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INCLUSION RELATIONS BETWEEN POWER METHODS OF LIMITATION

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Let $p(x) = \sum p_k x^k$ be a power series with $p_k (k = 0, 1, \cdots)$ complex numbers and $0 < \rho_p \leqq \infty$ its radius of convergence, and assume that $P(x) \neq 0$ for $0 \leqq \alpha_p \leqq x < \rho_p$. The power method of limitation, P, is defined by

$$\lim_p s = \lim_{x \to \rho_p -} \sum_{k=0}^{\infty} p_k s_k x^k / P(x) \quad (x \text{ real})$$

(provided the series converges in $[\alpha_p, \rho_p)$ and the limit exists and is finite). Abel and Borel methods are the best known power methods. In this article inclusion relations between two power methods are investigated. Several theorems are proved, which lead to necessary and sufficient conditions, for inclusion, that are correct under some fairly moderate restrictions.

1. Introduction. Let $P(x) = \sum p_k x^k$ be a power series with $p_k(k = 0, 1, \cdots)$ complex numbers and $0 < \rho_p \leqq \infty$ its radius of convergence, and assume that $P(x) \neq 0$ for $0 \leqq \alpha_p \leqq x < \rho_p$. The *power method of limitation*, P (see Włodarski [19] and Birkholc [2]), is defined by

$$\lim_p s = \lim_{x \to \rho_p -} \sum_{k=0}^{\infty} p_k s_k x^k / P(x) \qquad (x \text{ real})$$

(provided the series convergences in $[\alpha_p, \rho_p)$ and the limit exists and is finite).

The power method Q is defined analogously by $Q(x) = \sum q_k x^k$ and parameters α_q, ρ_q.

The best known power methods are the Abel method and the Borel exponential method. Other power methods which appear in the literature are A_λ, L and (B, a, b) (for more details see next section).

We are concerned here with inclusion relations of the form $P \subseteqq Q$. There are several results in the literature in this direction. Thus, Borwein proved (see [4], [5] and [8]) that $A_\lambda \subseteqq A_\mu$, $A_\mu \nsubseteqq A_\lambda$ provided $-1 < \mu < \lambda$, that $A_\lambda \subseteqq L$, $L \nsubseteqq A_\lambda$ provided $\lambda > -1$ and that $(B, a, \beta) \subseteqq (B, a, b)$ provided $a > 0$, $-\infty < \beta \leqq b < +\infty$.

Other results, obtained by Borwein [4], [8] and Hoischen [12], are of a more general nature. Both authors investigated inclusion relations between power methods whose coefficients, $\{p_k\}$, $\{q_k\}$, are assumed, a priori, to be related by some particular cases of the relation

$$q_k = p_k \int_0^r \tau^{k-i} d\psi(\tau)\,, \qquad 0 < r < \infty, i \in \{0, 1, \cdots\}, \quad k = n, n+1, \cdots ,$$
$$\int_0^r |d\psi(\tau)| < \infty\ .$$

Assuming some more restrictive conditions (like $p_k \geqq 0$ or $p_k \neq 0$ or others) Borwein gets sufficient conditions and Hoischen necessary and sufficient conditions for inclusion.

In this article we are able to discuss the problem in greater generality. The single essential restriction which still remains necessary is:

$$\sum_{k\, p_k \neq 0} 1/k = \infty\ . \tag{1.1}$$

The main tools which make this discussion possible are taken from [21].

It seems that the problem is not simple enough to be solved by one or two theorems. A broader kind of investigation is needed. Actually it comes out that the case of power methods with finite radius of convergence should be separated from the case of infinite radius of convergence. The discussion of the first case provides results which are simpler to formulate and are more satisfactory.

The forthcoming results include, in particulr, necessary conditions, for inclusion, some combinations of which turn out to be also sufficient. So, necessary and sufficient conditions can be formulated, with (1.1) being the only pre-assumed restriction. Those conditions seem to be slightly complicated if $\rho_p = \infty$; so they are simplified for some restricted cases, where all the additional restrictions are sufficiently general to be automatically satisfied if P and Q are both regular power methods.

Few of the theorems are applied later to the above mentioned examples of power methods (all of which are regular) yielding some results of interest. Necessary and sufficient conditions for each of the inclusions $P \subseteqq A$, $A \subseteqq Q$, $P \subseteqq B$, $B \subseteqq Q$ (where A, B are the Abel and Borel methods and P, Q are some other power methods) are obtained as corollaries.

2. Definitions and statement of results.

2.1. *A definition and a convention.*

2.1.1. *Power methods of limitation.* Let $P(x) = \sum p_k x^k$, with complex coefficients, $p_k (k = 0, 1, \cdots)$, and radius of convergence

$0 < \rho_p \leqq \infty$, be some power series such that $P(x) \neq 0$, for $\alpha_p \leqq x < \rho_p$, where $0 \leqq \alpha_p < \rho_p$ is some real constant. A sequence of complex numbers $s = \{s_k\}$ is said to be P-convergent to σ if $\sum p_k s_k x^k$ is convergent for all $x \in [\alpha_p, \rho_p)$ and

$$\lim_{x \to \rho_p -} T_p(s, x) = \sigma\ , \qquad T_p(s, x) = \sum_{k=0}^{\infty} p_k s_k x^k / P(x) \quad (x \text{ real})\ .$$

$T_p(s, x)$ is called the *P-transform* of s and σ its *P-limit*. σ is denoted also by $\lim_p s$. By c_p we denote the field of the method P, i.e. the set of all complex sequences which are P-convergent to a finite limit. $c_p^{(0)}$ denotes the set of complex sequences which are P-limitable to zero, and m_p the set of all complex sequences whose P-transform exists and is bounded in $[\alpha_p, \rho_p)$.

In analogy with P the power method Q is defined by the series $Q(x) = \sum q_k x^k$ and parameters α_q, ρ_q. The Q-transform of a sequence s and its Q-limit are denoted by $T_q(s, x)$, $\lim_q s$. The field of Q and the other related sets are denoted by c_q, $c_q^{(0)}$, m_q.

We say that $P \subseteqq Q$ (i.e. P is included in Q) if $c_p \subseteqq c_q$ and P, Q are consistent (i.e. $\lim_q s = \lim_p s$ for all $s \in c_p$).

In many of the results of this paper, P is required to satisfy the additional condition

$$\sum_{k\, p_k \neq 0} 1/k = \infty\ . \tag{2.1.1}$$

The following are examples of well known power methods:

A-Abel's method: $P(x) = 1/(1 - x)$, $\alpha_p = 0$, $\rho_p = 1$.

B-Borel's exponential method: $P(x) = e^x$, $\alpha_p = 0$, $\rho_p = \infty$.

A_λ-Abel-type methods: $P(x) = (1 - x)^{-1-\lambda}$, $\lambda > -1$, $\alpha_p = 0$, $\rho_p = 1$ (see Jakimovski [1] and Borwein [5]).

L or A_{-1}-Logarithmic method: $P(x) = \log [1/(1 - x)]$, $\alpha_p > 0$, $\rho_p = 1$ (see Borwein [6]).

(B, a, b)-Borel-type methods: $P(x) = \sum_{k=N}^{\infty} x^k / \Gamma(ak + b) \sim a^{-1} x^{(1-b)/a} e^{x^{1/a}}$ $(x \to +\infty)$, $a > 0$, $-\infty < b < +\infty$, $aN + b > 0$, $\alpha_p > 0$, $\rho_p = \infty$ (see Borwein [8]).

2.1.2. *A convention about functions of bounded variation.* Every complex valued function whose variation is bounded in some finite or infinite interval is assumed throughout to be continuous to the right at all points of this interval, with the possible exception of the interval's ends.

2.2. *Theory which is restricted only by condition* (2.1.1).

REMARK 2.2.1. It should be observed that if $P(x)$ is a polynomial then every sequence is P-convergent. This is, then, a trivial case. For this reason it is assumed throughout that both $P(x)$ and $Q(x)$ are not polynomials.

REMARK 2.2.2. If $c_p^{(0)} \subseteqq m_q$ then the set $J = \{k \mid p_k = 0, q_k \neq 0\}$ is finite.

2.2.3. DEFINITION OF r_{pq}. We define

$$r_{pq} = \overline{\lim_{k \to \infty}} \mid q_k/p_k \mid^{1/k} ,$$

where k is considered, in the limiting process, only if $p_k \neq 0$.

THEOREM 2.2.4. *Let* (2.1.1) *be satisfied. If* $c_p^{(0)} \subseteqq m_q$ *then each of the following must be satisfied*:

(i) $0 < r_{pq} < \infty$

(ii) $\rho_q r_{pq} = \rho_p$

(iii) *The limit*

$$\lim_{k \to \infty} q_k/(p_k r_{pq}^k) \quad (k \text{ is considered only if } p_k \neq 0)$$

exists and is finite.

