PACIFIC JOURNAL OF MATHEMATICS
Vol 68, No 1, 1977

SKEW LINEAR VECTOR FIELDS ON SPHERES IN THE STABLE RANGE

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THEOREM. *Assume $n > 2k$. Then every $(k-1)$-field on S^{n-1} is skew linear.*

1. Introduction.

Skew linear vector fields on spheres have been studied by Strutt [6], Zvengrowski [8] and Milgram and Zvengrowski [4, 5]. Extensive calculation of projective homotopy classes in [5] led Milgram and Zvengrowski to conjecture that every r-field on S^{n-1} is skew linear. Here we will prove this conjecture in the stable range, as stated above.

After a reformulation using a construction of L. Woodward [7] and the results of [1], the theorem will follow from the Kahn–Priddy theorem [3].

Since proving this theorem I have learned that Milgram and Zvengrowski had already obtained the result using different methods [9]. They have also shown that 7 and 8-fields on S^{15} are skew linear, the two remaining cases excluded by the condition $n > 2k$ and not already dealt with in [8]. L. Woodward has also proved the theorem by methods similar to those used here.

2. Proof of the theorem.

If $p: E \to B$ is a fibration let $C(B; E)$ denote the set of vertical homotopy classes of cross sections to p. If Z_2 acts freely on B and E in such a way that p is equivariant let $C_{Z_2}(B; E)$ denote the set of equivariant vertical homotopy classes of equivariant cross sections to p.

Let $V_{n,k}$ denote the Stiefel manifold of k-frames in R^n with the involution $[v_1, \cdots, v_k] \to [-v_1, \cdots, -v_k]$. Recall that a *skew linear* $(k-1)$-field on S^{n-1} is a cross section to the bundle $V_{n,k} \to S^{n-1}$ which is vertically homotopic to an equivariant cross section. Let $L_{n,k}$ denote the space of equivariant maps $S^{k-1} \to S^{n-1}$. Fixing $x_0 = (1, 0, \cdots, 0) \in S^{k-1}$ as base point we have a fibration $L_{n,k} \to S^{n-1}$ by evaluating at x_0 and a commutative square

$$\begin{tikzcd} V_{n,k} \arrow[r, "\sigma"] \arrow[d] & L_{n,k} \arrow[d] \\ S^{n-1} \arrow[r, leftrightarrow] & S^{n-1} \end{tikzcd}$$

where σ is the natural inclusion. The antipodal map on S^{n-1} induces an involution on $L_{n,k}$ such that the maps in the above diagram are equivariant. As is well known **[2]**, σ is a $(2(n-k)-1)$-equivalence. Hence

$$C(S^{n-1}; V_{n,k}) \simeq C(S^{n-1}; L_{n,k})$$

and

$$C_{Z_2}(S^{n-1}; V_{n,k}) \simeq C_{Z_2}(S^{n-1}; L_{n,k}).$$

Let P_k denote $(k-1)$-dimensional real projective space and η_k the Hopf bundle over P_k. Let $\mathrm{Tr}(n\eta_k)$ (respectively, $\mathrm{Tr}_{Z_2}(n\eta_k)$) denote the set of fiber homotopy classes of fiber preserving maps (respectively, equivariant fiber homotopy classes of equivariant fiber preserving maps) $P_k \times S^{n-1} \to S(n\eta_k)$, whose restriction to the fiber over $[x_0]$ is the identity map. Here $S(n\eta_k)$ is the unit sphere bundle of $n\eta_k$. Define a map

$$\mu: C(S^{n-1}; L_{n,k}) \to \mathrm{Tr}(n\eta_k)$$

by $\Delta \to \tilde{\Delta}$ where $\tilde{\Delta}([x], y) = [x, \Delta(y)(x)]$, $x \in S^{k-1}$, $y \in S^{n-1}$. This map is a bijection; in fact the underlying function spaces are homeomorphic (see Woodward [7, Lemma 1,2]). Similarly we have a bijection

$$\mu_{Z_2}: C_{Z_2}(S^{n-1}; L_{n,k}) \to \mathrm{Tr}_{Z_2}(n\eta_k).$$

Let $G(S^{n-1})$ denote the identity component of the space of maps $S^{n-1} \to S^{n-1}$ and let $G = \text{inj. lim. } G(S^{n-1})$. Let $G_{Z_2} = \text{inj. lim. } G_{Z_2}(S^{n-1})$ where $G_{Z_2}(S^{n-1})$ is the identity component of the space of equivariant maps $S^{n-1} \to S^{n-1}$. Fixing an equivariant fiber map $f: S(n\eta_k) \to P_k \times S^{n-1}$ whose restriction to the fiber over $[x_0]$ is the identity, we have equivalences

