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ABELIAN GROUPS IN WHICH EVERY ENDOMORPHISM IS A LEFT MULTIPLICATION

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Let $\langle G+\rangle$ be an abelian group. With each multiplication on G (binary operation $*$ such that $\langle G+*\rangle$ is a ring) and each $g \in G$ is associated the endomorphism g_l^* of left multiplication by g. Let $L(G)=\{g_l^* \mid g \in G, * \varepsilon \text{ Mult } G\}$. Abelian groups G such that $L(G)=E(G)$ are studied. Such groups G are characterized if G is torsion, reduced algebraically compact, completely decomposable, or almost completely decomposable of rank two. A partial results is obtained for mixed groups.

Let $\langle G+\rangle$ be an abelian group. With each multiplication on G (binary operation $*$ such that $\langle G+*\rangle$ is a ring) and each $g \in G$ is associated the endomorphism g_l^* of left multiplication by g given by $g_l^*(x)=g*x$, $x \in G$. Let $L(G)$ be the set of all such endomorphisms, i.e., $L(G)=\{g_l^* \mid g \in G, * \varepsilon \text{ Mult}(G)\}$. In general all one can say is that $L(G)$ is a subset of the endomorphism ring $E(G)$. In this paper we consider abelian groups G such that every endomorphism is a left multiplication.

DEFINITION 1. An abelian group G is multiplicatively faithful iff $L(G)=E(G)$.

We mostly follow the notations in [2]. Specifically: all groups are abelian, rings are not necessarily associative, $\otimes$ denotes the tensor product over Z and $g\otimes\_$ the natural map $x \to g \otimes x$ from G into $G \otimes G$, $o(x)$ is the order of an element x, $Z(d)$ is the cyclic group of order d and $Z(d)^*$ is the multiplicative group of units in $Z(d)$. For a prime p, we write Z_p for the localization of Z at p and $\hat{Z}_p$ for the ring (or group) of p-adic integers. We use $t(A)[t(x)]$ for the type of a rank one torsion free group A [element x] and $h(x)$ for the height sequence. Finally, $\langle S\rangle[\langle S\rangle_*]$ is the subgroup [pure subgroup] generated by S.

We begin by listing some simple results.

A. Let $\theta_g\colon \operatorname{Hom}(G\otimes G, G) \to E(G)$ be given by $\theta_g(\Delta)=\Delta\circ(g\otimes\_)$, $\Delta \in \operatorname{Hom}(G\otimes G, G)$, $g \in G$. Then G is multiplicatively faithful iff $\bigcup_{g\in G} \operatorname{Image}\theta_g = E(G)$.

Proof. Mult G, the group of all multiplications on G, is isomorphic to $\mathrm{Hom}(G \otimes G, G)$. Under this identification $\Delta \circ (g \otimes \_) = g_l$.

B. G is multiplicatively faithful iff for each $\theta \in E(G)$, there exists $u \in G$, $\sigma \in \mathrm{Mult}\, G$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[rr, "u \otimes \_"] \arrow[dr, "\theta"'] & & G \otimes G \arrow[dl, "\sigma"] \\ & G & \end{tikzcd}$$

Proof. Obvious.

C. A divisible group is multiplicatively faithful iff it is torsion free. More generally, if $G = D \oplus R$, D the maximal divisible subgroup of G with D torsion free, then $L(G) = E(G)$ iff $L(R) = E(R)$.

Proof. This follows directly from (B) and elementary properties of the tensor product.

D. If Z is a direct summand of G, then $L(G) = E(G)$. More generally, if A is a ring, $1 \in A$, and H is a unital A module, then $A \oplus H$ is multiplicatively faithful.