COROLLARY 2.2.5. *Let* (2.1.1) *be satisfied. If* $c_p^{(0)} \subseteqq m_q$ *then either* $\rho_p < \infty$ *and* $\rho_q < \infty$ *or* $\rho_p = \rho_q = \infty$.

THEOREM 2.2.6. *Let* (2.1.1) *be satisfied and assume that* $\rho_p < \infty$. *If* $c_p^{(0)} \subseteqq m_q$ *then a function* ϕ *of bounded variation and constants* $0 < \theta < 1$, $m \in \{0, 1, \cdots\}$ *exist, which satisfy*:

$$q_k = p_k r_{pq}^k \left(\int_0^1 \tau^k d\phi(\tau) + O(\theta^k) \right) \quad (k \to \infty, k = m, m+1, \cdots) \tag{2.2.1}$$

$$\int_0^1 \mid P(x r_{pq} \tau) \mid \mid d\phi(\tau) \mid = O(Q(x)) \quad (x \to \rho_q^-, x \text{ real}) . \tag{2.2.2}$$

THEOREM 2.2.7. *Let* (2.1.1) *be satisfied and assume that* $\rho_p < \infty$. *If* $c_p \subseteqq c_q$ *then the limit* $1/Q(\rho_q - 0)$ *must exist and be finite, unless* $Q(x) \equiv \kappa P(x r_{pq})$, $\kappa \neq 0$ (in which case P and Q are trivially equivalent). *If, further,* $P \subseteqq Q$ (and $Q(x) \not\equiv \kappa P(x r_{pq})$) *then the said limit must be zero.*

REMARK 2.2.8. Theorem 2.2.7 cannot be extended to the case $\rho_p = \infty$. In fact an example is given in Section 3.1.15 of two essentially different power methods, P, Q, with $\rho_p = \rho_q = \infty$ which satisfy $P \subseteqq Q$, while the limit $1/Q(+\infty)$ does not exist.

THEOREM 2.2.9. *Let* (2.1.1) *be satisfied and assume that* $\rho_p = \infty$. *If* $c_p^{(0)} \subseteqq m_q$ *then there exist: a function* ϕ, *whose variation is bounded in* $[\varepsilon, 1]$, *for every* $\varepsilon > 0$, *constants* $0 \leqq R < +\infty$, $u(\alpha_q < u < +\infty$, $ur_{pq} > \alpha_p)$, $m \in \{0, 1, \cdots\}$ *and a matrix* (e_{xk}) $(u \leqq x < +\infty$, $k = 0, 1, \cdots)$ *such that*

(i) $q_k = p_k r_{pq}^k \int_{u/x}^1 \tau^k d\phi(\tau) + Q(x) e_{xk}/x^k$,

$$(u \leqq x < +\infty,\ k = 0, 1, \cdots).$$

(ii) $\int_{u/x}^1 |P(xr_{pq}\tau)| \, |d\phi(\tau)| = O(Q(x))$, $(x \to +\infty,\ x \text{ real})$.

(iii) $\begin{cases} |e_{xk}| \leqq R\,|p_k|\, u^k r_{pq}^k, & (u \leqq x < +\infty,\ k = m, m+1, \cdots) \\ |e_{xk}| \leqq R, & (u \leqq x < +\infty,\ k = 0, 1, \cdots). \end{cases}$

If, further, $c_p \subseteqq c_q$ *and we denote*

$$e'_{xk} = e_{xk} - [p_k u^k r_{pq}^k / P(ur_{pq})] \sum_{\imath=0}^{\infty} e_{x\imath} \quad (u \leqq x < +\infty,\ k = 0, 1, \cdots)$$

then, in addition,

(iv) *The limits*

$$e'_k = \lim_{x \to +\infty} e'_{xk} \quad (x \text{ real},\ k = 0, 1, \cdots)$$

all exist and are finite.

(v) *The limit*

$$\gamma(T) = \lim_{x \to +\infty} \int_{u/x}^1 P(xr_{pq}\tau) T(x\tau) d\phi(\tau)/Q(x) \quad (x \text{ real})$$

exists and is finite, for every function $T(t)$, *which is continuous in the interval* $[u, +\infty)$, *vanishes at its left end and has a finite limit* $T(+\infty)$.

If, further, $P \subseteqq Q$ *then, in addition,*

(iv)′ $e'_k = 0 \quad (k = 0, 1 \cdots)$

(v)′ $\gamma(T) = T(+\infty)$.

REMARK 2.2.10. It is of interest to note that ϕ is not constant in $[1 - \delta, 1]$ for any $\delta > 0$. This is true in Theorem 2.2.6 as well as in Theorem 2.2.9. In Theorem 2.2.6 ϕ is uniquely determined (up to an additive constant) in $[\theta, 1]$ and in Theorem 2.2.9 it is uniquely determined in $(0, 1]$.

THEOREM 2.2.11. *Assume that* $\rho_p < \infty$. *If* $0 < r_{pq} < \infty$, $\rho_q r_{pq} = \rho_p$, $1/Q(\rho_q - 0) = 0$ *and* (2.2.1), (2.2.2) *are satisfied, with* ϕ *of bounded variation and some constants* $0 < \theta < 1$, $m \in \{0, 1, \cdots\}$ *then* $P \subseteqq Q$.

COROLLARY 2.2.12. *Assume that* (2.1.1) *is satisfied,* $\rho_p < \infty$ *and* $1/Q(\rho_q - 0) = 0$. *If* $c_p^{(0)} \subseteqq m_q$ *then* $P \subseteqq Q$.

THEOREM 2.2.13. *Assume that* $\rho_p = \infty$. *If* $0 < r_{pq} < \infty$ *and conditions* (i), (iii), (iv), (v) *of Theorem* 2.2.9 *are satisfied, then* $c_p \subseteqq c_q$. *If conditions* (iv)′, (v)′ *of Theorem* 2.2.9 *are satisfied also then* $P \subseteqq Q$.

REMARK 2.2.14. In Theorems 2.2.11 and 2.2.13 it need not be pre-assumed that r_{pq} satisfies Definition 2.2.3. The rest of the requirements mentioned in the theorems suffice.

2.3. *Restricted results.* Power methods P with $p_k \geqq 0$ $(k = 0, 1, \cdots)$ were investigated more than others in the past. Therefore they are of special interest, and an attempt to speciallize some theorems for them is worthwhile. It happens that the results of this attempt gave rise to theorems which are applicable to conservative and regular power methods in general, and therefore we begin with the characterization of these kinds of methods:

2.3.1. DEFINITION OF $\hat{P}(x)$. Given $P(x) = \sum p_k x^k$, we define

$$\hat{P}(x) = \sum_{k=0}^{\infty} | p_k | x^k , \qquad (\alpha_p \leqq x < \rho_p) .$$

Obviously $| P(x) | \leqq \hat{P}(x)$, $\alpha_p \leqq x < \rho_p$.

THEOREM 2.3.2. *P is conservative if and only if*

(i) *A constant* $L > 0$ *exists such that*

$$L\hat{P}(x) \leqq | P(x) | \leqq \hat{P}(x) \qquad (\alpha_p \leqq x < \rho_p) \tag{2.3.1}$$

and

(ii) *The limit* $1/P(\rho_p - 0)$ *exists and is finite.*

P is regular if and only if, in addition

(ii)′ $1/P(\rho_p - 0) = 0$ *and*

(iii) $P(x)$ *is not a polynomial.*

REMARK 2.3.3. It should be observed that (iii) of Theorem 2.3.2 is automatically satisfied if $\rho_p < \infty$ and that (i) $\Rightarrow$ (ii)′ in case $\rho_p = \infty$ (unless $P(x) \equiv$ const.).

THEOREM 2.3.4. *Let P be conservative and satisfy* (2.1.1). *If* $c_p^{(0)} \subseteqq m_q$ *then the limit*

$$\lim_{x \to \rho_q -} Q(x)/P(xr_{pq}) \quad (x \text{ real})$$

exists and is finite. If, further, P is regular and $\rho_p = \infty$ *then in addition*

$$\lim_{x\to+\infty} | Q(x)/P(xr) | = \begin{cases} +\infty, & 0 < r < r_{pq} \\ 0 \quad , & r_{pq} < r < +\infty \end{cases}, \quad (x \text{ real}).$$

THEOREM 2.3.5. *Let* (2.1.1) *be satisfied and assume that* $\rho_p < \infty$ *and that the limit* $1/P(\rho_p - 0)$ *exists and is finite. If* $0 < r_{pq} < \infty$, $\rho_q r_{pq} = \rho_p$, *the limit* $1/Q(\rho_q - 0)$ *exists and is finite and* (2.2.1), (2.2.2) *are satisfied, with some* ϕ *of bounded variation,* $0 < \theta < 1$ *and some* $m \in \{0, 1, \cdots\}$ *then* $c_p \subseteqq c_q$.

