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GENERALISED QUASI-NÖRLUND SUMMABILITY

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Just as (N, p, q) generalises Nörlund methods, so also, in this paper we define generalised quasi-Nörlund Method (N^*, p, q) generalising the quasi-Nörlund method due to Thorpe.

To begin with, we have determined the inverse of a generalised quasi-Nörlund matrix in a limited case. Besides, limitation Theorems for both ordinary and absolute (N^*, p, q) summability have been established.

Finally we have established an Abelian Theorem (the main theorem) for $(N^*, p, q) \Rightarrow (J, q)$, where (J, q) is a power series method which reduces to the Abel method (A) for $q_n = 1$ (all n).

1. Vermes [10] pointed out that there is a close relation between the summability properties of a matrix $A = (a_{nk})$ regarded as a sequence to sequence transformation and those of its transpose $A^* = (a_{kn})$ regarded as a series to series transformation.

Suppose that A is a sequence to sequence transformation and further that

$$\sum_{k=0}^{\infty} a_{nk} = 1 \quad \text{for all } n,$$

then by using Theorems of regularity (see Hardy [5], Theorem 2) and absolute regularity (see Knopp and Lorentz [6]) we see that A^* is an absolutely regular series to series transformation.

Conversely, given any absolutely regular series to series method $C = (c_{nk})$, its transpose C^* is regular as a sequence to sequence method provided that

$$c_{nk} \to 0 \quad \text{as} \quad k \to \infty \quad \text{for fixed } n.$$

We can also see that if A is absolutely regular and the above condition is satisfied then A^* is regular and the converse also holds.

We shall call A^* the quasi-method associated with A and remember that, it is a series to series transformation.

Kuttner [7] defined quasi-Cesàro summability and investigated its main properties as a quasi-Hausdorff transformation (see also Ramunujan [8] and White [11]. Thorpe [9] defined quasi-Nörlund (quasi-Riesz) summability.

Just as (N, p, q) generalises Nörlund methods, so also we can define generalised quasi-Nörlund method (N^*, p, q) generalising the quasi-Nörlund methods. We give the definition in the following manner:

Given p_n and q_n we define $r_n = \Sigma_{v=0}^{n} p_{n-v}q_v$ and suppose that $r_n \neq 0$ for $n \geqq 0$. We say that the (N^*, p, q) method is applicable to the given infinite series $\Sigma\, a_n$ if

$$b_n = q_n \sum_{k=n}^{\infty} \frac{p_{k-n}a_k}{r_k} \tag{1.1}$$

exists for each $n \geqq 0$. If further, $\Sigma\, b_n = s$, then we say that $\Sigma\, a_n$ is summable by (N^*, p, q) method to sum s and if $\Sigma |b_n| < \infty$ then $\Sigma\, a_n$ is said to be absolutely summable by $|N^*, p, q|$ method.

The method (N^*, p, q) reduces to the quasi-Nörlund method (N^*, p) if $q_n = 1$, to the quasi-Riesz method $(\bar{N}^*, q)$ if $p_n = 1$, to (say) quasi-Euler-Knopp method (E^*, σ) when

$$p_n = \frac{\alpha^n \sigma^n}{n!}, \quad q_n = \frac{\alpha^n}{n!} \quad (\alpha > 0,\ \sigma > 0),$$

to the (say) (C^*, α, β) method (let us call it generalised quasi-Cesàro method) when

$$p_n = \binom{n+\alpha-1}{\alpha}, \qquad q_n = \binom{n+\beta}{\beta}.$$

It may be recalled that (N, p, q) matrix is given by

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}q_k}{r_n} & (k \leqq n), \\ 0 & (k > n). \end{cases}$$

and the (N^*, p, q) is given by its transpose matrix:

$$a^*_{nk} = \begin{cases} \dfrac{q_n p_{k-n}}{r_k} & (k \geqq n), \\ 0 & (k < n). \end{cases}$$

Since for the (a_{nk}) defined above we have

$$\sum_{k=0}^{n} a_{nk} = 1,$$

it follows from the above discussion that if

$$p_{k-n} = o(r_k) \quad \text{as} \quad k \to \infty,$$

for each fixed n, then (N^*, p, q) is regular if and only if (N, p, q) is absolutely regular, and (N^*, p, q) is absolutely regular if and only if (N, p, q) is regular.

The main object of this paper is to obtain certain conditions for which $\Sigma a_n \in (N^*, p, q) \Rightarrow \Sigma a_n \in (J, q)$.

The method (J, q) is defined as follows. Suppose that $q_n \geqq 0$ and $q_n \neq 0$ for an infinity values of n. Let ρ_q $(\rho_q < \infty)$ be the radius of convergence of the power series

$$q(z) = \sum_{n=0}^{\infty} q_n z^n.$$

If the sequence to function transformation,

$$J(x) = \frac{\sum_{n=0}^{\infty} q_n s_n x^n}{\sum_{n=0}^{\infty} q_n x^n}$$

exists for $0 \leqq x \leqq \rho_q$, we say that (J, q) method is applicable to Σa_n (or $\{s_n\}$), and if further $J(x) \to s$ as $x \to \rho_q - 0$, we say that Σa_n (or $\{s_n\}$) is summable (J, q) to s. See Hardy **[5]**, Das **[4]**.

