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AUTOMORPHISM GROUPS OF UNIPOTENT GROUPS OF CHEVALLEY TYPE

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Let G be a quasi-simple algebraic group defined and split over the field k. Let V be a maximal k-split unipotent subgroup of G and $\mathrm{Aut}(V)$ the group of k-automorphism of V. The structure of $\mathrm{Aut}(V)$ is determined and the obstructions to making $\mathrm{Aut}(V)$ algebraic when $\mathrm{char}\, k > 3$ are made explicit. If G is not of type A_2, then $\mathrm{Aut}(V)$ is solvable.

Introduction. In [5] Hochschild and Mostow showed that the automorphism group of a unipotent algebraic group defined over a field k of characteristic zero carries the structure of an algebraic k-group. For example if V is a vector group over $\mathbf{C}$, then $\mathrm{Aut}_{\mathbf{C}}(V) = \mathrm{GL}(n, \mathbf{C})$. For more complicated unipotent groups—even over $\mathbf{C}$—little seems to be known about the actual structure of the automorphism group. On the other hand, it was shown by Sullivan in [8] and again by this author in [3] that the Hochschild–Mostow result never holds in positive characteristics when the dimension of the given unipotent group is greater than one.

In [4] Gibbs determined generators for the (abstract) automorphism group of $V(k)$—the k-rational points of a maximal k-split unipotent subgroup V of any k-split simple algebraic group. The characteristic of the field k was assumed distinct from 2 or 3, but no other assumptions on the field k were made. We refer to such groups V as *unipotent groups of Chevalley type.* The purpose of this paper is two-fold:

1. To determine the automorphism groups in characteristic zero of unipotent groups of Chevalley type; and
2. To exhibit the obstructions to making these groups algebraic in positive characteristics.

Let $Aut_V(k)$ denote the group of k-automorphisms of the unipotent k-group of Chevalley type V. We show (2.9) that there is an exact sequence

$$1 \to N(k) \to Aut_V(k) \to H(k) \to 1$$

such that

(i) $H(k)$ is the group of k-rational points of an algebraic k-group H.

(ii) $N(k) = 0$ if $\mathrm{char}\, k = 0$, and $N(k) = \coprod_{n=1}^{\infty} G_a(k)$ if $\mathrm{char}\, k > 3$.

(iii) The above sequence splits and $Aut_V(k)$ is the semi-direct product of $N(k)$ and $H(k)$.

Moreover, if the quasi-simple group containing V is not isogenous to PGL_2, then $Aut_V(k)$ is solvable.

Our treatment of this problem is slightly more general than required to prove the desired results over fields. In fact if $A = \mathbf{Z}[1/2, 1/3]$, then all schemes (no separation property is implied here) are assumed to be A-schemes and the problem we discuss is that of representing the functor $S \to Aut_V(S) := \mathrm{Aut}_{S\text{-}gr}(V \times_A S)$ for S a reduced A-scheme. The results for fields then follow by base change.

After setting up some notation and discussing preliminaries in §0, we proceed in §1 to give a functorial description of the generators described by Gibbs. Section 2 is devoted to the computation of Aut_V and §3 to the special case of groups of type A_2.

The author's debt to Gibbs' work will become clear soon. It is also a pleasure to thank J. Tits for several useful comments on an earlier version of this paper.

0. Preliminaries.

0.1. Let $\mathscr{L}$ be a simple complex Lie algebra with root system Σ and fundamental roots $\Phi \subset \Sigma$. There is a unique (up to isomorphism) smooth algebraic group scheme $\mathrm{ch}(\Phi, \Sigma)$, defined over the integers, corresponding to $\mathscr{L}$ called the Chevalley group of type Σ [**2**: Vol III, Exp. XXV]. If the set of positive roots relative to Φ is denoted Σ^+, then Σ^+ determines a unique Borel subgroup $B = B(\Sigma^+)$ of $\mathrm{ch}(\Phi, \Sigma)$. Let $V = V(\Phi, \Sigma)$ be the unipotent radical of B. We call V a *unipotent group scheme* of *Chevalley type.*

0.2. A unipotent group scheme of Chevalley type, say V, is completely determined by the group valued functor it represents. We recall the definition here. Let S be a scheme. Then $V(S)$ as an abstract group is generated by symbols $x_r(t)$, $r \in \Sigma^+$, $t \in \Gamma(S, \theta_S) := G_a(S)$ subject to the relations

$$\text{(R)}\qquad \begin{aligned} &\text{R.1: } x_r(t)x_r(u) = x_r(t+u) \\ &\text{R.2: } [x_s(u), x_r(t)] = \begin{cases} 1 & r+s \notin \Sigma^+ \\ \prod_{i,j} x_{ir+js}(C_{ij,rs}(-t)^i u^j) & r+s \in \Sigma^+ \end{cases} \end{aligned}$$

where the product in R.2 is taken over all pairs of positive integers (i, j) such that $ir + js \in \Sigma^+$. We use the conventions: $[a, b] = a^{-1}b^{-1}ab$ and $(\mathrm{int}\, a)b = a^{-1}ba$.

0.3. With respect to a given ordering of Σ^+, every element of $V(S)$ can be written uniquely in the form

$$x_{r_1}(t_1) \cdot x_{r_2}(t_2) \cdots x_{r_N}(t_N)$$

where $N = |\Sigma^+|$ and $r_1 < r_2 < \cdots < r_N$. Moreover, the constants $C_{ij,rs}$ are integers and $|C_{ij,rs}| < 4$. Recall that $N_{rs} = C_{11,rs}$, and also that $C_{12,rs} = \frac{1}{2}N_{sr}N_{s,r+s}$, $C_{21,rs} = \frac{1}{2}N_{rs}N_{r,r+s}$, $N_{rs} = -N_{sr}$ and $N_{rs} = \pm 1$ all r, s in Σ.

0.4. The subgroups X_r, $r \in \Sigma^+$ given by $X_r(S) = \{x_r(t)\colon t \in G_a(S)\}$ are called *root subgroups* and each $X_r \simeq G_a$. The uniqueness assertion of 0.3 says that the product morphism $X_{r_1} \times \cdots \times X_{r_N} \to V$ is an isomorphism.

0.5. Put $V_m = \Pi_{h(r) \geqq m} X_r$ for $1 < m \leqq h = h(r_N)$ where $h(r)$ is the height function (c.f. [**4**; §0]).

PROPOSITION. [**4**: 5.4] *The series*

$$V = V_1 > V_2 > \cdots > V_h > 1$$

is both the upper and lower central series for V.

Moreover, for each i, V_i/V_{i+1} has a canonical structure of a vector group (i.e. is isomorphic to G_a^n for some integer n).