2.3.6. DEFINITION OF e, $e^{(l)}$, U. We denote $e = (1, 1, 1, \cdots)$ and $e^{(l)} = (0, \cdots, 0, 1, 0, \cdots)$, where the single 1 is at the lth place. Also $U = \{e\} \cup \{e^{(l)} \mid l = 0, 1, \cdots\}$.

Conditions under which U forms a fundamental set in c_p are to be found in [21] (see also Włodarski [19] and Birkholc [2], [3]). In particular U is foundamental in the fields of the Abel and Borel methods (see Zeller [20] and Ryll-Nardzewski [15]).

THEOREM 2.3.7. *Assume that* $\rho_p = \infty$ *and that either* U *is fundamental in* c_p *or* $Q(\varepsilon x)/Q(x) \to 0$ $(x \to +\infty$, x real$)$ *for every sufficiently small* $\varepsilon > 0$. *If* $0 < r_{pq} < \infty$ *and conditions* (i), (ii), (iii) *of Theorem* 2.2.9 *are satisfied then* $P \subseteqq Q$.

REMARK 2.3.8. Theorem 2.3.7 is applicable to the case that Q is regular for in this case $Q(\varepsilon x)/Q(x) \to 0$ for every $0 < \varepsilon < 1$.

COROLLARY 2.3.9. *Let* (2.1.1) *be satisfied and assume that* $\rho_p = \infty$ *and that either* U *is fundamental in* c_p *or* $Q(\varepsilon x) /Q(x) \to 0$ $(x \to +\infty$, x real$)$ *for every sufficiently small* $\varepsilon > 0$. *If* $c_p^{(0)} \subseteqq m_q$ *then* $P \subseteqq Q$.

The following theorem provides an easy means of producing examples of inclusions between non-regular power methods, with $\rho_p = \infty$.

THEOREM 2.3.10. *Assume that* $\rho_p = \infty$ *and that the limit* $P(+\infty)$ *exists and is finite. If*

$$q_k = p_k r^k \int_0^1 \tau^k d\phi(\tau) \quad (k = 0, 1, \cdots)\,, \tag{2.3.2}$$

where $0 < r < \infty$ *and* ϕ *is of bounded variation, then* $\rho_q = \infty$ *and the limit* $Q(+\infty)$ *exists and is finite. If, further,* $Q(+\infty) \neq 0$ *then* $c_p \subseteqq c_q$. *If, in addition,* $P(0)[\phi(0+) - \phi(0)] = 0$ *then* $P \subseteqq Q$.

2.4. *Examples.* In this section we present some results of applying the general theorems of the previous sections to particular power methods.

2.4.1. *A simple test of inclusion.* Let us investigate among the particular power methods defined at the end of § 2.1.1, which pairs may satisfy an inclusion relation. An easy preliminary check can be performed by application of Theorem 2.2.4 and its Corollaries 2.2.5 and 2.3.4.

Thus, calculation of r_{pq} for the relevant pairs of methods immediately excludes, by (i) of Theorem 2.2.4, the possibility of inclusion between A_λ or L and (B, a, b). It also proves the impossibility of inclusion between (B, a, b) and (B, α, β) in case $a \neq \alpha$. (In fact no direct calculation of r_{pq} is needed. One can use Corollary 2.2.5 and Theorem 2.3.4 instead.)

No further conclusions can be drawn by (i) of Theorem 2.2.4. However (iii) of this theorem or, its corollary, the first part of Theorem 2.3.4, provides a finer test and its application shows that $L \not\subseteqq A_\mu \not\subseteqq A_\lambda$ if $\lambda > \mu > -1$ and that $(B, a, b) \not\subseteqq (B, a, \beta)$ if $a > 0$, $-\infty < \beta < b < +\infty$. So after a complete check with Theorem 2.2.4 it seems that the only possible inclusions are $A_\lambda \subseteqq A_\mu \subseteqq L$ $(\lambda \geqq \mu > -1)$ and $(B, a, \beta) \subseteqq (B, a, b)$ $(a > 0, -\infty < \beta \leqq b < +\infty)$. Those inclusions are, in fact, known to be valid and were proved by D. Borwein (see [4] and [8]) using methods which could be interpreted as applications of Theorems 2.2.11 and 2.3.7.

The results of the previous sections make it possible to solve completely certain inclusion problems. As examples we formulate necessary and sufficient conditions for the inclusions $A \subseteqq Q$, $P \subseteqq A$, $B \subseteqq Q$, $P \subseteqq B$ where A, B are Abel and Borel methods and P, Q are any power methods (not restricted in any sense).

In the following c_a, $c_b^{(0)}$, m_b denote, respectively, the field of the Abel method and the appropriate sets which are related to Borel method.

COROLLARY 2.4.2. *In order that $c_a \subseteqq c_q$ it is necessary and sufficient that the following is satisfied: $\rho_q < \infty$, the limit $1/Q(\rho_q - 0)$ exists and is finite and*

$$q_k \rho_q^k = \int_0^1 \tau^k d\phi(\tau) + O(\theta^k) \quad (k \to \infty)$$

where

$$\int_0^1 |d\phi(\tau)| < \infty\,, \qquad 0 < \theta < 1\,, \qquad \int_0^1 [\rho_q/(\rho_q - x\tau)]\,|d\phi(\tau)| = O(Q(x))$$
$$(x \to \rho_q^-,\ x \text{ real})\,.$$

The same, with the additional condition $1/Q(\rho_q - 0) = 0$, is necessary and sufficient for $A \subseteqq Q$.

COROLLARY 2.4.3. *In order that $c_p \subseteqq c_a$ it is necessary and sufficient that the following is satisfied: $\rho_p < \infty$, $p_k \neq 0$, for all sufficiently large k, and*

$$1/(p_k \rho_p^k) = \int_0^1 \tau^k d\phi(\tau) + O(\theta^k) \quad (k \to \infty)$$

where

$$\int_0^1 |d\phi(\tau)| < \infty , \quad 0 < \theta < 1 , \quad \int_0^1 |P(x\rho_p\tau)|\,|d\phi(\tau)| = O[1/(1-x)] ,$$
$$(x \to 1-, \ x \text{ real}) .$$

The same is necessary and sufficient for $P \subseteqq A$.

COROLLARY 2.4.4. *In order that $c_b^{(0)} \subseteqq m_q$ it is necessary and sufficient that*

$$0 < r_{bq} = \overline{\lim_{k\to\infty}} |q_k k!|^{1/k} < +\infty$$

and that

$$q_k = (r_{bq}^k/k!) \int_{u/x}^1 \tau^k d\phi(\tau) + Q(x) e_{xk}/x^k , \quad (u \leqq x < +\infty, k = 0, 1, \cdots)$$

where

$$\int_{u/x}^1 |d\phi(\tau)| < \infty , \quad (u \leqq x < +\infty)$$

$$\int_{u/x}^1 e^{xr_{bq}\tau} |d\phi(\tau)| = O(Q(x)) , \quad (x \to +\infty, \ x \text{ real})$$

$$|e_{xk}| \leqq R r_{bq}^k u^k/k! \quad (u \leqq x < +\infty, \ k = m, m+1, \cdots)$$

$$|e_{xk}| \leqq R , \quad (u \leqq x < +\infty, \ k = 0, 1, \cdots)$$

$$\alpha_q < u < +\infty , \quad 0 \leqq R < +\infty , \quad m \in \{0, 1, \cdots\} .$$

The same is necessary and sufficient for $B \subseteqq Q$.

COROLLARY 2.4.5. *In order that $c_p^{(0)} \subseteqq m_b$ it is necessary and sufficient that the following is satisfied: $p_k \neq 0$ for all sufficiently large k,*

$$0 < r_{pb} = \overline{\lim_{k\to\infty}} |p_k k!|^{-1/k} < \infty$$

and

$$1/k! = p_k r_{pb}^k \int_{u/x}^1 \tau^k d\phi(\tau) + e^x e_{xk}/x^k , \quad (u \leqq x < +\infty, k = 0, 1, \cdots)$$

where

$$\int_{u/x}^{1} |d\phi(\tau)| < \infty$$

$$|e_{xk}| \leqq R\,|p_k|\,u^k r_{pb}^k\,, \quad (u \leqq x < +\infty, k = m, m+1, \cdots)$$

$$|e_{xk}| \leqq R\,, \quad (u \leqq x < +\infty, k = 0, 1, \cdots)$$

$$\int_{u/x}^{1} |P(xr_{pb}\tau)|\,|d\phi(\tau)| = O(e^x) \quad (x \to +\infty, x \text{ real})$$

$$0 < u < +\infty\,, \quad 0 \leqq R < +\infty\,, \quad m \in \{0, 1, \cdots\}\,.$$

The same is necessary and sufficient for $P \subseteqq B$.

3. Proofs.

3.1. *Proofs of the results in* § 2.2.