As well-known particular cases of the (J, q) method, we have the Abel method when $q_n = 1$, the logarithmic method or (L) method when $q_n = 1/n + 1$ (Borwein **[1]**, Hardy **[5]** p. 81), the A_α method when $q_n = \binom{n+\alpha}{\alpha}$ (Borwein **[2]** (A_0 is the same as Abel method A), the Borel method where $q_n = 1/n\,!$ (see Hardy **[5]**). We write $p_n \in \mathfrak{M}$, when $p_n > 0$ and $p_n/p_{n-1} \leqq p_{n+1}/p_n \leqq 1$ $(n > 0)$.

Let $P_n = \Sigma_{v=0}^{n} p_v$, $Q_n = \Sigma_{v=0}^{n} q_v$.

Let c_n be defined formally by the identity,

$$\left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) = 1.$$

2. Statements of the theorems. As in the case of quasi-Nörlund, it is not always possible to obtain an inverse to the transformation (1.1) but we have succeeded in getting an inverse for a class of sequences $p_n \in \mathfrak{M}$ and $q_n \neq 0$ $(n \geqq 0)$.

This is embodied in.

THEOREM 1. *Suppose that* $p_n \in \mathfrak{M}$ *and* $q_n \neq 0$ $(n \geqq 0)$. *Then* (N^*, p, q) (where applicable) *has an inverse transformation, whose matrix* *is given by the transpose of the inverse of* (N, p, q), *that is, if* b_n *is given by transformation* (1.1), *then*

$$a_n = r_n \sum_{k=n}^{\infty} \frac{b_k c_{k-n}}{q_k}. \tag{2.1}$$

This is our basic theorem in the sense that it is widely used here and elsewhere and it may be noted that this theorem yields a result due to Thorpe **[8]** *in the case* $q_n = 1$.

The next couple of theorems are limitation theorems which assert that the method can not sum too rapidly divergent series.

THEOREM 2. *Suppose* $p_n \in \mathfrak{M}$, $q_n \neq 0$ $(n \geqq 0)$ *and that* $|q_n|$ *is non-decreasing. If* $\Sigma\, a_n$ *be summable* (N^*, p, q) *to* s *then*

$$a_n = o\left(\frac{|r_n|}{|q_n|}\right).$$

If further $r_n \geqq 0$, *then*

$$s_n = s + o(Q_n / |q_n|).$$

THEOREM 3. *Suppose* $p_n \in \mathfrak{M}$, q_n *is positive,* $\{q_n\}$ *is nondecreasing and* $\{q_n / r_n\}$ *is nonincreasing. Then if* $\Sigma\, a_n$ *is summable* $|N^*, p, q|$, *then*

$$\left\{\frac{q_n s_n}{r_n}\right\} \in BV.$$

The main theorem in this paper is the Abelian theorem which is stated as:

THEOREM 4. *Suppose* $p_n \in \mathfrak{M}$, $q_n > 0$ *and that* $\{q_n\}$ *and* $\{q_n / q_{n+1}\}$ *are nondecreasing. Also let*

$$r_n (q_{n+1} - q_n) = O(q_{n+1}(r_{n+1} - r_n)). \tag{2.2}$$

Then

$$\Sigma\, a_n = s(N^*, p, q) \Rightarrow \Sigma\, a_n = s(J, q).$$

It may be remarked that the relationship between (N, p, q) and (J, q) was studied by Das (4). Putting $q_n = 1$ in Theorem 4, we obtain the result of Thorpe regarding $(N^*, p) \Rightarrow$ (A). We need the following lemma for the proof of the theorem.

LEMMA 1. *Let* $p_n \in \mathfrak{M}$. *Then*
(i) $\sum_{n=0}^{\infty} |c_n| < \infty$,
(ii) $c_0 > 0$, $c_n \leqq 0$ $(n \geqq 1)$,
(iii) $\sum c_n \geqq 0$,
(iv) $\sum c_n = 0$, *if and only if* $P_n \to \infty$ *as* $n \to \infty$.

The above theorem is due to Kaluza. The proof of the theorem appears in Hardy (5), Theorem 22.

3. *Proof of Theorem* 1. We know from the identity:

$$(\sum c_n x^n)(\sum p_n x^n) = 1$$

that

$$\sum_{n=0}^{k} p_n c_{k-n} = \begin{cases} 1 & (k=0), \\ 0 & (k>0). \end{cases} \tag{3.1}$$

Hence

$$\sum_{k=n}^{N} c_{k-n} p_{v-k} = -\sum_{k=N+1}^{v} c_{k-n} p_{v-k} \qquad (v > n). \tag{3.2}$$