0.6. In the following we exclude the trivial case $\Sigma = A_1$.

PROPOSITION. *Let (Σ, Φ, Σ^+) be a system of roots, fundamental roots and positive roots in a complex simple Lie algebra. Then*

(i) *There exist a unique root $r_N \in \Sigma^+$ of maximal height h.*

(ii) *If $r \in \Sigma^+$ and $h(r) \geqq 2$ then there exist $r_i \in \Phi$ such that $r - r_i \in \Sigma^+$.*

(iii) *If $r \in \Sigma^+$ and $r \neq r_N$, $r \notin \Phi$ then there exist $r_i \in \Phi$ such that $r + r_i \in \Sigma^+$.*

(iv) *If the combination $ir + js$ for $i > 0$, $j > 0$, $r, s \in \Sigma^+$ lies in Σ^+ then $r + s \in \Sigma^+$.*

(v) *There exist $r_i \in \Phi$ such that $r_N - r_i \in \Sigma^+$.*

We make some remarks concerning (v). It is easy to see by examining the root systems that, if Σ is not of type A_l, then there is a unique fundamental root $r_i \in \Phi$ with $r_N - r_i \in \Sigma^+$. If Σ is of type C_l and $r_N - r_i \in \Sigma^+$, then $r_N - 2r_i$ is also in Σ^+. In particular, if Σ is not of type A_l, then there is a unique root of height $h-1$. For if $h(r) = h - 1$, $r \in \Sigma^+$, then $h(r + r_j) = h$ some $r_j \in \Sigma^+$ by (iii) so that $r + r_j = r_N$ by (i). Thus $r_N - r_j = r \in \Sigma^+$, so $r_j = r_i$.

If Σ is of type C_l and $h(s) = h - 2$, then X_s and $X_{r_N - 2r_i}$ commute. For $h(s + r_N - 2r_i) = 2h - 4$. But in this case $h = h(r_N) \geqq 5$

so that $2h-4>h$ and, hence, $s+r_N-2r_i \notin \Sigma^+$. So by (iv) and R.2, X_s and $X_{r_N-2r_i}$ commute. It follows that we can assume that, in case C_l, the roots are ordered so that r_N-2r_i, r_N-r_i and r_N are the last three roots. We keep this assumption throughout.

If Σ is of type A_l with $l>2$, then there are just two roots $r_1, r_l \in \Phi$ such that $r_N - r_i \in \Sigma^+$, $i=1,l$. In this case, since $h(r_N)>2$, $r_N-r_1+r_N-r_l$ is not a root so that $X_{r_N-r_1}$ and $X_{r_N-r_l}$ commute. This fails of course if $\Sigma = A_2$ for then $\Sigma^+=\{r,s,r+s\}$.

In what follows we set $A := Z[1/2, 1/3]$. All schemes are A-schemes. We assume no separation property for A-schemes.

1. The Gibbs subfunctors. In [**4**] Gibbs listed six types of automorphisms of the abstract group $V(k)$ for any field and showed that every automorphism was a product of these six types. We shall give here five of those six types—described functorially—which will in fact generate $Aut_V(S)$ for any reduced A-scheme S.

1.1. *Diagonal automorphisms.* Let M be the free abelian group generated by the set of roots $\Phi \subset \Sigma$ determining $V=V(\Phi,\Sigma)$. Let D be the A-torus representing $D_A(M)$. Recall $D_A(M)$ is the functor defined by

$$D_A(M)(S)=\mathrm{Hom}_{A\text{-alg}}(A[M],\Gamma(S,O_S))$$

where $A[M]$ is the group algebra of M over A.

We define a homomorphism of group valued functors

$$w_D: h_D = D(M) \rightarrow Aut_V$$

as follows: For any A-scheme S and $\lambda \in h_D(S)$ let $w_D(\lambda)$ be the map which sends $x_r(t)$ to $x_r(\lambda(r)t)$ all $r\in\Sigma^+$. As in [**4**: §4], it is easily verified that this determines an automorphism of $V(S)$.

Since $\lambda(r)\in\Gamma(S,O_S)^*$, it is also easy to see that $w_D(\lambda)$ is in fact an automorphism of $V\times_A S$ over S. Indeed, if B is the standard Borel subgroup of $\mathrm{Ch}(\Phi,\Sigma)$ determined by Σ^+ and T is its maximal torus, then (cf. [**2**: Vol. III, Exp. XXII, 1.13]) $D\cong T$ and w is just the map induced by T acting on $V\cong B_u$ via conjugation. In particular, w_D is a homomorphism of group valued functors which (since $\mathrm{Ch}(\Phi,\Sigma)$ is of adjoint type) is in fact a monomorphism.

1.2. *Inner automorphisms.* Let $I=V/Z(V)$ where $Z(V)$ is the center of V. Then we have the natural functorial monomorphism of group valued functors $I\rightarrow Aut_V$.

1.3. *Central automorphisms.* Let Λ be the functor from Sch/A to abelian groups given by $\Lambda(S) = \mathrm{Hom}_{S\text{-gr}}(G_a \times S, G_a \times S)$ and let $l = |\Phi|$. Put $\mathbf{C} = \Lambda \times \cdots \times \Lambda$ (l copies). We define a monomorphism $w_{\mathbf{C}}$: $\mathbf{C} \to Aut_V$ of group valued functors as follows:

For $S \in \mathrm{Sch}/A$ and $c = (c_1, \cdots, c_l) \in \mathbf{C}(S)$ we define $w_C(c)$ by

$$w_{\mathrm{C}}(c)[x_{r_1}(t_1)\cdots x_{r_N}(t_N)] = x_{r_1}(t_1)\cdots x_{r_{N-1}}(t_{N-1})x_{r_N}\Bigl(t_N + \sum_{i=1}^{l} c_i(t_i)\Bigr)$$

for all $t_i \in \Gamma(S, O_S)$, $1 \leqq i \leqq N$ and $c_i \in \mathrm{End}_{S\text{-gr}}(G_a \times_A S)$ where $c_i(t_i)$ is given by the canonical action of c_i on $\Gamma(S, O_S) = G_a(S)$. Since the subgroup X_{r_N} is central, it follows immediately that $w_{\mathrm{C}}(c)$ is an automorphism of $V(S)$ and it is equally clear that $w_{\mathrm{C}}(c)$ yields an element of $Aut_V(S)$.

Finally, recall that the abelian group structure on $\Lambda(S)$ is given by $(c_1 + c_2)(t) = c_1(t) + c_2(t)$ all $t \in G_a(S)$; so w_C is in fact a monomorphism of group valued functors.

1.4. *Graph automorphisms.* Let Π be a finite group of automorphisms of M which stabilize Φ and hence Σ^+. Then an element $g \in \Pi$ determines a graph automorphism $w_\Pi(g)$ of $V \times_A S$ over S for an A-scheme S via the assignments $x_r(t) \to x_{g(r)}(t)$ all $r \in \Sigma^+$ and $t \in \Gamma(S, O_S)$. Such automorphisms arise as certain graph automorphisms of $\mathrm{Ch}(\Phi, \Sigma)$ determined by Steinberg [**1**: 12.2] which stabilize Φ.