3.1.1. *Proof of Remark* 2.2.2. Define

$$s_k = \begin{cases} 0 & , \ k \notin J \\ 1/(q_k a^k)\,, & k \in J \end{cases}, \quad \alpha_q < a < \rho_q\,, \qquad k = 0, 1, \cdots\,.$$

Obviously $s \in c_p^{(0)}$. However, $\sum q_k s_k a^k$ would not converge if J is infinite. Hence $s \notin m_q$, in such a case, which means that $c_p^{(0)} \subseteqq m_q$ cannot hold.

In the forthcoming proofs we use several results, which are cited below. The first is a generalization, due to R. Trautner [16] of a well known theorem of J. G. Mikusinski [14]:

TRAUTNER'S THEOREM 3.1.2 (R. Trautner [16]). *Let*

$$\mu_k = \int_0^1 \tau^k d\chi(\tau)\,, \quad \int_0^1 |d\chi(\tau)| < \infty \qquad (k = 0, 1, \cdots)$$

be Hausdorff moments. If

$$\mu_{k_i} = O(r^{k_i}) \quad (i \to \infty, 0 < r < 1, k_i \uparrow \infty, k_i \text{ natural numbers})$$

then either $\chi = \text{const.}$ *in* $[r, 1]$ *or* $\sum_i 1/k_i < \infty$.

An immediate corollary of Trautner's theorem is:

LEMMA 3.1.3. *Let* (2.1.1) *be satisfied and assume that* g *is a function of bounded variation in* $[\alpha_p, \rho_p)$, *which is constant in* $[r', \rho_p)$, *for some* $r' < \rho_p$. *If*

$$\left|P_k \int_{\alpha_p}^{\rho_p -} [t^k/P(t)]dg(t)\right| \leqq R\,|p_k|\,r^k\,, \quad \alpha_p < r < \rho_p \qquad (k = m, m+1, \cdots)$$

then $g = \text{const.}$ *in* $[r, \rho_p)$.

The rest of the cited results are taken from A. Ziv [21]:

LEMMA 3.1.4 (A. Ziv [21]). *Let* (2.1.1) *be satisfied and let* $n \in \{0, 1, \cdots\}$, $\varepsilon > 0$, $\alpha_p \leqq r < \rho_p$, $S_0, S_1, \cdots S_{n-1}$ *be given. Assume that* $T(x)$ *is any function, continuous in* $[r, \rho_p)$, *having a finite limit* $T(\rho_p - 0)$ *and satisfying*

$$T(r) = \sum_{k=0}^{n-1} p_k S_k r^k / P(r)$$

and let $\delta(x) > 0$ *be continuous in* $[r, \rho_p)$. *There exists a sequence* $s \in c_p$ *which satisfies*

$$\text{(3.1.1)} \quad \begin{cases} s_j = S_j \quad (j = 0, 1, \cdots, n-1)\,; & \sum_{k=n}^{\infty} |p_k s_k| r^k < \varepsilon \\ |T_p(s, x) - T(x)| < \delta(x) \quad (r \leqq x < \rho_p)\,, & \lim_p s = T(\rho_p - 0)\,. \end{cases}$$

THEOREM 3.1.5 (A. Ziv [21]). *Let* (2.1.1) *be satisfied. If* $\sum \beta_k s_k$ *converges for all* $s \in c_p^{(0)}$ *then*

$$\text{(3.1.2)} \quad |\beta_k| \leqq R|p_k| r^k, \quad r < \rho_p, \quad k = m, m+1, \cdots, m \in \{0, 1, \cdots\}\,.$$

3.1.6. *Matrix methods of limitation* (see Lazić [13] and Ziv [21]). Let $W = \{w_{xk}\}(x \in I, k = 0, 1, \cdots)$ be an infinite matrix of complex numbers with I a subset of some topological space. Let $x_0 \notin I$ be a point of accumulation of I, which has a denumerable basis of neighborhoods. A sequence $s = \{s_k\}$ of complex numbers is said to be *W-ocnvergent* to σ if its *W-transform*,

$$T_w(s, x) = \sum_{k=0}^{\infty} w_{xk} s_k\,,$$

exists for all $x \in I$ and $\lim_{x \to x_0} T_w(s, x) = \sigma$ $(x \in I)$. We denote by m_w the set of all sequences whose W-transforms exist and are bounded in I.

THEOREM 3.1.7 (A. Ziv [21]). *Let* (2.1.1) *be satisfied. If* $c_p^{(0)} \subseteqq m_w$ *then the matrix* W *may be decomposed in the form* $W = C + D$, *where the matrices* $C = (c_{xk})$, $D = (d_{xk})$ $(x \in I, k = 0, 1, \cdots)$ *satisfy*

(i) $c_{xk} = p_k \int_{\alpha_p}^{r_x} [t^k / P(t)] dg_x(t)$, $\int_{\alpha_p}^{r_x} |dg_x(t)| \leqq R < \infty$, $\alpha_p < r_x < \rho_p$, $(x \in I, k = 0, 1, \cdots)$.

(ii) $|d_{xk}| \leqq R|p_k| r^k$, $\alpha_p < r < \rho_p$, $(x \in I, k = m, m+1, \cdots, m \in \{0, 1, \cdots\})$; $|d_{xk}| \leqq R < \infty$ $(x \in I, k = 0, 1, \cdots)$.

r_x, g_x are independent of k and R, r, m are independent of both k and x.

3.1.8. *Proof of* (i) *and* (ii) *of Theorem* 2.2.4. If $c_p^{(0)} \subseteqq m_q$ then $\sum q_k s_k x^k$ converges for all $s \in c_p^{(0)}$. Therefore it follows, from Theorem 3.1.5, that

$$xr_{pq} < \rho_p \quad \text{for all} \quad x \in [\alpha_q, \rho_q) \tag{3.1.3}$$

which implies that

$$r_{pq} < \infty \;, \quad \rho_q r_{pq} \leqq \rho_p \;. \tag{3.1.4}$$

Considering only values of k such that $p_k \neq 0$ we get

$$\rho_p^{-1} r_{pq} = (\overline{\lim_{k\to\infty}} \, |p_k|^{1/k})(\overline{\lim_{k\to\infty}} \, |q_k/p_k|^{1/k}) \geqq \overline{\lim_{k\to\infty}} \, |q_k|^{1/k} \;;$$

so by Remark 2.2.2

$$r_{pq}/\rho_p \geqq 1/\rho_q \;. \tag{3.1.5}$$

Now we divide the discussion into two cases. First the case $\rho_q < \infty$: In this case (3.1.5) implies—since $r_{pq} < \infty$-that $\rho_p < \infty$ and that $r_{pq} > 0$. This by (3.1.4) and (3.1.5) completes the proof.

Next assume that $\rho_q = \infty$. Had we shown that $r_{pq} > 0$ we would get from (3.1.3) that $\rho_p = \infty$ which completes the proof. So let us try to reach a contradiction while assuming that $r_{pq} = 0$.

The method Q is a matrix method (see § 3.1.6) with $w_{xk} = q_k x^k / Q(x)$, $(x \in [\alpha_q, +\infty), k = 0, 1, \cdots)$. The assumption $r_{pq} = 0$ means that

$$\lim_{k\to\infty} |w_{xk}/p_k|^{1/k} = 0 \quad (k \text{ is considered only if } p_k \neq 0) \;.$$

Hence from Theorem 3.1.7 we get

$$\left| p_k \int_{\alpha_p}^{r_x} [t^k/P(t)] dg_x(t) \right| = p_k O(r^k) \;, \quad (k \to \infty)$$

which implies, by Lemma 3.1.3, that r_x need not exceed r. Estimating w_{xk} by Theorem 3.1.7, we get, therefore,

$$|q_k x^k / Q(x)| \leqq M |p_k| r^k \;, \quad r < \rho_p \;, \qquad (\alpha_q \leqq x < +\infty, k = m, m+1, \cdots) \tag{3.1.6}$$

where, M and r are independent of both x and k. Let $l \in \{0, 1, \cdots\}$ be larger than both m and k and such that $q_l \neq 0$ (see Remark 2.2.1). We get

$$|x^k/Q(x)| = |q_l x^{l-k}|^{-1} |q_l x^l / Q(x)| \leqq |q_l x^{l-k}|^{-1} M |p_l| r^l \to 0, \; (x \to +\infty)$$

which leads to the following absurdity:

$$1 = \lim_{x\to+\infty} \sum_{k=0}^{\infty} q_k x^k/Q(x) = \sum_{k=0}^{\infty} \lim_{x\to+\infty} q_k x^k/Q(x) = 0 .$$

The summation-limitation exchange is permitted because the series is majorized by $\sum M|p_k|r^k$ (see (3.1.6)).