Now for $N > n$ and by (1.1) we have,

$$\begin{aligned} r_n \sum_{k=n}^{N} \frac{b_k c_{k-n}}{q_k} &= r_n \sum_{k=n}^{N} \frac{c_{k-n}}{q_k} q_k \sum_{v=k}^{\infty} \frac{a_v p_{k-v}}{r_v} \\ &= r_n \sum_{k=n}^{N} c_{k-n} \left(\sum_{v=k}^{N} + \sum_{v=N+1}^{\infty} \right) \frac{a_v p_{v-k}}{r_v} \\ &= r_n \sum_{v=n}^{N} \frac{a_v}{r_v} \sum_{k=n}^{v} c_{k-n} p_{v-k} \\ &\quad + r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} p_{v-k} \\ &= a_n + r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} p_{v-k} \end{aligned}$$

by (3.1). Thus the necessary and sufficient condition for the validity of (2.1) is that, for each fixed n,

$$\sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} p_{v-k} \to 0, \quad \text{as} \quad N \to \infty,$$

which is the same thing as, for each fixed n,

$$\phi_N = \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=N+1}^{v} c_{k-n} p_{v-k} \to 0, \quad \text{as} \quad N \to \infty \tag{3.3}$$

in view of (3.2).

Let us write

$$\begin{aligned} b_0 &= q_0 \sum_{k=0}^{\infty} \frac{p_k a_k}{r_k}, \\ \omega_v &= q_0 \sum_{k=v}^{\infty} \frac{p_k a_k}{r_k}. \end{aligned} \tag{3.4}$$

Since (N^*, p, q) method is applicable to $\Sigma\, a_n$, b_0 is finite and hence, ω_v is well defined and tends to zero as $v \to \infty$. Now from (3.4)

$$\frac{a_v}{r_v} = \frac{\omega_v - \omega_{v+1}}{q_0 p_v}.$$

Hence

$$\phi_N = \frac{1}{q_0} \sum_{v=N+1}^{\infty} \frac{\omega_v - \omega_{v+1}}{q_0 p_v} \sum_{k=N+1}^{v} c_{k-n} p_{v-k}.$$

Now for $M > N$,

$$\begin{aligned} &\frac{1}{q_0} \sum_{v=N+1}^{M} \frac{\omega_v - \omega_{v+1}}{p_v} \sum_{k=N+1}^{v} c_{k-n} p_{v-k} \\ &= \frac{1}{q_0} \sum_{v=N+1}^{M} \omega_v \left[\sum_{k=N+1}^{v} \frac{p_{v-k} c_{k-v}}{p_v} - \sum_{k=N+1}^{v-1} \frac{p_{v-k-1} c_{k-n}}{p_{v-1}} \right] \\ &\quad - \frac{1}{q_0} \frac{\omega_{M+1}}{p_M} \sum_{k=N+1}^{M} p_{M-k} c_{k-n}. \end{aligned}$$

Since $p_n \in \mathfrak{M}$ (by Lemma 1)

$$\left| \sum_{k=N+1}^{M} p_{M-k} c_{k-n} \right| = O(1), \quad \text{as} \quad M \to \infty,$$

and by definition,

$$\omega_M = o(1), \quad \text{as} \quad M \to \infty,$$

we see that,

$$\phi_N = \frac{1}{q_0} \sum_{v=N+1}^{\infty} \omega_v \sum_{k=N+1}^{v} c_{k-n} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right).$$

Since $\{\omega_v\}$ is an arbitrary sequence tending to 0, hence (3.3) is valid, that is, $\phi_N \to 0$ if and only if, (see Hardy (5), Theorem 8) for fixed n,

$$J_N = \sum_{v=N+1}^{\infty} \left| \sum_{k=N+1}^{v} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} \right| = O(1)$$

as $N \to \infty$. But by virtue of (3.1)

$$\sum_{k=N+1}^{v} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} = -\sum_{k=n}^{N} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n}$$

for $v > n$ and also,

$$\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \leqq 1, \quad \text{for} \quad k \leqq v-1.$$

Hence

$$\begin{aligned} J_N &= \sum_{v=N+1}^{\infty} \left| \sum_{k=n}^{N} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} \right| \\ &\leqq \sum_{v=N+1}^{\infty} c_0 \left| \frac{p_{v-n}}{p_v} - \frac{p_{v-n-1}}{p_{v-1}} \right| \\ &\quad + \sum_{v=N+1}^{\infty} \sum_{k=n+1}^{N} \left| c_{k-n} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) \right| \\ &= J_N^{(1)} + J_N^{(2)}, \quad \text{(say)}. \end{aligned}$$

Since $p_n \in \mathfrak{M}$, $\{p_n/p_{n+1}\}$ is nonincreasing and so,

$$J_N^{(1)} = O(1), \quad \text{as} \quad N \to \infty.$$

Since $p_n/p_{n+1} \geqq 1$ and $\{p_n/p_{n+1}\}$ is nonincreasing it follows that, $\lim p_n/p_{n+1}$ exists and

$$A = \lim p_n/p_{n+1} \geqq 1.$$

Hence,

$$\begin{aligned} &\sum_{v=N+1}^{\infty} \left(\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) \\ &= \lim_{v\to\infty} \frac{p_{v-k}}{p_v} - \frac{p_{N-k}}{p_N} \\ &= \lim_{v\to\infty} \left(\frac{p_{v-k}}{p_{v+1-k}} \frac{p_{v+1-k}}{p_{v+2-k}} \cdots \frac{p_{v-1}}{p_v} \right) - \frac{p_{N-k}}{p_N} \\ &= A^k - \frac{p_{N-k}}{p_N}. \end{aligned}$$