If Σ is of type A_l ($l > 1$), D_l ($l > 4$) or E_6 then $\Pi = Z_2$. If Σ is of type D_4, then $\Pi = S_3$. These are the only graph automorphisms which occur. We identify Π with the constant functor and see immediately that w_Π is a monomorphism of group valued functors.

1.5. *Extremal automorphisms.* Let r_N be the unique root in Σ^+ of maximal height (cf. 0.6). Then there is a root $r_i \in \Phi$ such that $r_N - r_i \in \Sigma^+$. Suppose temporarily that Σ is not of type A_l or C_l. Put $E = G_a$ and define a map w_E: $E \to Aut_V$ as follows: For any A-scheme S and $u \in E(S) = \Gamma(S, O_S)$ let $w_E(u)$ be the map which acts trivially on $x_r(t)$, $r \neq r_i$ and which sends $x_{r_i}(t)$ to

$$x_{r_i}(t)x_{r_N - r_i}(ut)x_{r_N}(\tfrac{1}{2}N_{r_N - r_i, r_i}ut^2)$$

for all $t \in \Gamma(S, O_S)$. As in [**4**: §4] one verifies that $w_E(u)$ determines an automorphism of $V(S)$. It is also seen that this induces a (unique) automorphism of $V \times_A S$ over S. The following lemma is immediate.

LEMMA 1.6. *The map w_E: $E \to Aut_V$ is a monomorphism of group valued functors.*

Suppose now that Σ is of type A_l $(l > 1)$. Then there are two fundamental roots, say r_1 and r_l, such that $r_N - r_i \in \Sigma^+$. The lemma shows that for each r_i, $i = 1, l$ we have a monomorphism $w_i\colon G_a \to Aut_V$. If $l > 2$ then $r_N - r_1 \neq r_l$ and $r_N - r_l \neq r_1$. Since $X_{r_n - r_2}$ and $X_{r_n - r_l}$ commute if $l > 2$, for all $u, v \in G_a(S)$ we have $w_1(u)w_l(v) = w_l(v)w_1(u)$. We thus obtain a monomorphism $w_E\colon E = G_a \times G_a \to Aut_V$. When $l = 2$ we have two monomorphisms w_1 and w_2 (recall in this case $\Sigma = \{r, s, r+s\}$) and we denote by E the subfunctor of Aut_V which the images of these two homomorphisms generate. We will see in §3 that this case gives rise to the only exception to the solvability of Aut_V.

Now suppose Σ is of type C_l. Then if $r_N - r_i \in \Sigma^+$, we also have $r_N - 2r_i \in \Sigma^+$. We have just as above $w_1\colon G_a \to Aut_V$; but we can also define another map $w_2\colon G_a \to Aut_V$ by $w_2(u)[x_r(t)] = x_r(t)$, $r \neq r_i$, and

$$w_2(u)[x_{r_i}(t)] = x_{r_i}(t)x_{r_N - 2r_i}(ut)x_{r_N - r_i}(\tfrac{1}{2}N_{r_N - 2r_i, r_i}ut^2) \cdot x_{r_N}(\tfrac{1}{3}C_{12, r_N - 2r_i, r_i}ut^3).$$

As above, the following lemma follows from straight forward computations.

Lemma 1.7. *The morphism w_2 is a monomorphism of group valued functors and $w_2(u)w_1(v) = w_1(v)w_2(u)$ for all $u, v \in G_a(S)$, $S \in$ Sch/k. In particular there exist a monomorphism of group valued functors*

$$w_E\colon E = G_a \times G_a \to Aut_V$$

such that $w_E(u, 0) = w_1(u)$ and $w_E(0, v) = w_2(v)$.

2. The structure of Aut_V. The purpose of this section is to describe the functor Aut_V in terms of the Gibbs subgroups and prove that these generate when Aut_V is restricted to the category $(\mathrm{Sch}/A)_{\mathrm{red}}$ of reduced A-schemes. Let $\mathbf{A}_V$ be the subgroup of Aut_V generated by π, D, E, I and C. We assume throughout this section that $\Sigma \neq A_2, B_2$. We let V be a unipotent group of Chevalley type over A and S an A-scheme.

Proposition 2.1. *The subgroup* $\mathbf{C}$ *is normal in* Aut_V.

Proof. Recall that if U is any abstract group, U' its commutator subgroup and Z its center then there is a well-known homomorphism of monoids $\alpha\colon \mathrm{Hom}(U/U', Z) \to \mathrm{End}(U)$ defined by $\alpha(h)(u) = uh(uU')$. Since $Aut(U)$ operates on all objects involved and clearly preserves α, the intersection $\mathrm{Im}\,\alpha \cap \mathrm{Aut}(U)$ is a normal subgroup of $\mathrm{Aut}(U)$.

Applying this to our case set $U = V(S)$, $U' = V_2(S)$ $Z = V_h(S)$ (c.f. 0.5). Then $U/U' \simeq G_a(S)^l$ and the map α is just the map w_C defined in 2.3. Since $Z \subset U'$ it follows that $\operatorname{Im}\alpha \cap \operatorname{Aut} U = C(S)$ is normal in $\operatorname{Aut} U$.

PROPOSITION 2.2. *Let $\bar{W}$ be the subfunctor of $\mathbf{A}_V$ generated by E, I and $\mathbf{C}$. Then D normalizes $\bar{W}$ and $D\bar{W}$ is a semidirect product.*

Proof. Since D normalizes $\mathbf{C}$ and I is normal in Aut_V it suffices to show D normalizes E.

Let $r_i \in \Phi$ and $r_N - r_i \in \Sigma^+$. Let $e \in E(S)$ be the extremal automorphism determined by $x_{r_i}(t) \to x_{r_i}(t)x_{r_N - r_i}(ut)x_{r_N}(Kut^2)$, K an appropriate constant and suppose $d \in D(S)$ corresponds to the character α.

Now $ded^{-1}[x_{r_i}(t)] = de[x_{r_i}(\alpha(r_i)^{-1}t)]$. Put $s = \alpha(r_i)^{-1}t$. Then

$$\begin{aligned} de[x_{r_i}(s)] &= d[x_{r_i}(s)x_{r_N - r_i}(us)x_{r_N}(Kus^2)] \\ &= x_{r_i}(\alpha(r_i)s)x_{r_N - r_i}(\alpha(r_N - r_i)us)x_{r_N}(K\alpha(r_N)us^2). \end{aligned}$$

Since α is a character $\alpha(r_N - r_i) = \alpha(r_N)\alpha(r_i)^{-1}$. Thus

$$ded^{-1}[x_{r_i}(t)] = x_{r_i}(t)x_{r_N - r_i}(u't)x_{r_N}(Ku't^2)$$

where $u' = (\alpha(r_N)/[\alpha(r_i)]^2) \cdot u$.

