})$ is built on the dyadic tree of sets $(A_{n,i})$. Then $[h_{k,i}, 1 \leqq i \leqq 2^{k-1}, 0 \leqq k \leqq n] = [\chi_{A_{n,i}}; i \leqq 2^n]$ and any operator on this space attains its norm at one of the $\chi_{A_{n,i}}$, since the convex hull of $\{\pm\chi_{A_{n,i}}/m_{(A_{n,i})}, i \leqq 2^n\}$ is the unit ball of $[\chi_{A_{n,i}}; i \leqq 2^n]$. But each $\chi_{A_{n,i}}$ is contained in the span of a chain.

Thus it is enough to consider the equivalence of chains. For simplicity we shall consider only the chain $(h_{n,1})$, however the results obviously apply to any chain.

THEOREM 3.2. *Let $(h_{n,i})$ be the generalized Haar system based on the dyadic tree of sets $(A_{n,i})$, and let $(g_{n,i})$ be the generalized Haar system based on $(B_{n,i})$. Let $Th_{n,i} = g_{n,i}$, and define*

$$P_n = \int_{B_{n,1}} T(\chi_{A_{n,1}}/m_{(A_{n,1})}); \qquad q_n = \int_{A_{n,1}} T^{-1}(\chi_{B_{n,1}}/m_{(B_{n,1})}).$$

Then $(h_{n,1}) \sim (g_{n,1})$ iff $M = \max\{\mathrm{var}(p_n), \mathrm{var}(q_n)\} < \infty$, and the equivalence constant K satisfies: $M \leqq K \leqq 2M + 3$

$$\left(\textit{where, as usual, } \mathrm{val}(p_n) = \sum_{n=1}^{\infty} |p_n - p_{n+1}|\right).$$

Proof. Let $e_{n,i} = \chi_{A_{n,i}}/m_{(A_{n,i})}$. We have:

(3) $$e_{n,1} = e_{n-1,1} + 2c_n h_{n,1}, \quad \text{where}$$

(4) $$c_n = m(A_{n,2})/m(A_{n-1,1}).$$

(check their integrals on $A_{n-1,1}$ and on $A_{n,2}$).

Thus for any $k \leqq n-1$ and $i \leqq 2^k$, we have:

$$\int_{B_{k,i}} Te_{n,1} = \int_{B_{k,i}} Te_{n-1,1} + 2c_n \int_{B_{k,i}} g_{n,1} = \int_{B_{k,i}} Te_{n-1,1} = \int_{B_{k,i}} Te_{k,1},$$

and so

$$\int_{B_{k,2}} Te_{n,1} = \int_{B_{k-1,1}} Te_{n,1} - \int_{B_{k,1}} Te_{n,1} = p_{k-1} - p_k.$$

Now, $Te_{n,1}$ is constant on $B_{k,2}$, $(k \leqq n)$, and $B_{n,1}$, so that:

$$\| Te_{n,1} \| = \sum_{k=1}^{n} \left| \int_{B_{k,2}} Te_{n,1} \right| + \left| \int_{B_{n,1}} Te_{n,1} \right| = \sum_{k=1}^{n} | p_{k-1} - p_k | + | p_n |.$$

Finally, $e_{n,2} = e_{n-1,1} - 2(1-c_n)h_{n,1}$, similarly to (3), so $\| Te_{n,2} \| \leqq \| Te_{n-1,1} \| + 2$, and the unit ball of $[h_{n,1}, n = 0, 1, \cdots]$ is the closed convex hull of the set $\{\pm e_{n,2}, n = 1, 2, \cdots\}$.

From (3) and the definition of the Haar functions $g_{n,i}$ we get that

(5) $$p_n = m(B_{n,1})\left\{1 + \sum_{k=1}^{n} c_k / m(B_{k,1})\right\}.$$

Applying Stolz's theorem (i.e. the discrete version of L'Hospital's rule, cf. **[8]** p. 77, Remark 5) to p_n, and putting;

(6) $$d_n = m(B_{n,2})/m(B_{n-1,1}),$$

we see that if $\lim_n c_n/d_n = \lambda$ exists then $\lim_n p_n = \lambda$. Given a sequence (c_n), there is a generalized Haar system $(h_{n,i})$ for which (4) holds provided that:

(7) $$0 < c_n < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = \infty$$

(The latter condition ensures that $m(A_{n,1}) = \Pi_{i=1}^{n}(1-c_i) \xrightarrow{n} 0$). In particular, if we take $c_n = (n+1)^{-\alpha}$, for fixed $0 < \alpha \leqq 1$, then different values of α give mutually non-equivalent generalized Haar bases.