Therefore, by (3.1)

$$
\begin{aligned}
J_N^{(2)} &= \sum_{k=n+1}^{N} c_{k-n} A^k - \sum_{k=n+1}^{N} c_{k-n} \frac{p_{N-k}}{p_N} \\
&= \sum_{k=n+1}^{N} c_{k-n} A^k - \frac{1}{p_N}\left[\sum_{k=n}^{N} c_{k-n} p_{N-k} - c_0 p_{N-n}\right] \\
&= \sum_{k=n+1}^{N} c_{k-n} A^k + c_0 \frac{p_{N-n}}{p_N}.
\end{aligned}
$$

Since,

$$
\sum_{k=n+1}^{N} c_{k-n} A^k \leqq 0,
$$

we get,

$$
\begin{aligned}
J_N^{(2)} &\leqq \frac{c_0 p_{N-n}}{p_N} \\
&= O(1), \quad \text{as} \quad N \to \infty.
\end{aligned}
$$

This completes the proof of the theorem.

4. Proof of Theorem 2. Since Σa_n is (N^*, p, q) summable, Σb_n is convergent and hence $b_n = o(1)$. By using the inversion formula as given in Theorem 1 we obtain, by using hypotheses,

$$
\begin{aligned}
|a_n| &= \left| r_n \sum_{k=n}^{\infty} \frac{b_k c_{k-n}}{q_k} \right| \\
&\leqq \frac{|r_n|}{|q_n|} \sum_{k=n}^{\infty} |b_k c_{k-n}| \\
&= \frac{|r_n|}{|q_n|} \sum_{k=n}^{\infty} o(1) |c_{k-n}| \\
&= o\left(\frac{|r_n|}{|q_n|}\right),
\end{aligned}
$$

since $\Sigma |c_n| < \infty$ and $b_n = o(1)$.

Next, suppose that $\Sigma b_n = s$. Since

$$
\begin{aligned}
(\Sigma c_n x^n)(\Sigma r_n x^n) &= \Sigma q_n x^n, \\
(\Sigma c_n^{(1)} x^n)(\Sigma r_n x^n) &= \Sigma Q_n x^n,
\end{aligned}
$$

it follows that

$$\sum_{v=0}^{n} r_v c_{n-v} = q_n, \tag{4.1}$$

$$\sum_{v=0}^{n} r_v c_{n-v}^{(1)} = Q_n. \tag{4.2}$$

Thus, when $p_n \in \mathfrak{M}$ we have $c_n^{(1)} \geqq 0$ and if $r_n \geqq 0$, it follows from (4.2) that $Q_n \geqq 0$ whether or not q_n is positive.

Now by (4.1)

$$\begin{aligned} s_m &= \sum_{n=0}^{m} r_n \sum_{k=n}^{\infty} \frac{b_k c_{k-n}}{q_k} \\ &= \sum_{n=0}^{m} r_n \left(\sum_{k=n}^{m} + \sum_{k=m+1}^{\infty} \right) \frac{b_k c_{k-n}}{q_k} \\ &= \sum_{k=0}^{m} \frac{b_k}{q_k} \sum_{n=0}^{k} r_n c_{k-n} + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{q_k} \\ &= \sum_{k=0}^{m} b_k + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{q_k}. \end{aligned}$$

Hence, as $b_k = o(1)$,

$$\begin{aligned} \left| s_m - \sum_{k=0}^{m} b_k \right| &\leqq \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} o(1) \frac{|c_{k-n}|}{q_k} \\ &= o(1) \frac{1}{|q_m|} \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} |c_{k-n}|. \end{aligned}$$

But when $p_n \in \mathfrak{M}$, by Lemma 1, we have

$$\sum_{k=m+1}^{\infty} |c_{k-n}| \leqq c_{m-n}^{(1)}; \tag{4.3}$$

and hence, by identity (4.2)

$$\begin{aligned} \left| s_m - \sum_{k=0}^{m} b_k \right| &= o(1) \frac{1}{|q_m|} \sum_{n=0}^{m} r_n c_{m-n}^{(1)} \\ &= o(1) \frac{Q_m}{|q_m|}. \end{aligned}$$

This completes the proof.

Proof of Theorem 3. We have

$$\sum_{n=0}^{\infty}\left|\frac{s_nq_n}{r_n}-\frac{s_{n+1}q_{n+1}}{r_{n+1}}\right| = \sum_{n=0}^{\infty}\left|\Delta\left(\frac{s_nq_n}{r_n}\right)\right|$$

$$\leqq \sum_{n=0}^{\infty}|a_{n+1}|\frac{q_{n+1}}{r_{n+1}}+\sum_{n=0}^{\infty}|s_n|\Delta\left|\frac{q_n}{r_n}\right|$$

$$= L_n + M_n, \quad \text{(say)}.$$

By using (2.1), we get (as q_n is nondecreasing)

$$L_n \leqq \sum_{n=0}^{\infty}\frac{q_{n+1}}{r_{n+1}}r_{n+1}\sum_{k=n+1}^{\infty}\frac{|b_k||c_{k-n-1}|}{q_k}$$

$$\leqq \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}|b_k||c_{k-n-1}|$$

$$= \sum_{k=0}^{\infty}|b_k|\sum_{n=0}^{k-1}|c_{k-n-1}|$$

$$= O(1),$$

since $\Sigma|b_k|<\infty$ and $\Sigma|c_n|<\infty$ as $p_n \in \mathfrak{M}$. Since $\{q_n/r_n\}$ is decreasing we have,

$$\sum_{n=v}^{\infty}\left|\Delta\frac{q_n}{r_n}\right| = \sum_{n=v}^{\infty}\left(\frac{q_n}{r_n}-\frac{q_{n+1}}{r_{n+1}}\right)\leqq\frac{q_v}{r_v}.$$