Now suppose Σ is of type C_l $(l \geqq 3)$ and e is given by $x_{r_i}(t) \to x_{r_i}(t)x_{r_N - 2r_i}(ut)x_{r_N - r_i}(Kut^2) \cdot x_{r_N}(Lut^3)$—again for appropriate K and L. Then computing as above we find ded^{-1} is the extremal automorphism

$$x_{r_i}(t) \to x_{r_i}(t)x_{r_N - 2r_i}(u't)x_{r_N - r_i}(Ku't^2)x_{r_N}(Lu't^2)$$

where $u' = (\alpha(r_N)/\alpha(r_i)^3) \cdot u$. Hence, in any case, $D(S)$ normalizes $E(S)$ and hence $W(S)$.

The last assertion of the proposition will follow if we can show that $D \cap \bar{W} = \{1\}$. Let $w \in \bar{W}(S)$ and $d \in D(S)$ corresponding to the character α. Write $w = eic$ with $e \in E(S)$, $c \in C(S)$ and $i \in I(S)$. Then if $d = w$ we have $c = i^{-1}e^{-1}d$. Let $r_i \in \Phi$. Then for all $t \in \Gamma(S, O_S)$ we have

$$\begin{aligned} c[x_{r_i}(t)] = x_{r_i}(t)x_{r_N}(c_i(t)) &= i^{-1}e^{-1}d[x_{r_i}(t)] \\ &= i^{-1}e^{-1}[x_{r_i}(\alpha(r_i)t)] \\ &= x_{r_i}(\alpha(r_i)t) \cdot x_s(t) \cdots \\ &\quad \text{with } r_i < s. \end{aligned}$$

Thus $\alpha(r_i) = 1$ for all $r_i \in \Phi$ hence $d = 1$.

Let $\mathbf{G} = D\bar{W}$. We call G the *connected component* of $\mathbf{A}_V$. The next proposition justifies this terminology.

PROPOSITION 2.3. *The subgroup* Π *normalizes* $\mathbf{G}$ *and* $\mathbf{A}_V$ *is the semi-direct product of* Π *and* $\mathbf{G}$.

Proof. We have seen in 2.1 that Π normalizes $\mathbf{C}$ and it clearly normalizes I, so it remains to show that Π normalizes D and E.

Let $d \in D(S)$ correspond to the character α. Then if $\rho \in \Pi$, $\rho d \rho^{-1}[x_r(t)] = x_r(\alpha(\rho^{-1}(r))t)$. Hence Π normalizes D.

Now if $r_i \in \Phi$ and $r_N - r_i \in \Sigma^+$, then $\rho(r_N - r_i) = r_N - \rho(r_i) \in \Sigma^+$, for $\rho \in \Pi$. Thus $\rho(r_i) = r_i$ if Σ is not of type A_l and $\rho(r_1) = r_l$ if Σ is of type A_l and $r_N - r_i \in \Sigma^+$, $i = 1, l$. A straightforward computation now shows that, if $e \in E(R)$ and $e[x_{r_i}(t)] = x_{r_i}(t)x_{r_N - r_i}(ut) \cdot x_{r_N}(Kut^2)$, then $\rho e \rho^{-1} = e' \in E(R)$. In fact, if Σ is not of type A_l, then $\rho e \rho^{-1} = e$. If Σ is of type C_l $(l \geqq 3)$, then $\pi = 1$ and there is nothing to prove.

Now suppose $\rho \in \Pi$ and $\rho = $ diec, $d \in D(S)$, $i \in I(S)$, $e \in E(S)$ and $c \in C(S)$. Then $d^{-1}\rho$ maps $X_{r_i}(S)$ onto $X_{\rho(r_i)}(S)$. But *iec* acts trivially on $V(S)/V_2(S)$, since $\Sigma \neq A_2$ or B_2. Hence $\rho(r_i) = r_i$ and ρ is trivial. This completes the proof.

We need the following well known lemma.

LEMMA 2.4. *Let* $v = x_{s_1}(t_1) \cdot x_{s_2}(t_2) \cdot \cdots \cdot x_{s_K}(t_K)$ *be an element of* $V(S)$ *with* $s_1 < s_2 < \cdots < s_K$. *Then for all* $r \in \Sigma^+$ *and all* $t \in \Gamma(S, O_S)$

$$(\operatorname{int} v)x_r(t) = x_r(t)x_{r+s_i}(C_{11rs_i}(-t_it)) \cdots$$

where $i \leqq K$ *is the least integer such that* $r + s_i \in \Sigma^+$.

Proof. If $K = 1$ the result follows from Chevalley's commutator formula [**1**: 5.2.2]. The general case follows by a straight forward induction argument.

PROPOSITION 2.5. *An element* $c \in \mathbf{C}(S)$ *lies in* $E(S) \cdot I(S)$ *if and only if it lies in* $I(S)$ *and is conjugation by an element of* $V_{h-1}(S)$, *where* $h = h(r_N)$.

Proof. Suppose $c = e \cdot \gamma \neq 1$. Let $r_i \in \Phi$ with $r_N - r_i \notin \Sigma^+$. Then

$$\begin{aligned} c[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N}(c_i(t)) \\ &= e\gamma[x_{r_i}(t)] \\ &= x_{r_i}(t)x_{r_i+s_j}(N_{r_is_j}(-tt_j) \cdots \end{aligned}$$

where $\gamma = \operatorname{int} x_{s_1}(t_1)\cdots x_{s_K}(t_K)$ and j is the least integer such that $r_i + s_j \in \Sigma^+$. Equating terms, we see that $r_i + s_j = r_N$; i.e., $s_j = r_N - r_i \in \Sigma^+$. This is a contradiction unless γ acts trivially on $x_{r_i}(t)$ and $c_i(t) \equiv 0$. Hence $c[x_{r_i}(t)] = x_{r_i}(t)$ unless $r_N - r_i \in \Sigma^+$.

Now suppose $r_N - r_i \in \Sigma^+$, $r_i \in \Phi$ and Σ is not of type C_l. Then

$$\begin{aligned} c[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N}(c_i(t)) \\ &= e[x_{r_i}(t)x_{r_i+s_j}(N_{r_is_j}(-tt_j))\cdots] \\ &= x_{r_i}(t)x_{r_N-r_i}(ut)x_{r_N}(\tfrac{1}{2}N_{r_N-r_i,r_i}ut^2) \\ &\qquad \cdot x_{r_i+s_j}(N_{r_is_j}(-tt_j))\cdots. \end{aligned}$$

We must have $r_i + s_j = r_N - r_i$. But then $s_j = r_N - 2r_i \in \Sigma^+$ a contradiction, thus we conclude $e = 1$.

If Σ is of type C_l, then we have

$$\begin{aligned} c[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N}(c_i(t)) \\ &= e[x_{r_i}(t)x_{r_i+s_j}(N_{r_is_j}(-tt_j))\cdots \\ &= x_{r_i}(t)x_{r_N-2r_i}(f(t))x_{r_N-r_i}(g(t))x_{r_N}(h(t)) \\ &\qquad \cdot x_{r_i+s_j}(N_{r_is_j}(-tt_j))\cdots. \end{aligned}$$

Now if f is not identically zero, we have $r_i + s_j = r_N - 2r_i$ so that $s_j = r_N - 3r_i \in \Sigma^+$. But this is impossible since the Cartan matrix for C_l shows that the r_i chain of roots through r_N has length 2. Thus f is identically zero. It must happen, then, that $r_i + s_j = r_N - r_i$ so that $s_j = r_N - 2r_i$ and $g(t) = N_{r_i,r_N-2r_i}(tt_j)$.