Proof. Let $\theta \in E(A \oplus H)$. Set $u = 1 \in A$, and define $\sigma \in \mathrm{Mult}\, G$ by $\sigma(\Sigma a_i \otimes x_i \oplus y) = \Sigma a_i \theta(x_i)$; $a_i \in A$, $x_i \in A \oplus H$, $y \in H \otimes (A \oplus H)$. Then

$$\begin{tikzcd} A \oplus H \arrow[rr, "1 \otimes \_"] \arrow[dr, "\theta"'] & & (A \oplus H) \otimes (A \oplus H) \arrow[dl, "\sigma"] \\ & A \oplus H & \end{tikzcd}$$

commutes.

E. Let $R(G)$ be the set of all right multiplications by elements of G for all rings on G. Then $L(G) = E(G)$ iff $R(G) = E(G)$.

Proof. This follows from considering opposite rings.

Multiplicatively faithful torsion groups are easily characterized.

THEOREM 1. *Let G be a torsion group. Then G is multiplicatively faithful iff G is bounded.*

Proof. If $L(G) = E(G)$, then there exists $u \in G$, $\sigma \in \mathrm{Mult}\, G$ such that $\sigma \circ (u \otimes \_) = 1_G$, where 1_G is the identity endomorphism. It follows that $nG = (0)$, where $n = o(u)$. If $nG = (0)$, $n \in Z^+$, we can write $G = Z(n) \oplus H$. (D) applies to give $L(G) = E(G)$.

We next consider mixed groups, and characterize the multiplicatively faithful ones in one special case.

THEOREM 2. *Let G be mixed with maximal torsion subgroup $T = \bigoplus_p T_p$. Suppose that $T_p \neq (0)$ for only a finite number of primes p, and also that G/T is homogeneous completely decomposable. Then $L(G) = E(G)$ iff* (1) $G = T \oplus F$, (2) *each rank* 1 *summand of G/T has idempotent type,* (3) $p(G/T) = G/T$ *implies T_p is bounded.*

Proof. Suppose (1), (2) and (3) hold for G as above. Let $T = T_1 \oplus T_2$, where T_1 is the sum of the bounded and T_2 the sum of the unbounded p components of T. Since T_1 is bounded, write $T_1 = Z(n) \oplus X$ with X a unital $Z(n)$ module. $F \cong G/T$ is homogeneous, completely decomposable and nonzero. Say $F = A \oplus B$ where A is torsion free of rank one and $B = \bigoplus_{\alpha \in I} (A)_\alpha$. ($I = \varnothing$ is allowed.) Since $t(A)$ is idempotent, A is (may be regarded as) a subring with identity of Q ([2], Th. 121.1). Moreover, since $pA = A$ only when $(T_2)_p = (0)$, $B \oplus T_2$ may be made into a unital A module in the natural way. Thus, $X \oplus B \oplus T_2$ is a unital $Z(n) \oplus A$ module and (D) applies to show G is multiplicatively faithful.

Conversely, let $L(G) = E(G)$ for G satisfying the conditions of our theorem. Let $u \in G$ be such that $u_l^* = 1_G$, $*$ some multiplication on G. If $u \in pG$, clearly $T_p = (0)$.

Now consider a prime p such that $u + T \in p(G/T)$. Since $(u + T)_l$ induces the identity endomorphism on G/T, it follows immediately that $u + T \in p^n(G/T)$ for all $n \in Z^+$. Write $u = pg + t = pg + t_1 + t_2$, where $o(t_1) = p^k$, $(o(t_2), p) = 1$. If $t_1 = 0$, then $u \in pG$ and $T_p = (0)$. If $t_1 \neq 0$, then, for all $x \in T_p$,

$$x = u * x = (pg + t_1 + t_2) * x = p(g * x) + t_1 * x\ .$$

(Since $(o(t_2), p) = 1$ and $x \in T_p$, $t_2 * x = 0$.) But $o[p(g * x)] < o(x)$, $o(t_1 * x) \leqq o(x)$, so $o(x) = o(t_1 * x) \leqq o(t_1)$. Thus T_p is bounded.