Hence,

$$M_n = \sum_{n=0}^{\infty}\left|\Delta\frac{q_n}{r_n}\right|\left|\sum_{v=0}^{n}r_v\sum_{k=v}^{\infty}\frac{b_kc_{k-v}}{q_k}\right|$$

$$\leqq \sum_{n=0}^{\infty}\left|\Delta\frac{q_n}{r_n}\right|\sum_{v=0}^{n}r_v\sum_{k=v}^{\infty}\frac{|b_k||c_{k-v}|}{q_k}$$

$$= \sum_{v=0}^{\infty}r_v\sum_{n=v}^{\infty}\left|\Delta\frac{q_n}{r_n}\right|\sum_{k=n}^{\infty}\frac{|b_k||c_{k-v}|}{q_k}$$

$$= \sum_{v=0}^{\infty}r_v\sum_{k=v}^{\infty}\frac{|b_k||c_{k-v}|}{q_k}\sum_{n=v}^{\infty}\left|\Delta\frac{q_n}{r_n}\right|$$

$$\leqq \sum_{v=0}^{\infty}\frac{r_v}{q_v}\sum_{k=v}^{\infty}|b_k||c_{k-v}|\frac{q_v}{r_v}$$

$$= \sum_{v=0}^{\infty}\sum_{k=v}^{\infty}|b_k||c_{k-v}|$$

$$= \sum_{k=0}^{\infty}|b_k|\sum_{v=0}^{k}|c_{k-v}|$$

$$<\infty,$$

by hypothesis. Hence

$$\Sigma \left| \Delta \left(\frac{s_n q_n}{r_n} \right) \right| \leqq L_n + M_n = O(1) \quad \text{as} \quad n \to \infty$$

and therefore

$$\{s_n q_n / r_n\} \in BV.$$

This completes the proof of Theorem 3.

5. Now we will prove our main theorem and for this, we require the following lemma.

LEMMA 2. *Let $p_n \in \mathfrak{M}$, $q_n > 0$ and nondecreasing. Then* (2.2) *implies that*

$$0 \leqq q_k^2 \leqq \sum_{v=0}^{k} q_v r_v c_{k-v} = O(q_k^2).$$

Proof. Since $q_n > 0$ and nondecreasing and $p_n > 0$, it follows that $r_n > 0$ and nondecreasing. Since, as $p_n \in \mathfrak{M}$, by Lemma 1, $c_0 > 0$, $c_n \leqq 0$ $(n \geqq 1)$, when we get

$$\sum_{v=0}^{k} q_v r_v c_{k-v} \geqq q_k \sum_{v=0}^{k} r_v c_{k-v} = q_k^2 \geqq 0,$$

by identity (4.1). Now

$$\begin{aligned}\sum_{v=0}^{k} q_v r_v c_{k-v} &= \sum_{v=0}^{k} \Delta_v (q_{k-v} r_{k-v}) c_v(1) \\ &= \sum_{v=0}^{k} q_{k-v}(r_{k-v} - r_{k-v-1}) c_v(1) \\ &\quad + \sum_{v=0}^{k} r_{k-v-1}(q_{k-v} - q_{k-v-1}) c_v(1).\end{aligned}$$

Hence, as $c_n^{(1)} \geqq 0$, we get by (4.2)

$$\sum_{v=0}^{k} q_{k-v}(r_{k-v} - r_{k-v-1}) c_v^{(1)} \leqq q_k (Q_k - Q_{k-1}) = q_k^2.$$

Again by (2.2)

$$\begin{aligned}0 &\leqq \sum_{v=0}^{k} r_{k-v-1}(q_{k-v} - q_{k-v-1})c_v^{(1)} \\ &= O(1)\sum_{v=0}^{k} q_{k-v}(r_{k-v} - r_{k-v-1})c_v^{(1)} \\ &= O(1)q_k^2,\end{aligned}$$

as in the previous case.
Hence

$$0 \leqq \sum_{v=0}^{k} q_v r_v c_{k-v} = O(q_k^2).$$

This completes the proof of the lemma.

Proof of Theorem 4. We shall first prove that whenever Σa_n is summable (N^*, p, q), then (J, q) method is applicable to Σa_n.