We have shown that e is given by

$$x_{r_i}(t) \to x_{r_i}(t)x_{r_N-r_i}(ut)x_{r_N}(\tfrac{1}{2}N_{r_N-r_i,r_i}ut^2)$$

where $u = N_{r_i,r_N-2r_i}t_j = -N_{r_N-2r_i,r_i}t_j$. (Recall $N_{rs} = -N_{sr}$ all $r, s \in \Sigma^+$.) Consider now the effect of $\operatorname{int} x_{r_N-2r_i}(t_j)$. Recall $r_N - 2r_i + r \in \Sigma^+$ only if $r = r_i$. Thus we need only consider $\operatorname{int} x_{r_N-2r_i}(t_j)$ acting on $x_{r_i}(t)$:

$$\begin{aligned} \operatorname{int} x_{r_N-2r_i}(t_j)[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N-r_i}(N_{r_i,r_N-2r_i}(-tt_j)) \\ &\qquad \cdot x_{r_N}(C_{21,r_i,r_N-2r_i}t^2t_j) \\ &= x_{r_i}(t)x_{r_N-r_i}(-ut)x_{r_N}(C_{21,r_i,r_N-2r_i}t_jt^2). \end{aligned}$$

But

$$\begin{aligned} C_{21,r_i,r_N-2r_i}t_j &= \tfrac{1}{2}N_{r_i,r_N-2r_i}N_{r_i,r_N-r_i}t_j \\ &= \tfrac{1}{2}N_{r_i,r_N-r_i}u = -\tfrac{1}{2}N_{r_N-r_i,r_i}u. \end{aligned}$$

Thus $\operatorname{int} x_{r_N-2r_i}(t_j) = e^{-1}$ and, hence, if $c \in C(S) \cap E(S)I(S)$, then $c \in I(S)$.

It follows that for $r_i \in \Phi$, $r_N - r_i \in \Sigma^+$

$$\begin{aligned} c[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N}(c_i(t)) \\ &= x_{r_i}(t)x_{r_i+s_j}(N_{r_i s_j}(-tt_j)) \cdots . \end{aligned}$$

Equating terms we see that $r_i + s_j = r_N$ so $s_j = r_N - r_i$. If Σ is not of type A_l, then we could only have $\gamma = \operatorname{int} x_{r_N-r_i}(\alpha)$, since there is just one root of height $h-1$. In this case, we have

$$\operatorname{int} x_{r_N-r_i}(\alpha)[x_{r_i}(t)] = x_{r_i}(t)x_{r_N}(N_{r_i,r_N-r_i}(-\alpha t)).$$

Again equating terms we see $N_{r_N-r_i,r_i}\alpha t = c_i(t)$.

If Σ is of type A_l, then there are two roots $r_1, r_l \in \Phi$ with $r_N - r_i \in \Sigma^+$. Just as above, we see that conjugation by a suitable element of $X_{r_N-r_i}(S)$, $i = 1, l$, has the same effect as a central automorphism on $X_{r_i}(S)$, $i = 1, l$. In particular, these two types of automorphisms commute with each other since Σ is not of type A_2 and thus their product gives an element of $C(S)$. Since the $x_{r_i}(t)$, $r_i \in \Phi$, $t \in \Gamma(S, O_S)$ generate $V(S)$ as a group, the proof is complete.

PROPOSITION 2.6. *The subfunctor $W = EI$ is representable.*

Proof. We shall show that W is the semi-direct product of two representable functors. Suppose first that Σ is not of type C_l. We claim that, in this case, W is the semi-direct product of E and I.

Let $e \in E(S) \cap I(S)$, say $e = \gamma$ with $\gamma = \operatorname{int} x_{s_1}(t_1) \cdots x_{s_K}(t_K)$. Then if $r_N - r_i \in \Sigma^+$, $r_i \in \Phi$, using Lemma 2.4 we have

$$\begin{aligned} e[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N-r_i}(ut)x_{r_N}(\tfrac{1}{2}N_{r_N-r_i,r_i}ut^2) \\ &= \gamma[x_{r_i}(t)] = x_{r_i}(t)x_{r_i+s_j}(N_{r_i,r_i+s_j} - t_1 t) \cdots . \end{aligned}$$

Equating terms we see that $r_i + s_j = r_N - r_i$; hence $s_j = r_N - 2r_i \in \Sigma^+$, a contradiction. Hence $e = \gamma = 1$.

Now suppose Σ is of type C_l. Then

$$\begin{aligned} e[x_{r_i}(t)] &= x_{r_i}(t)x_{r_N-2r_i}(ut) \cdots = \gamma[x_{r_i}(t)] \\ &= x_{r_i}(t)x_{r_i+s_j}(N_{r_i,r_i+s_j}(tt_j) \cdots . \end{aligned}$$

Thus $r_i + s_j = r_N - 2r_i$ so $s_j = r_N - 3r_i \in \Sigma^+$, a contradiction. Hence e must be of the type considered in the first part of the proof. Equating terms again, we find that $s_j = r_N - 2r_i$. As in the proof of Proposition 2.5, we can find $\alpha \in \Gamma(S, O_S)$ such that $\operatorname{int} x_{r_N-2r_i}(\alpha) = e$. Now put

$$E_1 = \{e \in E(S) \mid e[x_{r_i}(t)] = x_{r_i}(t)x_{r_N-2r_i}(ut)\cdots,\ e[x_r(t)] = x_r(t),\ r \neq r_i\}.$$

Then $E_1 \cap I(S) = 1$ so W is the semi-direct product of E_1 and $I(S)$.

Finally, since E, E_1 and I are representable, so is W.

Let S be a reduced A-algebra and c an element of $\Lambda(S) = \operatorname{Hom}_{S\text{-}gr}(G_a \times_A S, G_a \times_A S)$. Then, since $G_a \times_A S = \operatorname{Spec} O_S[T]$, the element c is completely determined by a polynomial $c(T)$ in $O_S[T]$. We can write $c(T) = c_1(T) + c_2(T)$, where $c_1(T)$ has degree $\leqq 1$ and $c_2(T)$ contains no terms of degree less than 2. It follows that $\mathbf{C}(S) = C_1(S) \times C_2(S)$, where $C_1(S) = \{c \in C(S)\colon \text{degree } c_1 \leqq 1\}$ and $C_2(S) = \{c \in C(S)\colon c \text{ has no terms of degree less than } 2\}$. It is also clear that if $c \in C_1(S)$, then c has no constant term. Thus $C = C_1 \times C_2$ where $C_1 \simeq G_a^l$. An entirely straight forward computation shows that C_1 and C_2 are both normal subfunctors of $\mathbf{A}_V$. In fact, E and I centralize C so one need only check conjugation by elements of D and π and the result for these subgroups follows essentially from the definitions.