Thus, for each p such that $u + T \in p(G/T)$, we have $u + T \in p^n(G/T)$ for all $n \in Z^+$, and T_p is bounded. Since $t(u + T)$ is the type of each rank 1 summand of G/T—(recall G/T is homogeneous)—(2) and (3) hold. Let T_1, T_2 be as before. Since T_1 is bounded, $G = T_1 \oplus H$ with $T_2 \subseteqq H$.

To establish (1), we must show that T_2 is a direct summand of H. Write H/T_2 as a direct sum of isomorphic rank one groups, $H/T_2 = \bigoplus A_i$, and let $A_i = \langle a_i + T_2 \rangle_*$ where $h(a_i + T_2) = (m_{ij})$, $m_{ij} = 0$ or ∞ for all i, j. Since $p(H/T_2) = H/T_2 \rightarrow (T_2)_p = (0)$, the following implication holds: $a_i + T_2 \in p(H/T_2) \rightarrow a_i \in pH$. From this one easily obtains $H = T_2 \oplus F$, where $F = \langle\{a_i\}\rangle_*$.

REMARK. The condition $T_p \neq (0)$ for only finitely many p is necessary for the theorem. Let $G = \prod_p Z(p)$. Then $T(G) = \bigoplus_p Z(p)$ is not a direct summand of G. However, $G/T(G)$ is homogeneous completely decomposable (torsion free divisible) and—as we shall see in Theorem 3 — $L(G) = E(G)$.

We next characterize reduced algebraically compact multiplicatively faithful groups. If G is reduced algebraically compact, then $G = \prod_p G_p$, where each G_p is a complete module over $\hat{Z}_p$. Since each G_p is fully invariant in G ($qG_p = G_p$ for all $q \neq p$) and since $\mathrm{Hom}(G_p \otimes G_q, G_r) = (0)$ unless $p = q = r$, it follows that $L(G) = E(G)$ iff $L(G_p) = E(G_p)$ for all p. Each G_p may be written as a completion: $G_p = (B_p^0 \oplus B_p)\hat{}$, where $B_p^0 = \bigoplus_{\alpha \in I} (\hat{Z}_p)_\alpha$, $B_p = \bigoplus_{\beta \in J} Z(p^{k_\beta})$, $0 < k_\beta < \infty$. (See [2], § 40 for details.)

THEOREM 3. *Let G be reduced algebraically compact. Then G is multiplicatively faithful iff, for each p, either $B_p^0 \neq (0)$ or G_p is bounded.*

Proof. If G_p is bounded, then $L(G_p) = E(G_p)$ by Theorem 1. If $B_p^0 \neq (0)$, write $B_p^0 = \hat{Z}_p \oplus B'$. Then $G_p = (\hat{Z}_p \oplus B' \oplus B_p)\hat{}$. Since $\hat{Z}_p$ is algebraically compact and pure in G_p ([2], Th. 41.7, 41.9), we have $G_p = \hat{Z}_p \oplus G'$. Since G_p is a unital $\hat{Z}_p$ module, (D) gives $L(G_p) = E(G_p)$.

Conversely, suppose G is reduced, algebraically compact and multiplicatively faithful. Then $L(G_p) = E(G_p)$ for all p. If for some p $B_p^0 = (0)$, then $B_p = \bigoplus_{\beta \in J} Z(p^{k_\beta}) \subseteqq T \subseteqq G_p \subseteqq \prod_{\beta \in J} Z(p^{k_\beta})$, where T is the torsion subgroup of the direct product. ($T \subseteqq \hat{B}_p = G_p$.) Now, G_p/T is torsion free divisible, thus homogeneous completely decomposable. Moreover, T is a p-group, and $L(G_p) = E(G_p)$. Theorem 2 applies to give a splitting $G_p = T \oplus F$. Since $G_p = \hat{T}$, $F = (0)$. Thus, G_p is a reduced algebraically compact torsion group, and is, therefore, bounded ([2], Cor. 40.3).