REMARK 3.1.9. In view of (i), (ii) of Theorem 2.2.4, it is clear that the variable x of $Q(x)$ may, in cases of inclusion, be scaled to yield $r_{pq} = 1$, $\rho_q = \rho_p = \rho$. We may also increase either α_p or α_q to get $\alpha_q = \alpha_p = \alpha$. It enables us to simplify later proofs by assuming those "normalizing conditions".

THE FUNDAMENTAL LEMMA 3.1.10. *Let* (2.1.1) *be satisfied and assume that* $r_{pq} = 1$, $\rho_q = \rho_p = \rho$, $\alpha_q = \alpha_p = \alpha$. *If* $c_p^{(0)} \subseteqq m_q$ *then there exist a function* $\chi(t)$, *a matrix* e_{xk} *and constants* $R < \infty$, $\alpha < u < \rho$, $m \in \{0, 1, \cdots\}$ *such that:*

$$\int_{u/x}^{1} |d\chi(\tau)| < \infty , \quad (u \leqq x < \rho) \tag{3.1.7}$$

$$q_k = p_k \int_{u/x}^{1} \tau^k d\chi(\tau) + Q(x)e_{xk}/x^k , \quad (u \leqq x < \rho,\ k = 0, 1, \cdots) \tag{3.1.8}$$

$$|Q(x)|^{-1} \int_{u/x}^{1} |P(x\tau)||d\chi(\tau)| \leqq R , \quad (u \leqq x < \rho) \tag{3.1.9}$$

$$\begin{cases} |e_{xk}| \leqq R|p_k|u^k , & (u \leqq x < \rho,\ k = m, m+1, \cdots) \\ |e_{xk}| \leqq R , & (u \leqq x < \rho,\ k = 0, 1, \cdots) . \end{cases} \tag{3.1.10}$$

Proof. Using Theorem 3.1.7 we see that

$$q_k x^k/Q(x) = c_{xk} + d_{xk} \quad (\alpha \leqq x < \rho,\ k = 0, 1, \cdots) \tag{3.1.11}$$

where c_{xk}, d_{xk} satisfy (i), (ii) of that theorem.

Define $u = r$ (hence $\alpha < u < \rho$). Since $r_{pq} = 1$ we have from (3.1.11), for all $r = u \leqq x < \rho$, $\varepsilon > 0$ and for $k \to \infty$,

$$c_{xk} = p_k \int_{\alpha}^{r_x} [t^k/P(t)]dg_x(t) = q_k x^k/Q(x) - d_{xk} = p_k O[(1+\varepsilon)^k x^k] .$$

Hence, defining $g_x = $ const. in $[r_x, \rho)$, we get, from Lemma 3.1.3, that $g_x = $ const. in $[x, \rho)$, so

$$c_{xk} = p_k \int_{\alpha}^{x} [t^k/P(t)]dg_x(t) , \quad (u \leqq x < \rho,\ k = 0, 1, \cdots) .$$

And if we define

$$\chi_x(\tau) = -Q(x)\int_{x\tau}^{x} [1/P(t)]dg_x(t)\ , \quad (\alpha/x \leqq \tau \leqq 1,\ u \leqq x < \rho)$$

we get from (3.1.11)

$$(3.1.12) \quad q_k = p_k \int_{\alpha/x}^{1} \tau^k d\chi_x(\tau) + Q(x)d_{xk}/x^k\ , \quad (u \leqq x < \rho,\ k = 0, 1, \cdots)\ .$$

Let $u \leqq x \leqq y < \rho$. From (3.1.12) we get

$$q_k = p_k \int_{\alpha/y}^{1} \tau^k d\chi_y(\tau) + Q(y)d_{yk}/y^k\ .$$

Comparing this with (3.1.12) and using (ii) of Theorem 3.1.7 we get for $k \longrightarrow \infty$

$$p_k \int_{\alpha/x}^{1} \tau^k d[\chi_x(\tau) - \chi_y(\tau)] = p_k O(u^k/x^k)\ , \quad (u \leqq x \leqq y < \rho)\ .$$

From Trautner's theorem (see 3.1.2), and because $\chi_x(1) = \chi_y(1) = 0$, we deduce that $\chi_y(\tau) = \chi_x(\tau)$ whenever $u \leqq x \leqq y < \rho$ and $u/x \leqq \tau \leqq 1$. This enables us to, uniquely, define a function $\chi(\tau)$ in $(u/\rho, 1]$ by:

$$\chi(\tau) = \chi_x(\tau)\ , \quad \tau \in [u/x, 1]\ , \qquad (u \leqq x < \rho)\ .$$

(3.1.7) follows now immediately from the definition of χ_x. Obviously

$$|Q(x)|^{-1}\int_{u/x}^{1} |P(x\tau)|\,|d\chi(\tau)| = |Q(x)|^{-1}\int_{u/x}^{1} |P(x\tau)|\,|d\chi_x(\tau)|$$
$$= \int_{u}^{x} |dg_x(t)| \leqq R\ .$$

So (3.1.9) is satisfied.

e_{xk} is defined now by (3.1.8). From (3.1.11) and (3.1.8) we get

$$|e_{xk}| = \left| q_k x^k/Q(x) - p_k[Q(x)]^{-1}\int_{u/x}^{1} (x\tau)^k d\chi_x(\tau) \right|$$
$$= \left| d_{xk} + p_k \int_{\alpha}^{u} [t^k/P(t)]dg_x(t) \right| \leqq |d_{xk}| + |p_k|\, u^k R \sup_{\alpha \leqq t \leqq u} |P(t)|^{-1}\ ,$$

which implies (3.1.10), by (ii) of Theorem 3.1.7.

3.1.11. *Proof of* (iii) *of Theorem* 2.2.4. By Remark 3.1.9 we may restrict the discussion to the case $r_{pq} = 1$. In this case (iii) follows immediately from Lemma 3.1.10. In fact we get $\lim_{k\to\infty} q_k/p_k = \chi(1) - \chi(1-0)$.

3.1.12. *Proof of Theorem* 2.2.6. We insert in Lemma 3.1.10 $u < y < \rho_q$ in place of x and define

$$\phi(\tau) = \begin{cases} \chi(\tau) , & u/y \leqq \tau \leqq 1 \\ \text{const.} , & 0 \leqq \tau \leqq u/y . \end{cases}$$

Then (2.2.1), with $\theta = u/y$, follows immediately from (3.1.8) and 3.1.10). (2.2.2) follows from (3.1.9).

LEMMA 3.1.13.

(a) *If in Lemma* 3.1.10 *we add the assumption* $c_p \subseteqq c_q$, *then the limit*

$$\gamma(T) = \lim_{x \to \rho -} [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau)T(x\tau)d\chi(\tau) , \quad (x \textit{ real})$$

must exist and be finite for every function $T(t)$ *which is continuous in* $[u, \rho)$, *vanishes at its left end and has a finite limit* $T(\rho - 0)$.

(b) *If in addition* $P \subseteqq Q$ *then* $\gamma(T) = T(\rho - 0)$ *for every such a function.*

Proof. Substituting $r = u$ in Lemma 3.1.4 we infer the existence of a sequence of sequences $s^{(l)} \in c_p (l = 1, 2, \cdots)$ which satisfy $\lim_p s^{(l)} = T(\rho - 0)$ and

$$\text{(3.1.13)} \qquad \begin{cases} s_0^{(l)} = s_1^{(l)} = \cdots = s_{m-1}^{(l)} = 0 , & \sum_{k=0}^{\infty} | p_k s_k^{(l)} | u^k < 1/l \\ | T_p(s^{(l)}, x) - T(x) | < 1/l , & (u \leqq x < \rho, l = 1, 2, \cdots) . \end{cases}$$

By (3.1.8)

$$T_q(s^{(l)}, x) = [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau)T_p(s^{(l)}, x\tau)d\chi(\tau) + \sum_{k=0}^{\infty} e_{xk} s_k^{(l)} .$$

We denote $\lim_q s^{(l)} = T_q(s^{(l)}, \rho - 0) = \beta_l$ (the limit exists since $s^{(l)} \in c_p \subseteqq c_q$). From (3.1.9) and (3.1.10) we have, then,

$$\begin{aligned} &\overline{\lim_{x \to \rho -}} \left| [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau)T(x\tau)d\chi(\tau) - \beta_l \right| \\ &\leqq \overline{\lim_{x \to \rho -}} | Q(x) |^{-1} \int_{u/x}^{1} | P(x\tau) | \, l^{-1} | d\chi(\tau) | + \overline{\lim_{x \to \rho -}} \sum_{k=0}^{\infty} | e_{xk} s_k^{(l)} | \\ &\leqq R/l + \sum_{k=0}^{\infty} R \, | p_k s_k^{(l)} | \, u^k \leqq 2R/l . \end{aligned}$$

From this it is easily deduced that $\{\beta_l\}$ is a Cauchy sequence-hence convergent. Denoting its limit by β we have

$$\overline{\lim_{x \to \rho -}} \left| [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau)T(x\tau)d\chi(\tau) - \beta \right| \leqq 2R/l + | \beta_l - \beta |$$

which yields, with $l \to \infty$, $\gamma(T) = \beta$.