By Theorem 2, we have

$$s_n = s + o\left(\frac{Q_n}{q_n}\right) = O\left(\frac{Q_n}{q_n}\right).$$

Hence

$$\begin{aligned}J(x) &= \frac{\Sigma q_n s_n x^n}{\Sigma q_n x^n} \\ &= O(1)\frac{\Sigma Q_n x^n}{\Sigma q_n x^n} \\ &= O(1)\Sigma x^n.\end{aligned}$$

Since $\Sigma x^n = 1/(1-x)$ for $|x| < 1$, it follows that $J(x)$ exists for $|x| < 1$ and hence (J, q) method is applicable. Now for $|x| < 1$,

$$\begin{aligned}J(x) &= \frac{1}{q(x)}\sum_{v=0}^{\infty} r_v \sum_{n=v}^{\infty} q_n x^n \sum_{k=v}^{\infty} \frac{b_k c_{k-v}}{q_k} \\ &= \frac{1}{q(x)}\sum_{v=0}^{\infty} r_v \sum_{k=v}^{\infty} \frac{b_k c_{k-v}}{q_k} \sum_{n=v}^{\infty} q_n x^n \\ &= \frac{1}{q(x)}\sum_{k=0}^{\infty} \frac{b_k}{q_k} \sum_{v=0}^{k} r_v c_{k-v} \sum_{n=v}^{\infty} q_n x^n \qquad (5.1) \\ &= \sum_{k=0}^{\infty} g_k(x) b_k,\end{aligned}$$

where,

$$g_k(x) = \frac{\sum_{v=0}^{k} r_v c_{k-v} \sum_{n=v}^{\infty} q_n x^n}{q_k q(x)}.$$

The change of order of summation involved in obtaining (5.1) is justified in the range $|x|<1$, by the absolute convergence of the double sum.

Now (5.1) is a series to function transformation, transforming the series Σb_n to the function $J(x)$. To prove the theorem, we have to show that the transformation (5.1) is regular, that is, we have to show that the conditions of regularity (see Cooke **[3]**, page 65) are satisfied. Note that

$$\begin{aligned} g_k(x) &= \frac{\sum_{v=0}^{k} r_v c_{k-v}\left(q(x) - \sum_{n=0}^{v-1} q_n x^n\right)}{q_k q(x)} \\ &= \frac{1}{q_k}\sum_{v=0}^{k} r_v c_{k-v}\left(1 - \sum_{n=0}^{v-1} q_n x^n / q(x)\right) \qquad (5.2) \\ &= 1 - \left(\sum_{v=0}^{k} r_v c_{k-v} \sum_{n=0}^{v-1} q_n x^n\right) \Big/ (q(x) q_k) \end{aligned}$$

by identity (4.1).

Since $q_n > 0$ is increasing, we have

$$\Sigma q_n x^n \geqq q_0 \Sigma x^n \to \infty \quad \text{as} \quad x \to 1-0.$$

Hence from (5.2), we obtain

$$g_k(x) \to 1, \quad \text{as} \quad x \to 1-0.$$

We have only to show that

$$\sum_{k=1}^{\infty} |g_k(x) - g_{k+1}(x)| \leqq M, \tag{5.3}$$

for $0 < x < 1$, where M is a positive number.

Now let us write

$$\phi_v(x) = \sum_{k=v}^{\infty} q_k x^k / q(x).$$

It is obvious that, $\phi_0(x) = 1$. Hence

$$\begin{aligned} g_k(x) - g_{k+1}(x) &= \sum_{v=0}^{k+1} \phi_v(x) r_v \left(\frac{c_{k-v}}{q_k} - \frac{c_{k+1-v}}{q_{k+1}}\right) \\ &= \sum_{v=0}^{k} c_{k-n}\left(\phi_v(x)\frac{r_v}{q_k} - \phi_{v+1}(x)\frac{r_{v+1}}{q_{k+1}}\right) - r_0 \frac{c_{k+1}}{q_{k+1}}. \end{aligned}$$

Since by hypothesis $\Sigma|c_n|<\infty$ and $\{1/q_n\}$ decreases as n increases, we have,

$$\sum_{k=0}^{\infty}\frac{|c_{k+1}|}{q_{k+1}}\leqq\frac{1}{q_0}\sum_{k=0}^{\infty}|c_{k+1}|<\infty.$$

Hence in order to show that (5.3) holds it is enough to show that,

$$\theta(x)=\sum_{k=0}^{\infty}\left|\sum_{v=0}^{k}c_{k-v}\left(\phi_v(x)\frac{r_v}{q_k}-\phi_{v+1}(x)\frac{r_{v+1}}{q_{k+1}}\right)\right|<M,$$

for $0<x<1$.

Now since

$$\phi_v(x)-\phi_{v+1}(x)=\frac{q_v x^v}{q(x)},$$

it follows that,

$$\text{(5.5)}\quad \theta(x)=\sum_{k=0}^{\infty}\left|\sum_{v=0}^{k}c_{k-v}\,(\phi_v(x)-\phi_{v+1}(x))\frac{r_v}{q_k}+\phi_{v+1}(x)\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right)\right|$$

$$\leqq M(x)+N(x),$$

where,

$$M(x)=\frac{1}{q(x)}\sum_{k=0}^{\infty}\frac{1}{q_k}\left|\sum_{v=0}^{k}c_{k-v}q_v r_v x^v\right|$$

$$N(x)=\sum_{k=0}^{\infty}\left|\sum_{v=0}^{k}c_{k-v}\phi_{v+1}(x)\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right)\right|.$$

Since

$$\sum_{v=0}^{k}c_{k-v}q_v r_v x^v=\sum_{v=0}^{k-1}c_{k-v}q_v r_v(x^v-x^k)+x^k\sum_{v=0}^{k}c_{k-v}q_v r_v,$$

to prove $M(x)=O(1)$ we need only show that,

$$M'(x)=\frac{1}{q(x)}\sum_{k=0}^{\infty}\frac{1}{q_k}\sum_{v=0}^{k-1}c_{k-v}q_v r_v(x^v-x^k)=O(1),$$

in view of Lemma 2.