For the rest of the paper, we consider torsion free groups. First, we do the completely decomposable case.

THEOREM 4. *Let $G = \bigoplus_{\lambda \in \Lambda} A_\lambda$, where each A_λ is torsion free rank one. Then $L(G) = E(G)$ iff there exist subsets $\Lambda_1, \cdots, \Lambda_n$ of the index set Λ and rank one groups $A_{\lambda_1}, \cdots, A_{\lambda_n}$, $\lambda_i \in \Lambda_i$, with* (1) $\Lambda = \bigcup_{i=1}^n \Lambda_i$ *and* (2) $t(A_{\lambda_i}) + t(A_{\lambda'}) \leqq t(A_{\lambda'})$ *for all $\lambda' \in \Lambda_i$, $i = 1, \cdots, n$.*

Proof. Suppose $\Lambda_1, \cdots, \Lambda_n$; $A_{\lambda_1}, \cdots, A_{\lambda_n}$ exist satisfying the above conditions. Without loss of generality, assume $\Lambda_1, \cdots, \Lambda_n$ are disjoint. Put $\lambda' = \lambda_i$ in (2) to see that each $t(A_{\lambda_i})$ is idempotent. Thus, each A_{λ_i} can be made into a rank one ring with identity. Let $G_i = \bigoplus_{\lambda \in \Lambda_i} G_\lambda$. Due to (2), each G_i can be regarded (in the natural way) as a unital A_{λ_i} module. So we have $G = \bigoplus_{i=1}^n G_i$ is a unital A module with $A = \bigoplus_{i=1}^n A_{\lambda_i}$ (ring direct sum). Since A is a (group) direct summand of G, (D) applies.

Now suppose $G = \bigoplus_{\lambda \in \Lambda} A_\lambda$ with $L(G) = E(G)$. Choose $u \in G$, $\sigma \in \operatorname{Mult} G$ such that $\sigma \circ (u \otimes \_) = 1_G$. Write $u = \sum_{i=1}^n a_{\lambda_i}$, $a_{\lambda_i} \in A_{\lambda_i}$. Then, for all $\lambda \in \Lambda$, $\pi\ \sigma(\bigoplus_{i=1}^n A_{\lambda_i} \otimes A_\lambda) = A_\lambda$ when π is the projection from G onto A_λ. Thus, for each λ, there exists at least one i, $1 \leqq i \leqq n$, with $t(A_{\lambda_i} \otimes A_\lambda) = t(A_{\lambda_i}) + t(A_\lambda) \leqq t(A_\lambda)$. The desired partition of Λ now easily can be constructed.

Let G be an almost completely decomposable rank two torsion free group, i.e., $G \supseteqq A \oplus B \supseteqq dG$ for some $d \in Z^+$ and rank one subgroups A, B of G. We will obtain a numerical condition to show when such a G is multiplicatively faithful. We may assume $t(A)$ and $t(B)$ are incomparable. (If $t(A)$ and $t(B)$ are comparable, then $G \cong A \oplus B$ by Theorem 9.6 of [1]. If $G \cong A \oplus B$, Theorem 4 gives a complete description of when G is multiplicatively faithful.)

Let $A = \langle a \rangle_*$, $B = \langle b \rangle_*$ and let d be the minimal positive integer with $dG \subseteqq A \oplus B$. It is easy to show that $G = \langle A \oplus B, a + nb/d \rangle \subseteqq Q \oplus Q$ where n is an integer with $(n, d) = 1$. $(G/A \oplus B \cong Z(d).)$

Let $h_p(x)$ be the p-component of the height sequence of x and let $\prod_A = \{p \mid h_p(a) = \infty\}$, $\prod_B = \{p \mid h_p(b) = \infty\}$. It is also easy to show that $p \in \prod_A \cup \prod_B \rightarrow (p, d) = 1$. Let S be the multiplicative subgroup of $Z(d)^*$ generated by $\prod_A \cup \prod_B$.