This completes the proof of part (a). Part (b) is proved in a very similar way. Only this time, since $s^{(l)} \in c_p$, $P \subseteqq Q$, we have

$$\beta_l = \lim_q s^{(l)} = \lim_p s^{(l)} = T(\rho - 0) \,; \quad \text{so} \quad \gamma(T) = \beta = T(\rho - 0) \,.$$

3.1.14. *Proof of Theorem* 2.2.7. By Remark 3.1.9 we may consider the case $r_{pq} = 1$, $\rho_q = \rho_p = \rho$, $\alpha_q = \alpha_p = \alpha$ only. Assume, first, that $c_p \subseteqq c_q$ and that the limit $1/Q(\rho_q - 0)$ does not exist. We have to show that $Q(x) = \kappa P(x)$:

Let $u < t_0 < t_1 < \rho$ and let $h(t)$ be some continuous function in $[u, \rho)$, which vanishes in $[u, t_0]$ and in $[t_1, \rho)$. Denote $T(t) = h(t)/P(t)$. From Lemma 3.1.13 (a) we infer the existence and finiteness of the limit

$$\gamma(T) = \lim_{x \to \rho -} [Q(x)]^{-1} \int_{t=u}^{x} P(t)T(t)d\chi(t/x) = \lim_{x \to \rho -} [Q(x)]^{-1} \int_{t=t_0}^{t_1} h(t)d\chi(t/x) \,.$$

Denoting

$$\chi_x(t) = \chi(t/x) \,, \quad (t_0 \leqq t \leqq x,\ t_1 < x < \rho)$$

we get

$$\gamma(T) = \lim_{x \to \rho -} [Q(x)]^{-1} \int_{t_0}^{t_1} h(t)d\chi_x(t) \,. \tag{3.1.14}$$

The functions $\chi_x(t)$ $(t_1 \leqq x < \rho)$ have uniformly bounded variation in $[t_0, t_1]$. This is because denoting $u\rho/t_0 = y$ we have $u < y < \rho$ and

$$\int_{t_0}^{t_1} |d\chi_x(t)| = \int_{t=t_0}^{t_1} |d\chi(t/x)| \leqq \int_{t_0/x}^{1} |d\chi(\tau)| \leqq \int_{u/y}^{1} |d\chi(\tau)| < \infty \,, \qquad (t_1 \leqq x < \rho) \,.$$

Therefore the following limit exists (see Widder [17] Theorem 16.4 Ch. I):

$$\lim_{x \to \rho -} \int_{t_0}^{t_1} h(t)d\chi_x(t) = \int_{t_0}^{t_1} h(t)d[\lim_{x \to \rho -} \chi_x(t)] = \int_{t_0}^{t_1} h(t)d\chi(t/\rho) \,.$$

Since $1/Q(x)$ does not converge to a finite limit, as $x \to \rho^-$, we infer from the existence and finiteness of $\gamma(T)$ (see (3.1.14)) that

$$\int_{t_0}^{t_1} h(t)d\chi(t/\rho) = 0 \,.$$

The argument which led to this result is correct for every function $h(t)$, which is continuous in $[t_0, t_1]$ and vanishes at both its ends. Hence $\chi(t/\rho) = \text{const.}$ for $t \in (t_0, t_1)$. Since t_0 and t_1 may be taken as close as one wishes to u and ρ, respectively, we actually have $\chi(\tau) = \text{const.}$ for $\tau \in (u/\rho, 1)$. Hence, by (3.1.8)

$$q_k = \kappa p_k + Q(x)e_{xk}/x^k\ , \quad \kappa = \chi(1) - \chi(1-0)$$
$$(u \leqq x < \rho,\ k = 0, 1, \cdots)$$

which proves that $Q(x)e_{xk}/x^k$ is independent of x. We may therefore write

$$(3.1.15)\quad \begin{cases} q_k = \kappa p_k + q'_k\ , \quad (k = 0, 1, \cdots) \\ \text{where by (3.1.10),} \\ |q'_k x^k/Q(x)| \leqq R\,|p_k|\,u^k\ , \quad (u \leqq x < \rho,\ k = m, m+1, \cdots)\ . \end{cases}$$

From this it follows immediately that $\kappa \neq 0$ (otherwise we would have $r_{pq} < 1$).

We complete the proof by showing that $q'_k = 0$ $(k = 0, 1, \cdots)$:

Let $i \in \{0, 1, \cdots\}$ and choose $r = u$,

$$T(x) = \begin{cases} p_i r^i/p(r)\ , & x = r \\ \text{linear}\ , & r \leqq x \leqq r_1\ ; \quad r < r_1 < \rho \\ 0\ , & r_1 \leqq x < \rho \end{cases}$$

in Lemma 3.1.4. We infer the existence of a sequence of sequences, $s^{(l)} \in c_p^{(0)}$ $(l = 1, 2, \cdots)$, such that

$$(3.1.16)\quad \begin{cases} s_j^{(l)} = 0,\ (j \neq i,\ j < \max\{i, m\})\ , \quad s_i^{(l)} = 1 \\ \sum_{k=i+1}^{\infty} |p_k s_k^{(l)}|\,u^k < 1/l\ , \quad |T_p(s^{(l)}, x)| < \delta(x)/l \quad (r_1 \leqq x < \rho)\ . \end{cases}$$

From (3.1.15) we have

$$T_q(s^{(l)}, x) = \kappa[P(x)/Q(x)]T_p(s^{(l)}, x) + q'_i x^i/Q(x) + \sum_{k=i+1}^{\infty} q'_k s_k^{(l)} x^k/Q(x)$$

so, for $r_1 \leqq x < \rho$ we get from (3.1.15) and (3.1.16)

$$\begin{aligned} |q'_i x^i/Q(x) - T_q(s^{(l)}, x)| &\leqq |\kappa[P(x)/Q(x)]T_p(s^{(l)}, x)| + \sum_{k=i+1}^{\infty} |q'_k s_k^{(l)}|\,x^k/|Q(x)| \\ &\leqq |\kappa P(x)/Q(x)|\,\delta(x)/l + R/l\ . \end{aligned}$$

Choosing properly $\delta(x) > 0$ (e.g. $\delta(x) = |Q(x)/P(x)|$) we may then get,

$$|q'_i x^i/Q(x) - T_q(s^{(l)}, x)| < M/l\ , \quad M < \infty\ , \quad (r_1 \leqq x < \rho,\ l = 1, 2, \cdots)\ .$$

Since $s^{(l)} \in c_p^{(0)} \subseteqq c_q$ the limits $\beta_l = \lim_{x \to \rho-} T_q(s^{(l)}, x)$ $(l = 1, 2, \cdots)$ all exist and we get

$$\overline{\lim_{x \to \rho-}}\ |q'_i x^i/Q(x) - \beta_l| \leqq M/l \qquad (l = 1, 2, \cdots)\ .$$

This implies that β_l is a Cauchy sequence, thus having a limit β. Obviously

$$\overline{\lim_{x\to\rho-}}\, |q_i' x^i/Q(x) - \beta| \leqq M/l + |\beta_l - \beta| ,$$

so letting l tend to ∞ we infer the existence of the limit $\lim_{x\to\rho-} q_i' x^i/Q(x) = \beta$, which is impossible, since $1/Q(\rho - 0)$ does not exist, unless $q_i' = 0$. This completes the proof for the case $c_p \subseteqq c_q$.

The proof for the case $P \subseteqq Q$ is similar. We just have to use Lemma 3.1.13 (b) instead of (a) to infer that $\gamma(T) = 0$, and then remember that $s^{(l)} \in c_p^{(0)} \subseteqq c_q^{(0)}$, so $\beta_l = 0$ $(l = 1, 2, \cdots)$, which means that $\beta = 0$ and therefore $q_i' = 0$, unless $1/Q(\rho - 0) = 0$.

3.1.15. *Proof of Remark* 2.2.8. First let us show the existence of two entire functions $P(x) = \sum p_k x^k$, $Q(x) = \sum q_k x^k$ which satisfy

(a) $Q(x) = \left(\int_0^x P(t)dt\right)\Big/x = \sum_{k=0}^{\infty} p_k x^k/(k+1)$, $(0 < x < +\infty)$

(b) $P(x) \geqq 1$, $(0 \leqq x < +\infty)$

(c) The limit $1/Q(+\infty)$ does not exist.