The considerations above motivate:

THEOREM 3.3. *Let $(h_{n,i})$, $(g_{n,i})$ be two generalized Haar systems, built on the dyadic trees $(A_{n,i})$, $(B_{n,i})$ respectively. Let $c_n = m(A_{n,2})/m(A_{n-1,1})$, $d_n = m(B_{n,2})/m(B_{n-1,1})$. If*

$$\operatorname{var}\left(\frac{c_n}{d_n}\right),\ \operatorname{var}\left(\frac{d_n}{c_n}\right) \leqq M < \infty$$

then the chains $(h_{n,1})$ and $(g_{n,1})$ are equivalent (with constant $\leqq 2M+3$).

Proof. In formula (5), putting: $h_{k,i} = g_{k,i}$, we get

$$1 = m(B_{n,1})\left\{1 + \sum_{k=1}^{n} d_k / m(B_{k,1})\right\},$$

so

$$p_n = \left\{1 + \sum_{k=1}^{n} c_k / m(B_{k,1})\right\} \Big/ \left\{1 + \sum_{k=1}^{n} d_k / m(B_{k,1})\right\}.$$

It is enough therefore to apply the following:

LEMMA. *Let (a_n), (b_n) be sequences of reals with all $b_n > 0$, and let*

$$A_n = \sum_{k=1}^{n} a_k, \qquad B_n = \sum_{k=1}^{n} b_k.$$

Then

$$\operatorname{var}\left(\frac{A_n}{B_n}\right) \leqq \operatorname{var}\left(\frac{a_n}{b_n}\right).$$

Proof. Let $a_k = t_k b_k$. Using Abel's transform, we have:

$$A_n / B_n = B_n^{-1} \sum_{k=1}^{n} t_k b_k = B_n^{-1} \sum_{k=1}^{n-1} (t_k - t_{k+1}) B_k + t_n, \text{ which gives:}$$

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \left(\frac{1}{B_n} - \frac{1}{B_{n+1}}\right) \sum_{k=1}^{n} (t_{k+1} - t_k) B_k, \text{ so that:}$$

$$\begin{aligned}
\operatorname{var}\left(\frac{A_n}{B_n}\right) &= \sum_{n=1}^{\infty} \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| \\
&\leqq \sum_{n=1}^{\infty} \left(\frac{1}{B_n} - \frac{1}{B_{n+1}}\right) \sum_{k=1}^{n} | t_{k+1} - t_k | B_k \\
&= \sum_{k=1}^{\infty} | t_{k+1} - t_k | \cdot B_k \sum_{n=k}^{\infty} \left(\frac{1}{B_n} - \frac{1}{B_{n+1}}\right) \leqq \operatorname{var}(t_k).
\end{aligned}$$

REMARKS. (1) It is conceivable that the condition in Theorem 3.3 is also necessary. We can prove only that if $(h_{n,1})$ and $(g_{n,1})$ are equivalent and either $\inf c_n > 0$ or $\inf d_n > 0$, then $\operatorname{var}(c_n/d_n) < \infty$.

(2) If $(h_{n,i})$ is a generalized Haar basis for L_p, then the chain $(h_{n,1};\, n = 0, 1, \cdots)$ spans a space isometric to l_p. In L_1 these chains are conditional bases for l_1 (by (7), (3) and **[6]**, Lemma 2). As, shown above, there is an uncountable family of mutually non-equivalent such chains.

For $1 < p \neq 2$, we do not know if all chains in L_p are equivalent.

REFERENCES

1. T. Ando, *Contractive projections in L_p spaces*, Pacific J. Math., **17** (1966), 391–405.
2. D. L. Burkholder, *Martingale transforms*, Ann. Math. Stat., **37** (1966), 1494–1504.
3. H. B. Cohen, F. E. Sullivan, *Projections onto cycles in smooth reflexive Banach spaces*, Pacific J. Math., **34** (1970), 355–364.
4. R. G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math., **15** (1965), 443–462.
5. J. Hennefeld, *On nonequivalent normalized unconditional bases for Banach spaces*, Proc. Amer. Math. Soc., **41** (1973), 156–158.
6. W. B. Johnson, E. Odell, *Subspaces of L_p which imbed into l_p*, Composito Math., **28** (1974), 37–49.
7. M. I. Kadec, A. Pełczyński. *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math., **21** (1962), 161–176.
8. Knopp, *Theory and applications of infinite series*, Glasgow 1966.
9. H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer Verlag, 1974.
10. A. Pełczyński, H. P. Rosenthal, *Localization techniques in L^p spaces*, to appear.
11. L. Tzafriri, *Remarks on contractive projections in L_p-spaces*, Israel J. Math., **7** (1969), 9–15.
12. A. Zygmund, *Trigonometric series*, Cambridge 1968.

Received July 17, 1974.

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