$$\nu: \mathrm{Tr}(n\eta_k) \to [P_k; G]$$

and

$$\nu_{Z_2}: \mathrm{Tr}_{Z_2}(n\eta_k) \to [P_k; G_{Z_2}].$$

Each of these is defined by sending $h: P_k \times S^{n-1} \to S(n\eta_k)$ to the adjoint of

$$P_k \times S^{n-1} \xrightarrow{h} S(n\eta_k) \xrightarrow{f} P_k \times S^{n-1} \to S^{n-1}.$$

Here [;] denotes homotopy classes of base point preserving maps.
Summarizing, let

$$\psi\colon C(S^{n-1}; V_{n,k}) \to [P_k; G]$$

denote the composite

$$C(S^{n-1}; V_{n,k}) \xrightarrow{\sigma\#} C(S^{n-1}; L_{n,k}) \xrightarrow{\mu} \mathrm{Tr}\,(n\eta_k) \xrightarrow{\nu} [P_k; G]$$

and let ψ_{Z_2} denote its equivariant analogue.

LEMMA. *Assume $n > 2k$. There is a commutative square*

$$\begin{array}{ccc} C_{Z_2}(S^{n-1}; V_{n,k}) & \xrightarrow{\psi_{Z_2}} & [P_k; G_{Z_2}] \\ \Big\downarrow \phi & & \Big\downarrow \phi_* \\ C(S^{n-1}; V_{n,k}) & \xrightarrow{\psi} & [P_k; G] \end{array}$$

in which ψ and ψ_{Z_2} are equivalences and ϕ is the forgetful map.

If X is a connected space let $Q^0(X^+)$ denote the o-component of $Q(X^+) = \Omega^\infty S^\infty(X^+)$. By the main result of **[1]** there is a commutative square

$$\begin{array}{ccc} G_{Z_2} & \xrightarrow{\lambda} & Q^0(RP^{\infty +}) \\ \Big\downarrow \phi & & \Big\downarrow \tau \\ G & \xrightarrow{\lambda} & Q^0(S^0) \end{array}$$

in which the horizontal maps are homotopy equivalences and τ is the transfer map associated with the double cover $S^\infty \to RP^\infty$. In view of this and the above lemma, our theorem will follow by showing that

$$\tau_*\colon [P_k; Q^0(RP^{\infty +})] \to [P_k; Q^0(S^0)]$$

is epimorphic. This is a consequence of the Kahn–Priddy theorem **[3]**. First note that both of these groups are finite and τ_* is clearly onto the odd primary part. The Kahn–Priddy result states that τ_* also maps onto the 2–primary part. (Although they only consider the morphisms $\tau_*: [S^m; Q^0(RP^{\infty+})] \to [S^m; Q^0(S^0)]$, for all m, their proof is valid with S^m replaced by any finite complex.)

REFERENCES

1. J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory I*, Comm. Math. Helv., **49** (1974), 1–34.

2. A. Haefliger and M. W. Hirsch, *Immersions in the stable range*, Ann. Math., **75** (1962), 231–241.

3. D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory*, Bull. Amer. Math. Soc., **78** (1972), 981–987.

4. R. J. Milgram and P. Zvengrowski, *Projective Stiefel manifolds and skew linear vector fields*, Proc. London Math. Soc., **28** (1974), 671–682.

5. ———, *Stable projective homotopy and applications to skew linear vector fields*, to appear.

6. J. Strutt, *Projective homotopy of Stiefel manifolds*, Canad. J. Math., **24** (1972), 465–476.

7. L. M. Woodward, *Vector fields on spheres and a generalization*, Quart. J. Math., **24** (1973), 357–366.

8. P. Zvengrowski, *Skew linear vector fields on spheres*, J. London Math. Soc., **3** (1971), 625–632.

9. R. J. Milgram and P. Zvengrowski, *Skew linearity of r-fields on spheres*, Topology, to appear.

Received February 3, 1976. Research supported by the Science Research Council (Gt. Britain) and the National Science Foundation.

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