We define H to be the subfunctor of $\mathbf{A}_V$ generated by π, D, E, I and C_1 and set $N = C_2$.

PROPOSITION 2.7. *For any A-scheme S, $\mathbf{A}_V(S)$ is the semi-direct product of $H(S)$ and $N(S)$.*

Proof. Clearly $\langle H(S), N(S)\rangle = \mathbf{A}_V(S)$ and it suffices to show that $H(S) \cap N(S) = 1$. This follows from 2.2, 2.3 and 2.5.

COROLLARY 2.8. *The subgroup H is represented by a smooth solvable A-group scheme and $\mathbf{A}_V$ is a sheaf on the Zariski site* Sch/A.

Proof. Let $H_0 = \langle D, E, I, C_1\rangle$. Then

$$1 \to H_0 \to H \to \pi \to 1$$

is exact. But π is solvable by 1.4 and the solvability of H_0 follows from 2.2 and 2.5. That H is representable is a consequence of the semi-direct product decompositions $H = \pi \cdot H_0$, $H_0 = D \cdot EIC_1$, Proposition 2.6 and the representability of D, E, I, C_1 and π.

The second assertion will follow if we show N is a sheaf since $\mathbf{A}_V = H \cdot N$. But $N \cong \Pi_{n=1}^{\infty} G_a$ which is clearly a a sheaf on Sch/A.

THEOREM 2.8. *Let S be a reduced A-scheme. Then the canonical map $j: \mathbf{A}_V(S) \to Aut_V(S)$ is an isomorphism. Consequently $\mathbf{A}_V \simeq Aut_V$ on* $(\mathrm{Sch}/A)_{\mathrm{red}}$.

Proof. Let $S \in (\mathrm{Sch}/A)_{\mathrm{red}}$ be given and $\{S_i\}$ an open affine covering of S. Since $\mathbf{A}_V$ and Aut_V are sheaves we have the following commutative diagram

$$\begin{tikzcd} \mathbf{A}_V(S) \arrow[r] \arrow[d] & \prod_i \mathbf{A}_V(S_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} \mathbf{A}_V(S_i \cap S_j) \arrow[d] \\ Aut_V(S) \arrow[r] & \prod_i Aut_V(S_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} Aut_V(S_i \cap S_j) \end{tikzcd}$$

where the rows are exact. It follows that the theorem holds if and only if it holds for affine schemes S. Moreover, since connected components are open, we may also assume each S_i is also connected so it is enough to establish the result when $S = \operatorname{Spec} R$ is reduced and connected.

The theorem holds when R is a field by [**4**: Theorem 6.2]. We indicate now how each step used in the proof of the case for fields can be carried out over R. The proof proceeds through seven steps. We write $Aut_V(R)$ for $Aut_V(\operatorname{Spec} R)$.

I. Let $t_1, \cdots, t_l$ be arbitrary nonzero elements of R and suppose $\theta \in Aut_V(R)$. If $\theta[x_{r_i}(t_i)] = \Pi_{j=1}^{l} x_{r_j}(t_{ij}) \bmod V_2(R)$, then the matrix $T = [t_{ij}]$ is monomial; i.e., T has just one entry in each row and each column. Consequently, $\theta(X_{r_i}) = X_{r_{\rho(i)}} \pmod{V_2}$, where ρ is some permutation of $1, \cdots, l$.

Proof. Let $y_1, \cdots, y_l$ be indeterminants and put $A' = R[y_1, \cdots, y_l]$. Let $\theta_{A'}$ be the image of θ under the homomorphism $Aut_V(R) \to Aut_V(A')$. We confuse $\theta_{A'}$ with $\theta_{A'}(A') \in \mathrm{Aut}_{\mathrm{Gr}}(V(A))$. Suppose $\theta_{A'}(x_{r_i}(y_i)) \equiv \Pi_{j=1}^{l} x_{r_j}(y_{ij}) \bmod V_2(A')$ with $y_{ij} \in A'$, $1 \leqq i \leqq l$. Let $p \in \operatorname{Spec} R$. Then $\bar{y}_i \neq 0$ in $R/p[y_1, \cdots, y_l] = A'/p$ and we have $\theta_{A'/p}[\bar{y}_i] \equiv \Pi_{j=1}^{l} x_{r_j}(\bar{y}_{ij})$. The automorphism $\theta_{A'/P}$ extends uniquely to an automorphism of $V(K)$ where K is the quotient field of A/P. Then by [**4**:6.3] the matrix $T_{A/P} = [\bar{y}_{ij}]$ is monomial. Since this holds for all $p \in \operatorname{Spec} R$, R is reduced and S is connected we have shown that $T_{A'} = [y_{ij}]$ is monomial.

Now let $\varphi: A' \to R$ be the R-algebra homomorphism determined by $\varphi(Y_i) = t_i$, $1 \leqq i \leqq l$. This induces a group homomorphism $\bar{\varphi}: V(A') \to V(R)$ and moreover

$$\begin{tikzcd} V(A') \arrow[r, "\bar{\varphi}"] \arrow[d, "\theta_{A'}"] & V(R) \arrow[d, "\theta"] \\ V(A') \arrow[r, "\bar{\varphi}"] & V(R) \end{tikzcd}$$

commutes. It follows that $T = [t_{ij}] = \varphi[T_{A'}] = [\varphi(y_{ij})]$ is monomial and that $\theta[X_{r_i}] \equiv X_{r_{\rho(i)}}$ mod V_2 for some permutation ρ of $1, 2, \cdots, l$.

II. Let $\theta \in Aut_V(R)$. Then there exist a graph automorphism $g \in \Pi$ such that $g^{-1}\theta(X_{r_i}(R)) \equiv X_{r_i}(R)$ mod $V_2(R)$ for each fundamental root subgroup $X_{r_i}(R)$, $r_i \in \Phi$.

Proof. By step I, $\theta(X_{r_i}(R)) \equiv X_{r_{\rho(i)}}(R)$ mod $V_2(R)$ for $1 \leqq i \leqq l$, where ρ is some permutation of $1, 2, \cdots, l$. Then it follows as in [**4**:6.4] that ρ induces a symmetry of Σ—the set of all roots. The corresponding graph automorphism $g \in \Pi$ does the trick.

III. Let g and θ be as in II. Then there is a diagonal automorphism $d \in D(R)$ such that $d^{-1}g^{-1}\theta$ acts trivially on $V(R)$ mod $V_2(R)$.

Proof. By II, $g^{-1}\theta$ induces an automorphism of $V(R)/V_2(R)$ which sends $X_{r_i}(R)$ mod $V_2(R)$ into itself. Let $g^{-1}\theta[x_{r_i}(1)] \equiv x_{r_i}(u_i)$ mod $V_2(R)$. Then $g^{-1}\theta$ acts on $V(R)/V_2(R) \cong A^l(R)$ via the matrix $T = \operatorname{diag}(u_1, \cdots, u_l)$. Since $g^{-1}\theta$ is an automorphism, it follows that u_i is a unit in R all i. Let $d \in \mathbf{D}(R)$ be the diagonal automorphism determined by the homomorphism $\alpha: M \to R^*$ given by $\alpha(r_i) = u_i$, $1 \leqq i \leqq l$. Then $d^{-1}g^{-1}\theta$ acts trivially on $V(R)/V_2(R)$.

