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ON THE STRONG COMPACT-PORTED TOPOLOGY FOR SPACES OF HOLOMORPHIC MAPPINGS

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Suppose E is a separated complex locally convex space, U is non void open subset of E, F a complex normed space and $\mathscr{H}(U; F)$ the complex vector space of all holomorphic mappings from U into F. On $\mathscr{H}(U; F)$ we consider the following topologies; a) $\tau_{\omega s}$, the topology generated by the seminorms p which are $K-B$ ported for some $K \subset U$ compact and $B \subset E$ bounded. A seminorm p is $K-B$ ported if for every $\varepsilon > 0$, with $K+\varepsilon B \subset U$, there is $c(\varepsilon) > 0$, such that $p(f) \leqq c(\varepsilon) \sup\{\|f(x)\|; x \in K+\varepsilon B\}$ for all $f \in \mathscr{H}(U; F)$; b) τ_0, the compact open topology; c) $\tau_{\infty s}$ the topology defined by J. A. Barroso in "Topologias nos espaços de aplicações holomorfas entre espaços localmente convexos", An. Acad. Brasil. Ci, 43, 1971. The topology $\tau_{\omega s}$ is an generalization of the Nachbin topology (L. Nachbin, Topology on Spaces of Holomorphic Mappings, Springer-Verlag, 1968). The following results are valid: 1. $\mathscr{X} \subset \mathscr{H}(U; F)$ is τ_0-bounded if, and only if, $\mathscr{X}$ is $\tau_{\omega s}$-bounded. 2. $\mathscr{X} \subset \mathscr{H}(U; F)$ is $\tau_{\omega s}$-relatively compact if, and only if, $\mathscr{X}$ is $\tau_{\infty s}$-relatively compact. 3. Let E be a quasi complete space. Then $\tau_0 = \tau_{\omega s}$ on $\mathscr{H}(E; C)$ if, and only if E is a semi-Montel space. Moreover, the completion of $\mathscr{H}(E; C)$ on the $\tau_{\omega s}$ topology and the bornological topology associated to τ_0 are caracterized via the Silva-holomorphic mappings.

Throughout this article the following notations will be used. E is a complex separated locally convex space; U is a non void open subset of E; F is a complex normed space; $\mathscr{H}(U; F)$ is the complex vector space of all holomorphic mappings from U into F: $\mathscr{P}({}^nE; F)$ is the complex vector space of all continuous n-homogeneous polynomials from E into F; $(1/n!)\hat{d}^nf(t) \in \mathscr{P}({}^nE; F)$ is the nth coefficient of the Taylor series of f at $t, n = 0, 1, \cdots, f \in \mathscr{H}(U; F)$; τ_0 is the compact open topology on $\mathscr{H}(U; F)$; $\tau_{\infty s}$ is the locally convex topology on $\mathscr{H}(U; F)$ generated by all seminorms of the type

$$p_{K,n,B}(f) = \sup\left\{\left\|\frac{1}{n!}\hat{d}^nf(t)u\right\|; t \in K, u \in B\right\}$$

where $n = 0, 1, \cdots, K$ is a compact subset of U, B is a bounded balanced subset of E; $\mathscr{P}_s({}^nE; F)$ is $\mathscr{P}({}^nE; F)$ endowed with the locally convex topology of the uniform convergence on bounded subsets of E. We will introduce a new locally convex topology, $\tau_{\omega s}$, on $\mathscr{H}(U; F)$ which, in some cases, coincides with the Nachbin topology τ_ω (Nachbin [8]). The topology τ_ω has been extensively studied in the theory of infinite dimensional holomorphy. (Nachbin [7].) For example, $\tau_{\omega s} = \tau_\omega$ on $\mathscr{H}(E; C) = \mathscr{H}(E)$, if E is normed. Furthermore, $\tau_{\omega s} = \tau_0$ (the compact-open topology) on $\mathscr{H}(E)$, if E is a Montel space (see Corollary 1.14). In the § 1, the $\tau_{\omega s}$-continuous seminorms are characterized and we generalize for locally convex spaces a result of Dineen [5], which is true for Banach spaces. The $\tau_{\omega s}$-bounded subsets and the $\tau_{\omega s}$-relatively compact subsets of $\mathscr{H}(U; F)$ are studied. In the § 2, it is given a characterization of the completion of $(\mathscr{H}(E), \tau_{\omega s})$. In the § 3, it is given a characterization of the τ_{0b} (bornological topology associated with τ_0)-continuous seminorms on $H(E)$. (Here, $H(E)$ denotes the set of all functions $f: E \to C$, such that there is P_n in $\mathscr{P}(^nE)$, for $n = 0, 1, \cdots$, so that, for each $K \subset E$ compact, $B \subset E$ bounded, there is $\alpha = \alpha(B) > 0$, with $f = \sum_{n=0}^{\infty} P_n$, uniformly on $K + \alpha B$.)