THEOREM 5. *Let $G = \langle A \oplus B, a + nb/d \rangle$ be as above. Then $L(G) = E(G)$ iff $t(A)$ and $t(B)$ are idempotent and $n \in S$.*

Proof. Suppose $L(G) = E(G)$. If either A or B—A say—had nil type, then $AG = GA = (0)$ for any multiplication on G. (Recall that $t(A)$, $t(B)$ are incomparable.) Thus, 1_G could not be represented as a left multiplication for any ring on G. Since $L(G) = E(G)$ we must have $t(A)$, $t(B)$ idempotent.

Since $t(A)$, $t(B)$ are idempotent we can assume, without loss of generality, that $h_p(a) = 0$, $p \notin \prod_A$, $h_p(b) = 0$, $p \notin \prod_B$. Choose $\sigma \in \operatorname{Mult}(G)$, $x = \alpha a + \beta b \in G$, $\alpha, \beta \in Q$, such that the following is a commutative diagram:

$$\begin{tikzcd} G \arrow[r, "x \otimes \_"] \arrow[dr, "1_G"'] & G \otimes G \arrow[d, "\sigma"] \\ & G \end{tikzcd}$$

Let $\overline{\prod}_A = \{m \in Z \mid m = p_1^{e_1} \cdots p_k^{e_k},\ p_i \in \prod_A\}$ and define $\overline{\prod}_B$ similarly. Since $t(A)$, $t(B)$ are incomparable, we have $\sigma(a \otimes b) = \sigma(b \otimes a) = 0$; $\sigma(a \otimes a) = (c/h)a$, $h \in \overline{\prod}_A$; $\sigma(b \otimes b) = (e/k)b$, $k \in \overline{\prod}_B$. Let $y = a + nb/d$. Then

$$\sigma(y \otimes y) = \frac{1}{d^2}\left[\frac{c}{h}a + \frac{n^2 e}{k}b\right] \in G\ .$$

Since d is relatively prime both to n^2 and to anything in $\overline{\prod}_A \cup \overline{\prod}_B$, we must have $c = c'd$, $e = e'd$

$$\frac{1}{d}\left[\frac{c'}{h}a + \frac{n^2 e'}{k}b\right] \in G\ .$$

But $1/d[a + nb] \in G$. A short computation yields: $n^2e'/k - nc'/h \equiv 0\ (d)$. Since $(n, d) = 1$, we have $ne'h - c'k \equiv 0\ (d)$.

Now $\sigma[x \otimes a] = \sigma[(\alpha a + \beta b) \otimes a] = \alpha\sigma(a \otimes a) = \alpha(c'd/h)a = 1_G(a) = a$, so $\alpha = h/c'd$. Similarly, $\beta = k/e'd$. Since $\alpha a + \beta b \in G$, we must have $c' \in \overline{\prod}_A$, $e' \in \overline{\prod}_B$. But then $n \equiv c'k/e'h\ (d)$, so $n \in S$. This shows the two conditions of our theorem are necessary for $L(G) = E(G)$.

Conversely, suppose $t(A)$, $t(B)$ are idempotent and $n \in S$. Let a, b be as before. Let $\lambda \in E(G)$. Since $t(A)$, $t(B)$ are incomparable, $\lambda(a) = (m/h)a$, $\lambda(b) = (t/k)b$; $h \in \overline{\prod}_A$, $k \in \overline{\prod}_B$. Now $\lambda(y) = 1/d[(m/h)a + (nt/k)b] \in G$, so we must have $mk - th \equiv 0\ (d)$.

Since $n \in S$, it is easy to choose $c, c_1 \in \overline{\prod}_A$, $e, e_1 \in \overline{\prod}_B$ such that $nec_1 \equiv ce_1\ (d)$.