Since $c_n\leqq 0$ $(n\geqq 1)$ and $\{1/q_n\}$ is decreasing, we get,

$$\begin{aligned}
M'(x) &= -\frac{1}{q(x)}\sum_{k=0}^{\infty}\frac{1}{q_k}\sum_{v=0}^{k-1} q_v r_v c_{k-v}(x^v - x^k) \\
&= -\frac{1}{q(x)}\sum_{v=0}^{\infty} q_v r_v x^v \sum_{k=v+1}^{\infty} c_{k-v}\frac{(1-x^{k-v})}{q_k} \\
&\leqq -\frac{1}{q(x)}\sum_{v=0}^{\infty}\frac{q_v r_v x^v}{q_v}\sum_{k=v+1}^{\infty} c_{k-v}(1-x^{k-v}) \\
&= -\frac{1}{q(x)}\sum_{v=0}^{\infty} r_v x^v (c(1)-c(x)) \\
&\leqq \frac{1}{q(x)}\sum_{v=0}^{\infty} r_v x^v c(x) \\
&= \frac{r(x)c(x)}{q(x)} \\
&= 1.
\end{aligned}$$

Hence,

$$M(x) = O(1). \tag{5.6}$$

The inner sum of $N(x)$ can be written as,

$$\begin{aligned}
\phi_{k+1}(x)\sum_{v=0}^{k} c_{k-v}\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right) &+ \sum_{v=0}^{k} c_{k-v}(\phi_{v+1}(x)-\phi_{k+1}(x))\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right) \\
&= \phi_{k+1}(x)\sum_{v=0}^{k} c_{k-v}\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right) \\
&\quad + \sum_{v=0}^{k}\frac{c_{k-v}}{q(x)}\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right)\sum_{\mu=v+1}^{k} q_\mu x^\mu.
\end{aligned}$$

Hence,

$$N(x) \leqq N'(x) + N''(x), \tag{5.7}$$

where,

$$N'(x) = \sum_{k=0}^{\infty}\left|\phi_{k+1}(x)\sum_{v=0}^{k} c_{k-v}\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right)\right|,$$

and

$$N''(x) = \sum_{k=0}^{\infty}\left|\sum_{v=0}^{k} c_{k-v}\left(\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right)\frac{\sum_{\mu=v+1}^{k} q_\mu x^\mu}{q(x)}\right|.$$

By (4.1)

$$\sum_{v=0}^{k} c_{k-v}\frac{r_v}{q_k}-\sum_{v=0}^{k} c_{k-v}\frac{r_{v+1}}{q_{k+1}}$$

$$=1-\frac{1}{q_{k+1}}\sum_{v=0}^{k} c_{k-v}r_{v+1}$$

$$=1-\frac{1}{q_{k+1}}\left(\sum_{v=0}^{k+1} c_{k+1-v}r_v-c_{k+1}r_0\right)$$

$$=r_0\frac{c_{k+1}}{q_{k+1}}.$$

Hence,

$$N'(x)=r_0\sum_{k=0}^{\infty}\phi_{k+1}(x)\frac{|c_{k+1}|}{q_{k+1}}.$$

We know from the very definition of $\phi_k(x)$ that for $0<x<1$,

$$0\leqq \phi_k(x)\leqq 1.$$

Hence

$$N'(x)\leqq r_0\sum_{k=0}^{\infty}\frac{c_{k+1}}{q_{k+1}}\leqq\frac{r_0}{q_0}\Sigma\,|c_{k+1}|<\infty.$$

And

$$N''(x)\leqq\sum_{k=0}^{\infty}\sum_{v=0}^{k}|c_{k-v}|\left|\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right|\frac{\sum_{\mu=v+1}^{k} q_\mu x^\mu}{q(x)}$$

$$=\frac{1}{q(x)}\sum_{v=0}^{\infty}\sum_{k=v}^{\infty}|c_{k-v}|\left|\frac{r_v}{q_k}-\frac{r_{v+1}}{q_{k+1}}\right|\sum_{\mu=v+1}^{k} q_\mu x^\mu$$

$$=\frac{1}{q(x)}\sum_{v=0}^{\infty}\sum_{\mu=v+1}^{\infty} q_\mu x^\mu\sum_{k=\mu}^{\infty}|c_{k-v}|\left|r_v\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)+\frac{r_v-r_{v+1}}{q_{k+1}}\right|$$

$$\leqq\frac{1}{q(x)}\sum_{v=0}^{\infty} r_v\sum_{\mu=v+1}^{\infty} q_\mu x^\mu\sum_{k=\mu}^{\infty}|c_{k-v}|\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)$$