In order to prove this, notice that the function

$$f(x) = \int_0^x [(1 - e^{-t})/t]dt = x/1\cdot 1! - x^2/2\cdot 2! + x^3/3\cdot 3! - \cdots$$

is entire and satisfies $f(+\infty) = +\infty$. Hence the function $Q(x) = 3 + \sin[\theta f(x)]$ (where $0 < \theta \leqq 1$ is a constant to be specified later) is also entire and clearly satisfies (c). The relation (a), then, defines $P(x)$ to be:

$$P(x) = [xQ(x)]' = 3 + \sin[\theta f(x)] + \theta(1 - e^{-x})\cos[\theta f(x)] ,$$

which is obviously an entire function that satisfies (b). Inserting the power series expansion of $f(x)$ into the expansion of the sin, in order to obtain the expansion of $Q(x)$, we see that each of the coefficients q_k is a polynomial in θ which is not the null polynomial. One may choose θ to differ from all of the roots of these polynomials and get $q_k \neq 0$ $(k = 0, 1, \cdots)$. It follows, then, from (a), that (2.1.1) is satisfied also.

Now, from (a) it follows that

$$T_q(s, x) = \left[\int_0^x P(t)T_p(s, t)dt\right]\Big/\left[\int_0^x P(t)dt\right], \quad (0 < x < +\infty) .$$

Hence, by (b), $P \subseteqq Q$. This is so although (c) is satisfied.

It should be noticed that, by (a) and Theorem 2.2.4 (iii), $Q \nsubseteqq P$ so P and Q are essentially different power methods.

Examples of pairs of power methods, $P \subseteqq Q$ for which $1/Q(+\infty) \neq 0$, can be easily constructed by Theorem 2.3.10.

LEMMA 3.1.16.

(a) *If in Lemma* 3.1.10 *we add the assumption* $c_p \subseteqq c_q$ *and denote*

$$e'_{xk} = e_{xk} - [p_k u^k / P(u)] \sum_{i=0}^{\infty} e_{xi}\,, \quad (u \leqq x < \rho,\, k = 0, 1, \cdots) \tag{3.1.17}$$

then the limits

$$e'_k = \lim_{x \to \rho-} e'_{xk} \qquad (x \text{ real},\ k = 0, 1, \cdots) \tag{3.1.18}$$

all exist and are finite. (*b*) *If, in addition,* $P \subseteqq Q$ *then for all* $k \in \{0, 1, \cdots\}$ $e'_k = 0$.

Proof. First, we notice that by (3.1.10) and (3.1.17)

$$\begin{cases} |e'_{xk}| \leqq R' |p_k| u^k\,, & (u \leqq x < \rho,\, k = m, m+1, \cdots) \\ |e'_{xk}| \leqq R'\,, & (u \leqq x < \rho,\, k = 0, 1, \cdots)\,. \end{cases} \tag{3.1.19}$$

Next, we see from (3.1.8) that

$$Q(x) = \int_{u/x}^{1} P(x\tau) d\chi(\tau) + Q(x) \sum_{k=0}^{\infty} e_{xk}\,,$$

so,

$$\sum_{i=0}^{\infty} e_{xi} = 1 - [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau) d\chi(\tau)\,.$$

Substituting this into (3.1.17) and using (3.1.8) we get for every $s \in c_p$:

$$\begin{aligned} T_q(s, x) = T_p(s, u) + [Q(x)]^{-1} \int_{u/x}^{1} P(x\tau)[T_p(s, x\tau) \\ - T_p(s, u)] d\chi(\tau) + \sum_{k=0}^{\infty} e'_{xk} s_k\,. \end{aligned} \tag{3.1.20}$$

Let now $i \in \{0, 1, \cdots\}$ and choose in Lemma 3.1.4 $r = u$ and $T(x) \equiv p_i u^i / P(u)$. From Lemma 3.1.4 we infer the existence of a sequence of sequences $s^{(l)} \in c_p$ $(l = 1, 2, \cdots)$, that satisfy

$$s_j^{(l)} = 0 \quad (j \neq i,\, j < \max\{i, m\})\,, \quad s_i^{(l)} = 1\,, \quad \sum_{k=i+1}^{\infty} |p_k s_k^{(l)}| u^k < 1/l\,.$$

Since $s^{(l)} \in c_p \subseteqq c_q$ we infer, from (3.1.20) with the aid of Lemma 3.1.13(a), the existence and finiteness of the limits

$$\beta_l = \lim_{x \to \rho-} \sum_{k=0}^{\infty} e'_{xk} s_k^{(l)} \qquad (l = 1, 2, \cdots)\,.$$

By (3.1.19)

$$|e'_{xi} - \sum_{k=0}^{\infty} e'_{xk} s_k^{(l)}| = |-\sum_{k=i+1}^{\infty} e'_{xk} s_k^{(l)}| \leqq R'/l \ .$$

Therefore

$$\overline{\lim_{x \to \rho-}} |e'_{xi} - \beta_l| \leqq R'/l \ .$$

From this it follows that β_l is a Cauchy sequence. Denoting its limit by β we get then,

$$\overline{\lim_{x \to \rho-}} |e'_{xi} - \beta| \leqq R'/l + |\beta - \beta_l| \ ,$$

which yields, with $l \to \infty$, that actually $e'_i = \beta$.

Thus (a) is proved. The proof of (b) is similar. We just have to notice that in case $P \subseteqq Q$ we get from (3.1.20) and from Lemma 3.1.13(b) that $\beta_l = 0$ $(l = 1, 2, \cdots)$. Hence $e'_i = \beta = 0$.

3.1.17. *Proof of Theorem* 2.2.9. This is an immediate consequence of Lemmas 3.1.10, 3.1.13, and 3.1.16.

3.1.18. *Proof of Remark* 2.2.10. The fact that ϕ is not constant in $[1 - \delta, 1]$ follows immediately from the definition of r_{pq} (see § 2.2.3). The uniqueness of ϕ in certain intervals follows from Trautner's theorem (see § 3.1.2).

3.1.19. *Proof of Theorem* 2.2.11. It is sufficient to consider the case $r_{pq} = 1$, $\rho_q = \rho_p = \rho$, $\alpha_q = \alpha_p = \alpha$ and to show that $c_p^{(0)} \subseteqq c_q^{(0)}$. By (2.2.1) we may write

$$q_k = p_k\Big(\int_0^1 \tau^k d\phi(\tau) + \theta_k\Big) + q'_k \ , \quad |\theta_k| \leqq R\theta^k \qquad (k = 0, 1, \cdots)$$

where $q'_k = 0$ for $k \geqq m$. Hence for every $s \in c_p^{(0)}$

$$T_q(s, x) = [Q(x)]^{-1} \int_0^1 P(x\tau) T_p(s, x\tau) d\phi(\tau) + [Q(x)]^{-1} \sum_{k=0}^{\infty} \theta_k p_k s_k x^k$$
$$+ [Q(x)]^{-1} \sum_{k=0}^{m-1} q'_k s_k x^k = \sigma_1(x) + \sigma_2(x) + \sigma_3(x) \ .$$

It easily follows that $\sigma_1(x) \to 0$, $\sigma_2(x) \to 0$, $\sigma_3(x) \to 0$ $(x \to \rho-)$ for every $s \in c_p^{(0)}$; so $T_q(s, x) \to 0$, which completes the proof.

3.1.20. *Proof of Theorem* 2.2.13. It is sufficient to consider the case $\rho_q = \rho_p = \infty$, $r_{pq} = 1$, $\alpha_q = \alpha_p = \alpha$. The proof is based on (3.1.19) and (3.1.20), which are deduced here as in section 3.1.16.

It should be observed that (3.1.19) implies:

$$\lim_{x\to+\infty}\sum_{k=0}^{\infty} e'_{xk}s_k = \sum_{k=0}^{\infty}(\lim_{x\to+\infty} e'_{xk})s_k = \sum_{k=0}^{\infty} e'_k s_k .$$

3.1.21. *Proof of Remark* 2.2.14. Follows immediately from the proofs of Theorems 2.2.11 and 2.2.13.

3.2. *Proofs of the results in Section* 2.3.

3.2.1. *Proof of Theorem* 2.3.2. This follows in a straightforward way from Theorem III of Włodarski [19].

LEMMA 3.2.2. *If P is a regular power method with $\rho_p = \infty$ then* $P(\varepsilon x)/P(x) \to 0$ ($x \to +\infty$, x real) *for every* $0 < \varepsilon < 1$.