Let σ be defined by $\sigma(a \otimes a) = (dc/c_1)a$, $\sigma(b \otimes b) = (de/e_1)b$, $\sigma(a \otimes b) = \sigma(b \otimes a) = 0$. To show $\sigma[G \otimes G] \subseteq G$, it is enough to check that $\sigma(y \otimes a)$, $\sigma(a \otimes y)$, $\sigma(y \otimes b)$, $\sigma(b \otimes y)$ and $\sigma(y \otimes y)$ are all in G. All of these elements are obviously in G except the last one, and

$$\sigma(y \otimes y) = \frac{1}{d^2}\left[\frac{dc}{c_1}a + \frac{n^2 de}{e_1}b\right] = \frac{1}{d}\left[\frac{c}{c_1}a + n^2\frac{e}{e_1}b\right].$$

This is in G iff $n(c/c_1) \equiv n^2(e/e_1)(d)$, which is true by choice c, c_1, e, e_1. Thus, $\sigma \in \operatorname{Mult} G$.

Now let

$$g = \frac{1}{d}\left[\frac{c_1 m}{he}a + \frac{e_1 t}{ke}b\right].$$

It follows directly that $\sigma \circ (g \otimes \_) = \lambda$. (One need only check this identity on the independent set $\{a, b\}$.) It remains to show that $g \in G$. Now $g \in G$ iff $n[c_1m/hc] \equiv e_1t/ke\ (d)$. This congruence is easy to derive from $nec_1 \equiv ce_1\ (d)$ and $mk \equiv th\ (d)$, both of which are given. Thus, $g \in G$, $g_l^\sigma = \lambda$, and G is multiplicatively faithful.

The above theorem can be used to construct an example which shows that multiplicative faithfulness is not a quasi-isomorphism invariant for torsion free groups. Let $A = \{(m/3^k)a \mid m, k \in Z\}$, $B = \{(m/(11)^k)b \mid m, k \in Z\}$, and let $G = \langle A \oplus B, a + 2b/61\rangle$. Then $\Pi_A = \{3\}$, $\Pi_B = \{11\}$ and $2 \notin \langle \Pi_A \cup \Pi_B\rangle \subseteqq Z(61)^*$. G is not multiplicatively faithful by Theorem 5. $A \oplus B$ is multiplicatively faithful by Theorem 4. G is quasi-isomorphic to $A \oplus B$, since $G \supseteqq A \oplus B \supseteqq 61G$.

We give a name to a common occurence for torsion free groups.

DEFINITION 2. Let p be a prime and A a rank one subgroup of a torsion free group G. A is called p-dense in G iff $p(G/A) = G/A$ and G is p-reduced.

THEOREM 6. *Let A be p-dense in G for some prime p. Let $0 \neq a \in A$ and let $\Delta, \Gamma \in \operatorname{Mult} G$ be such that $a_l^\Delta = a_l^\Gamma$. Then $\Delta = \Gamma$.*

Proof. Since A is p-dense, $\operatorname{Hom}(G/A \otimes G, G) = (0)$. But then also $\operatorname{Hom}(G/\langle a\rangle \otimes G, G) = (0)$, since $A/\langle a\rangle \otimes G$ is the torsion subgroup of $G/\langle a\rangle \otimes G$ and G is torsion free.

The exact sequence: $0 \to G \xrightarrow{a\otimes\_} G \otimes G \to G/\langle a\rangle \otimes G \to 0$ yields: $0 \to \operatorname{Hom}(G/\langle a\rangle \otimes G, G) \to \operatorname{Mult} G \xrightarrow{\theta} E(G)$, where θ is given by $\theta(\Delta) = \Delta \circ (a \otimes \_) = a_l^\Delta \in E(G)$. Since $\operatorname{Hom}(G/\langle a\rangle \otimes G, G) = (0)$, θ is $1-1$.

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