$$+\frac{1}{q(x)}\sum_{v=0}^{\infty}(r_{v+1}-r_v)\sum_{\mu=v+1}^{\infty} q_\mu x^\mu\sum_{k=\mu}^{\infty}|c_{k-v}|\frac{1}{q_{k+1}}$$

$$=\alpha(x)+\beta(x),\quad\text{(say)}.$$

Now, since $\{q_n\}$ and $\{q_n/q_{n+1}\}$ are increasing with n we get, by using hypothesis (2.2) and (4.3)

$$
\begin{aligned}
\alpha(x) &\leqq \frac{1}{q(x)}\sum_{v=0}^{\infty} r_v \sum_{\mu=v+1}^{\infty} x^{\mu} \sum_{k=\mu}^{\infty} |c_{k-v}| \left(1-\frac{q_k}{q_{k+1}}\right) \\
&\leqq \frac{1}{q(x)}\sum_{v=0}^{\infty} \frac{r_v(q_{v+1}-q_v)}{q_{v+1}} \sum_{\mu=v+1}^{\infty} c_{\mu-v-1}^{(1)} x^{\mu} \\
&= \frac{1}{q(x)}\sum_{v=0}^{\infty} \frac{r_v(q_{v+1}-q_v)}{q_{v+1}} x^{v+1} \sum_{n=0}^{\infty} c_n^{(1)} x^n \\
&= \frac{1}{(1-x)q(x)p(x)}\sum_{v=0}^{\infty} \frac{r_v(q_{v+1}-q_v)}{q_{v+1}} x^{v+1} \\
&= \frac{1}{(1-x)r(x)} O(1) \sum_{v=0}^{\infty} (r_{v+1}-r_v) x^{v+1} \\
&= O(1),
\end{aligned}
$$

by using the identity,

$$(1-x)p(x)\Sigma c_n^{(1)} x^n = 1, \qquad (0<x<1).$$

Again since $\{r_n\}$ increases with n as $\{q_n\}$ increases, we get,

$$
\begin{aligned}
\beta(x) &\leqq \frac{1}{q(x)}\sum_{v=0}^{\infty} (r_{v+1}-r_v) \sum_{\mu=v+1}^{\infty} x^{\mu} \sum_{k=\mu}^{\infty} |c_{k-v}| \\
&\leqq \frac{1}{q(x)}\sum_{v=0}^{\infty} (r_{v+1}-r_v) \sum_{\mu=v+1}^{\infty} x^{\mu} c_{\mu-v-1}^{(1)} \\
&= \frac{1}{q(x)}\sum_{v=0}^{\infty} (r_{v+1}-r_v) x^{v+1} \sum_{n=0}^{\infty} c_n^{(1)} x^n \\
&= \frac{1}{(1-x)p(x)q(x)}\sum_{v=0}^{\infty} (r_{v+1}-r_v) x^{v+1} \\
&\leqq 1.
\end{aligned}
$$

Hence,

$$N''(x) = \alpha(x) + \beta(x) = O(1).$$

Hence by (5.7), (5.6) and (5.5)

$$\theta(x) \leqq M(x) + N(x) = O(1).$$

Hence (5.3) holds and this completes the proof of the theorem.

6. In this section, we now deduce some corollaries of Theorem 4.

COROLLARY 1. (Thorpe **[9]**). *Suppose* $p_n \in \mathfrak{M}$, *then* $\Sigma a_n \in (N^*, p) \Rightarrow \Sigma a_n \in (A)$, *where* (A) *is the Abel method.*

Proof. Put $q_n = 1$, for all n in Theorem 4.

COROLLARY 2. *Let* $q_n > 0$ *for all* n, $\{q_n\}$ *be increasing in* n, *such that* $\{q_n / q_{n+1}\}$ *is also increasing in* n *and,*

$$Q_n(q_{n+1} - q_n) = O(q_{n+1}^{(2)}). \tag{6.1}$$

Then,

$$\Sigma a_n \in (\bar{N}^*, q) \Rightarrow \Sigma a_n \in (J, q).$$

Proof. Put $p_n = 1$ for all n, in Theorem 4. In this case we have,

$$c_0 = 1, \quad c_1 = -1, \quad c_n = 0 \quad (n > 2).$$

COROLLARY 3. $(C^*, \alpha, \beta) \Rightarrow A_\beta$ *for* $0 < \alpha \leqq 1 \leqq \beta$.

Proof. Set

$$p_n = A_n^{\alpha - 1}, \qquad q_n = A_n^{\beta - 1} \quad \text{in Theorem 4.}$$

Then $r_n = A_n^{\alpha + \beta - 1}$ and condition (2.2) reduces to proving that

$$n^{\alpha + \beta - 1} n^{\beta - 2} = O(n^{\beta - 1} n^{\alpha + \beta - 2}),$$

which is valid in the present case. Also when $0 < \alpha \leqq 1$, then $p_n = A_n^{\alpha - 1} \in \mathfrak{M}$ and when $\beta \geqq 1$, then $q_n = A_n^{\beta - 1}$ is nondecreasing.

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