IV. Let $\theta \in Aut_V(R)$ and suppose θ acts trivially on V mod V_2. Then there is an inner automorphism $i \in I$ such that $i^{-1}\theta$ acts trivially on V mod V_m where $m = h - 1$ if Σ is not of type C_l and $m = h - 2$ if Σ is of type C_l and where $h = h(r_N)$ is the height of the highest root.

Proof. It clearly suffices to show that if $2 \leqq n \leqq m - 1$ and θ acts trivially on V mod V_n, then there exist an inner automorphism $i \in I$ such that $i^{-1}\theta$ acts trivially on V mod V_{n+1}.

Let $s \in \Sigma^+$ have height n and suppose that for some fundamental root $r_i \in \Phi$

$$\theta[x_{r_i}(t)] = x_{r_i}(t)x_s(f(t)) \cdots$$

all $t \in R$ with $f \in R[x]$ not identically zero. Then by arguments similar to those in [**4**:6.7], $s - r_i \in \Sigma^+$. For the remainder of the argument we replace $s - r_i$ by s so $h(s) = n - 1$ and

$$\theta[x_{r_i}(t)] = x_{r_i}(t)x_{s+r_i}(f(t)).$$

We claim there is an element $i_s \in I$ such that the element $i_s =$

$i_s(R) \in I(R)$ acts trivially on $V(R)$ mod $V_n(R)$ and $i_s[x_{r_i}(1)] = x_{r_i}(1)x_{s+r_i}(f(1))\cdots$.

To see this, suppose $i_s = \operatorname{int} x_s(\alpha)$ with $\alpha \in R$ to be determined. Then i_s acts trivially on $V(R)$ mod $V_n(R)$ by the Chevalley commutator formula [**1**:5.2.2]. Moreover

$$x_s(-\alpha)x_{r_i}(1)x_s(\alpha) = x_{r_i}(1)x_{s+r_i}(C_{11,s,r_i} - \alpha \cdot 1)\cdots.$$

Since $C_{11,s,r_i} \neq 0$ is an integer whose absolute value is less than 4, C_{11,s,r_i} is a unit in R and we take $\alpha = \pm C_{11,s,r_i}^{-1} f(1)$.

We now have $i_s^{-1}\theta[x_{r_i}(t)] = x_{r_i}(t)x_{s+r_i}(g(t))\cdots$ where $g(1) = 0$. We claim that g is identically zero and moreover if there is an $r_j \in \Phi$, $i \neq j$, such that $s + r_j \in \Sigma^+$ and $i_s^{-1}\theta[x_{r_j}(t)] = x_{r_j}(t)x_{s+r_j}(g'(t))\cdots$, then g' is also identically zero. But all of the above relations hold after reduction to R/P for any $p \in \operatorname{Spec} R$. If K is the quotient field of R/P, the above relations hold for the image of $i_s^{-1}\theta$ in $\operatorname{Aut}_{Gr}(V(K))$. Then by arguments similar to those given in [**4**:6.7] g and g' are identically zero.

Now we can find such an inner automorphism i_s for each root s such that $h(s) = n - 1$. If we put i equal to the product of all these inner automorphisms, then $i^{-1}\theta$ acts trivially on $V(R)/V_{n+1}(R)$ and the lemma is proved.

The last three stages of the argument consist in showing the following:

V. If V is of type C_l and $\theta \in Aut_V(R)$ acts trivially on $V(R)$ mod $V_{h-2}(R)$ then there are extremal and inner automorphisms i and e in $Aut_V(R)$ such that $e^{-1}i^{-1}\theta$ acts trivially on $V(R)$ mod $V_{h-1}(R)$.

VI. If $\theta \in Aut_V(R)$ acts trivially on $V(R)$ mod V_{h-1} then there exist an inner automorphism $i \in I(R)$ such that $i^{-1}\theta$ acts trivially on $V(R)$ mod $V_{h-1}(R)$ and on $V_2(R)$.

VII. If $\theta \in Aut_V(R)$ acts trivially on $V(R)$ mod $V_{h-1}(R)$ and on $V_2(R)$ then $\theta = iec$ where $i \in I(R)$, $e \in E(R)$ and $c \in \mathbf{C}(R)$.

The proofs of these assertions follow from adaptations of the proofs of Lemmas 6.8, 6.9 and 6.10 of [**4**] similar to the arguments we have given above.

We have shown that the subfunctors Π, D; I, E and $\mathbf{C}$ generate Aut_V when these functors are restricted to $(\mathrm{Sch}/A)_{\mathrm{red}}$ and V is not of type B_2 or A_2. These two special groups can be treated directly just as in [**4**:6.11]. We discuss them in §3.

Now let k be a field whose characteristic is different from 2 and 3. Applying the base change functor $-\times_A k$ we obtain V_k a unipotent k-group of Chevalley type. In this case we can summarize the above results as follows:

COROLLARY 2.9. *Let $V = V(\Phi, \Sigma)$ be a unipotent k-group of Chevalley type. Assume that* $\operatorname{char} k \neq 2, 3$ *and that Σ is not of type A_2 or B_2. Let $Aut_V(S) = \mathrm{Aut}_{S\text{-}gr}(V \times_k S)$ for all S in $(\mathrm{Sch}/k)_{\mathrm{red}}$. Then*

(i) *There exists an exact sequence of group valued functors on $(\mathrm{Sch}/k)_{\mathrm{red}}$*

$$1 \to N \to Aut_V \to H \to 1$$

making Aut_V the semi-direct product of $\mathbf{H}$ and N.

(ii) *The functor H is representable by a smooth solvable algebraic k-group scheme.*

(iii) *If* $\operatorname{char} k \neq 0$ *then $N \simeq \coprod_{n=1}^{\infty} G_a$.*

(iv) *If* $\operatorname{char} k = 0$ *then $N = 0$ and $Aut_V \simeq H$.*

3. The cases $\Sigma = A_2$ or B_2. Let $\Sigma = A_2$, $V = V(\Phi, \Sigma)$ and $\mathbf{A}_V$ be the subgroup of Aut_V generated by Π, D, E, I and C. Let $\Sigma = \{r, s, r+s\}$. An easy computation shows that $I \subset C$ in this case.