Proof. If $0 < \varepsilon < 1$ then $\varepsilon^k \to 0$ ($k \to \infty$). Hence the regularity of P yields

$$\lim_{x\to+\infty} P(\varepsilon x)/P(x) = \lim_{x\to+\infty}\sum_{k=0}^{\infty} p_k\varepsilon^k x^k/P(x) = \lim{}_p\,\{\varepsilon^k\} = 0 .$$

3.2.3. *Proof of Theorem* 2.3.4. Assume first that P is conservative. We define $s_j = 0$ if $j \in J$ (see Remark 2.2.2), $s_j = q_j/(p_j r_{pq}^j)$ if $p_j \neq 0$ and $s_j = \lim q_k/(p_k r_{pq}^k)$ if $p_j = 0$, $j \notin J$. From Theorem 2.2.4 (iii) we see that the sequence $s = \{s_k\}$ is well defined and convergent. Now,

$$\begin{aligned} Q(x)/P(xr_{pq}) &= [P(xr_{pq})]^{-1}\sum_{k=0}^{\infty} q_k x^k = [P(xr_{pq})]^{-1}\sum_{k=0}^{\infty} p_k s_k (xr_{pq})^k \\ &\quad + [P(xr_{pq})]^{-1}\sum_{k\in J} q_k x^k = T_p(s, xr_{pq}) + \sigma(x) . \end{aligned}$$

By Theorem 2.3.2 $\lim_{x\to\rho_q -}\sigma(x)$ exists, and from Theorem 2.2.4, and since P is conservative, $\lim_{x\to\rho_q -} T_p(s, xr_{pq})$ also exists. This establishes the existence of $\lim Q(x)/P(xr_{pq})$.

Now assume that $\rho_p = \infty$ and P is regular. We may write

$$Q(x)/P(rx) = [Q(x)/P(r_{pq}x)]\cdot[P(r_{pq}x)/P(rx)]$$

and this, by Lemma 3.2.2, tends to zero in case $r > r_{pq}$.

If $0 < r < r_{pq}$ we may choose $\tau_0 \in (r/r_{pq}, 1)$ and have, by Theorems 2.2.9 and 2.3.2, for $x \to +\infty$,

$$\begin{aligned} O(1) &= |Q(x)|^{-1}\int_{\tau_0}^{1} |P(xr_{pq}\tau)|\,|d\phi(\tau)| \geqq L\,|Q(x)|^{-1}\int_{\tau_0}^{1}\hat{P}(xr_{pq}\tau)\,|d\phi(\tau)| \\ &\geqq L\,|P(xr_{pq}\tau_0)/Q(x)|\int_{\tau_0}^{1}|d\phi(\tau)| . \end{aligned}$$

From Remark 2.2.10 $\int_{\tau_0}^{1}|d\phi(\tau)| \neq 0$; so $P(xr_{pq}\tau_0)/Q(x) = O(1)$ ($x \to +\infty$).

From Lemma 3.2.2 we have then,

$$P(rx)/Q(x) = [P(xr_{pq}\tau_0)/Q(x)]\cdot[P(rx)/P(xr_{pq}\tau_0)] \longrightarrow 0 \qquad (x \longrightarrow +\infty)$$

which completes the proof.

LEMMA 3.2.4. *Let E be an FK space and let $\{f_n\}$ be a sequence of continuous linear functionals on E. If $\lim_{n\to\infty} f_n(s)$ exists for all terms s of some fundamental set in E and if the sequence $\{f_n(s)\}$ is bounded for every $s \in E$ then the limit $f(s) = \lim_{n\to\infty} f_n(s)$ exists for every $s \in E$ and is a continuous linear functional in E.*

Proof. For a proof of a more general theorem, see Dunford & Schwartz [9] II 1.18.

3.2.5. *Proof of Theorem* 2.3.5 It is known that c_p is an FK space (see [2] and [21]), and it was proved in [21] that under the conditions of Theorem 2.3.5 U (see § 2.3.6) is fundamental in c_p. From Lemma 3.2.4, it is sufficient then to show that $c_p \subseteqq m_q$ and that $\lim_q s$ exists for every $s \in U$. But the existence of these limits follows from the existence of $1/Q(\rho_q - 0)$. The inclusion $c_p \subseteqq m_q$ follows from (2.2.2) and from the boundedness of $1/Q(x)$, after expressing $T_q(s, x)$ in terms of $T_p(s, x)$, via (2.2.1), as it was done at the beginning of § 3.1.19.

3.2.6. *Proof of Theorem* 2.3.7. If U is fundamental in c_p, then the proof is based on Lemma 3.2.4 in the same way as Theorem 2.3.5 (see § 3.2.5). This is possible because (i), (ii), (iii) imply $c_p \subseteqq m_q$, as is easily seen from the identity

$$T_q(s, x) = [Q(x)]^{-1}\int_{u/x}^{1} P(xr_{pq}\tau)T_p(s, xr_{pq}\tau)d\phi(\tau) + \sum_{k=0}^{\infty} e_{xk}s_k ,$$

which follows from (i). Also, the boundedness of

$$T_q(e^{(j+l)}, x) = q_{j+l}x^{j+l}/Q(x) \qquad (\alpha_q \leqq x < +\infty)$$

implies, when taking $l \geqq 1$, such that $q_{j+l} \neq 0$ (see Remark 2.2.1), that $x^j/Q(x) \longrightarrow 0$ $(x \longrightarrow +\infty)$ for all j. So $\lim_q e^{(j)} = 0$ for $j = 0, 1, \cdots$ (in a similar way $\lim_p e^{(j)} = 0$).

Thus $U \subseteqq c_q$ and Lemma 3.2.4 yields $c_p \subseteqq c_q$. The consistency of P and Q follows now because the continuous linear functionals $\lim_p$, $\lim_q$ coincide on the fundamental set U, hence all over c_p.

Consider now the case that $Q(\varepsilon x)/Q(x) \longrightarrow 0$ and assume, for the sake of simplicity, that $r_{pq} = 1$. We define $e_{xk}(v)$ by

(i)* $$q_k = p_k\int_{v/x}^{1} \tau^k d\phi(\tau) + Q(x)e_{xk}(v)/x^k ,$$

$$(u \leqq v \leqq x < +\infty, k = 0, 1, \cdots) .$$

From (ii) we infer the existence of an M, independent of both v and x, such that

$$\text{(ii)}^* \qquad |Q(x)|^{-1}\int_{v/x}^{1}|P(x\tau)||d\phi(\tau)| \leqq M \qquad (u \leqq v \leqq x < +\infty)\,.$$

Comparing (i) and (i)* and assuming $v/x = u/y$ we get

$$Q(x)e_{xk}(v)/x^k = Q(y)e_{yk}/y^k \Longrightarrow |e_{xk}(v)| \leqq |Q(ux/v)/Q(x)|(v/u)^k|e_{yk}|\,,$$

which implies, by (iii), for every sufficiently large v,

$$\text{(iii)}^* \qquad \begin{cases} |e_{xk}(v)| \leqq M_v|p_k|v^k & (v \leqq x < +\infty,\ k = m, m+1, \cdots) \\ |e_{xk}(v)| \leqq M_v & (v \leqq x < +\infty,\ k = 0, 1, \cdots) \end{cases}$$

where M_v is independent of x. Also, for sufficiently large v

$$\text{(iv)}^* \qquad \lim_{x\to+\infty} e_{xk}(v) = 0 \qquad (x \text{ real}, k = 0, 1, \cdots)\,.$$

Now, from (i)* we have

$$\begin{aligned} T_q(s, x) &= [Q(x)]^{-1}\int_{v/x}^{1} P(x\tau)T_p(s, x\tau)d\phi(\tau) + \sum_{k=0}^{\infty} e_{xk}(v)s_k \\ &= \sigma_1(v, x) + \sigma_2(v, x)\,. \end{aligned}$$

From (ii)* we have

$$\sup_{x\geqq X}|\sigma_1(v, x)| \leqq M\sup_{y\geqq v}|T_p(s, y)|$$

for all $X \geqq v$, and from (iii)*, (iv)*, for every sufficiently large v we have

$$\lim_{x\to+\infty}|\sigma_2(v, x)| = 0\,.$$

Hence for each v sufficiently large,

$$\varlimsup_{x\to+\infty}|T_q(s, x)| \leqq M\sup_{y\geqq v}|T_p(s, y)|\,.$$

Thus, if $s \in c_p^{(0)}$, it follows immediately that $\lim_q s = 0$. Therefore, $c_p^{(0)} \subseteqq c_q^{(0)}$ which implies $P \subseteqq Q$.

3.2.7. *Proof of Remark* 2.3.8. This follows immediately from Lemma 3.2.2.

3.2.8. *Proof of Theorem* 2.3.10. The proof is immediate if we use the identities

$$Q(x) = \int_0^1 P(xr\tau)d\phi(\tau)$$

$$T_q(s, x) = [Q(x)]^{-1}\int_0^1 P(xr\tau)T_p(s, xr\tau)d\phi(\tau) , \qquad (s \in c_p) ,$$

which follow from (2.3.2). Actually we infer that

$$\begin{aligned} Q(+\infty) &= P(0)[\phi(0+) - \phi(0)] + [\phi(1) - \phi(0+)]P(+\infty) \\ \lim_q s &= [Q(+\infty)]^{-1}P(0)[\phi(0+) - \phi(0)]s_0 \\ &\quad + [Q(+\infty)]^{-1}[\phi(1) - \phi(0+)]P(+\infty)\lim_p s \end{aligned}$$

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