THEOREM 3.1. *Let S be an A-scheme and $\rho\colon \mathbf{A}_V(S) \to \mathrm{Aut}_{S\text{-gr}}(V/X_{r+s} \times_A S)$ be the canonical homomorphism induced by passage to the quotient. Then the image of ρ is isomorphic to $\mathrm{GL}_2(S)$, the kernel of ρ is $\mathbf{C}(S)$ and the exact sequence*

$$1 \to \mathbf{C}(S) \to \mathbf{A}_V(S) \to \mathrm{GL}_2(S) \to 1$$

has a section making $\mathbf{A}_V(S)$ the semi-direct product of $\mathrm{GL}_2(S)$ and $\mathbf{C}(S)$. In particular $\mathbf{A}_V \simeq \mathrm{GL}_2 \cdot C$.

Proof. Let $\theta \in \mathbf{A}_V(S)$ and suppose θ acts trivially on $V/X_{r+s} \times_A S$. Then $\theta(x_r(t)) = x_r(t) x_{r+s}(f_r(t))$ and $\theta(x_s(t)) = x_s(t) x_{r+s}(f_s(t))$ for all $t \in \Gamma(S, O_S)$. It is easy to see that f_r and f_s give rise to a unique $c \in \mathbf{C}(S)$ and that $\theta = c$. This shows that $C(S)$ is the kernel of ρ.

Since $\mathbf{A}_V(S)$ is generated by $\Pi, D(S), E(S)$ and $C(S)$, and since $I(S) \subset C(S) = \operatorname{Ker} \rho$, to show that $\operatorname{Im} \rho = \mathrm{GL}_2(S)$ it suffices to show that the images of Π, $D(S)$ and $E(S)$ under ρ lie in $\mathrm{GL}_2(S)$. For Π and $D(S)$ this is clear. Recall that $E(S)$ is generated by two types of extremal automorphism e_1 and e_2: e_1 fixes $X_s(S)$ and $X_{r+s}(S)$ and maps $x_r(t)$ to $x_r(t) x_s(at) x_{r+s}(\frac{1}{2}at^2)$, $a, t \in G_a(S)$ and $e_2 = \sigma \cdot e_1 \sigma$ where σ is the generator of Π. Then $\rho(e_1)$ is represented by the matrix $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$ in $\mathrm{GL}_2(S)$

and it follows that $\rho(E(S)) \subset \mathrm{GL}_2(S)$ so $\mathrm{Im}\,\rho = \mathrm{GL}_2(S)$. For any matrix $M = \left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right|$ in $\mathrm{GL}_2(S)$ define $S'(M)$: $V(S) \to V(S)$ as follows

$$S'(M): \quad \begin{aligned} &x_r(t) \to x_r(at)x_s(bt)x_{r+s}(-abt^2) \\ &x_s(t) \to x_r(ct)x_s(dt)x_{r+s}(\tfrac{1}{2}cdt^2) \\ &x_{r+s}(t) \to x_{r+s}((ad-bc)t). \end{aligned}$$

It is an easy matter to verify that $S'(M)$ induces an automorphism of $V(S)$ hence also Vx_AS. It follows that ρ is surjective and as one sees easily S' is a homomorphism so that the remaining assertions of the theorem follow.

COROLLARY 3.2. *The group valued functors* $\mathbf{A}_V$ *and* Aut_V *are isomorphic on* $(\mathrm{Sch}/A)_{\mathrm{red}}$.

Proof. The corollary follows from a straight forward adaptation of Gibbs result [**4**:6.11] similar to the proof of Theorem 2.9 using the fact that $\mathbf{A}_V \simeq \mathrm{GL}_2 \cdot C$ is a sheaf.

COROLLARY 3.3. *Let* $\Sigma = A_2$ *and* k *be a field with* $\mathrm{char}\,k \neq 2, 3$. *Let* $V = V(\Phi, \Sigma)_k$ *and* Aut_V *the functor on* $(\mathrm{Sch}/k)_{\mathrm{red}}$ *given by* $S \to \mathrm{Aut}_{S\text{-}gr}(V \times_k S)$. *Then there exists an exact sequence of group valued functors*

$$1 \to N \to Aut_V \to H \to 1$$

on $(\mathrm{Sch}/k)_{\mathrm{red}}$ *such that*

(i) *H is representable by an affine algebraic k-group.*

(ii) $Aut_V \simeq H \cdot N$.

(iii) *There is a split exact sequence*

$$1 \to I \to H \to \mathrm{GL}_2 \to 1$$

(iv) *If* $\mathrm{char}\,k \neq 0$, $N \simeq \coprod_{n=1}^{\infty} G_a$ *and if* $\mathrm{char}\,k = 0$, $N = 0$.

Proof. Everything has been established except (iii) which follows from the fact that (using the notation of 2.7) $E_1 = I$.

When $\Sigma = B_2$, $\Pi = 1$ and $\mathbf{A}_V$ is generated by D, E, I and C. By methods entirely analogous to those above, we obtain the following result:

THEOREM 3.4. *Let* $V = V(\Phi, \Sigma)$ *be a unipotent* A*-group of Cheval-*

ley type with $\Sigma = B_2$. *Then there exists an exact sequence of group valued functors on* $(\mathrm{Sch}/A)_{\mathrm{red}}$.

$$1 \to N \to Aut_V \to H \to 1$$

such that

(i) $Aut_V = H \cdot N$ (semi-direct product).

(ii) *H is representable by a connected solvable affine k-group.*

The analogous results to 3.3 *hold for k a field,* $\mathrm{char}\, k \neq 2, 3$.

REMARKS. 1. At present we do not know whether the group functors $\mathbf{A}_V$ and Aut_V are isomorphic as functors on Sch/A. Even over an algebraically closed field of characteristic zero the situation is not clear.

We have excluded the trivial case $V = G_a$ throughout. However, in this case, we can see that $\mathbf{A}_V$ and Aut_V are distinct on Sch/k. In fact, $\mathbf{A}_V \simeq G_m$ in any characteristic. On the other hand, if R is a nonreduced k-algebra and $u \in R$, $u \neq 0$, $u^2 = 0$, then the map $R[x] \to R[x]$ determined by $x \to x + u \cdot f(x)$, $f(x)$ any additive polynomial gives an automorphism of G_a which is not necessarily in $\mathbf{A}_V(R)$.

2. If k is a field with $\mathrm{char}\, k \neq 2, 3$, then using Gibbs results one obtains immediately that the automorphism group of $V(k)$ (considered now as the k-rational points of $V(\bar{k})$, $\bar{k}$ an algebraic closure of k) is the semi-direct product of the group $Aut_V(k)$ and $\mathrm{Aut}(k)$. In particular, if $\mathrm{char}\, k = 0$, one obtains the expected result that $\mathrm{Aut}(V(k))$ is the semi-direct product of a group of matrices over k and the group of automorphisms of the field k.

3. Again, let k be a field with appropriate characteristic restrictions. Then it follows from what we have developed above that any finite set of automorphisms of $V \times_A k$ is contained in a subgroup of $Aut_V(k)$ which is represented by an algebraic k-group. This follows from an examination of the 'central components' of the given automorphisms — for if $\alpha \in N(k)$ then the degrees of the polynomials in $k[x]$ (c.f. 2.6 ff) determining α are bounded. This bound determines a certain representable subfunctor.

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