For basic material on Infinite Dimensional Holomorphy we refer to [6], [7], and [8].

1. The strong compact-ported topology.

Definition 1.1. Let B be a bounded balanced subset of E and K be a compact subset of U. A seminorm p on $\mathscr{H}(U; F)$ is $K - B$ ported or strongly ported by K if for each $\varepsilon > 0$, with $K + \varepsilon B \subset U$, there is $c(\varepsilon) > 0$ such that

$$p(f) \leqq c(\varepsilon) \sup \{\|f(t)\|; t \in K + \varepsilon B\}$$

for every $f \in \mathscr{H}(U; F)$. The locally convex topology $\tau_{\omega s}$ on $\mathscr{H}(U; F)$ is generated by all seminorms which are strongly ported by compact subsets of U. It is called the strong compact-ported topology.

Proposition 1.2. *If K is a compact subset of U, B is a balanced bounded subset of E and p is a seminorm on $\mathscr{H}(U; F)$, then the following conditions are equivalent:*

(1) *p is $K - B$ ported*

(2) *For each $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that*

$$p(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \sup \left\{ \left\| \frac{1}{n!} \hat{d}^n f(t) \right\|_B ; t \in K \right\}$$

for all

$$f \in \mathscr{H}(U; F) [\|P\|_B = \sup \{\|P(t)\|; t \in B\}] .$$

If U is balanced $\tau_{\omega s}$ is generated by all seminorms p such that for some $K \subset U$ compact and $B \subset E$ balanced and bounded, for each $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

(*) $$p(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(0) \right\|_{K+\varepsilon B}$$

for all $f \in \mathscr{H}(U; F)$.

Proof. Let p be a $K - B$ ported seminorm on $\mathscr{H}(U; F)$. Thus for each $\varepsilon > 0$ such that $K + \varepsilon B \subset U$ there exists $c(\varepsilon) > 0$ satisfying

$$p(f) \leqq c(\varepsilon) \sup \{\|f(t)\|; t \in K + \varepsilon B\}$$

for all $f \in \mathscr{H}(U; F)$. For $t = k + \varepsilon b \in K + \varepsilon B$ we have

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(k) \varepsilon b = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \hat{d}^n f(k) b \, .$$

Thus

$$\begin{aligned} p(f) &\leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \sup \left\{ \left\| \frac{1}{n!} \hat{d}^n f(k) b \right\|; k \in K, b \in B \right\} \\ &\leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \sup \left\{ \left\| \frac{1}{n!} \hat{d}^n f(k) \right\|_B; k \in K \right\} \end{aligned}$$

and (1) implies (2).

Conversely, suppose that p is a seminorm on $\mathscr{H}(U; F)$ as in (2). Then

$$p(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \sup \left\{ \left\| \frac{1}{n!} \hat{d}^n f(k)(\varepsilon b) \right\|; k \in K, b \in B \right\} .$$

Let $\varepsilon > 0$ be such that $K + \varepsilon B \subset U$. Let $\varepsilon' = \varepsilon/2 > 0$. By the Cauchy integral formulas we get

$$\begin{aligned} &\sup \left\{ \left\| \frac{1}{n!} \hat{d}^n f(k)(\varepsilon' b) \right\|; k \in K, b \in B \right\} \\ &\qquad \leqq \frac{1}{2^n} \sup \{\|f(k + \lambda \varepsilon' b)\|; k \in K, b \in B, |\lambda| = 2\} \\ &\qquad = \frac{1}{2^n} \sup \{\|f(y)\|; y \in K + \varepsilon B\} . \end{aligned}$$

Hence

$$\begin{aligned} p(f) &\leqq c\left(\frac{\varepsilon}{2}\right) \sum_{n=0}^{\infty} \frac{1}{2^n} \sup \{\|f(y)\|; y \in K + \varepsilon B\} \\ &= 2c\left(\frac{\varepsilon}{2}\right) \sup \{\|f(y)\|; y \in K + \varepsilon B\} \end{aligned}$$

and (2) implies (1).

Now we suppose that U is balanced and that p is $K - B$ ported. Thus for every $\varepsilon > 0$, with $K + \varepsilon B \subset U$, we have the existence of $c(\varepsilon) > 0$ such that

$$p(f) \leqq c(\varepsilon) \sup \{\|f(t)\|; t \in K + \varepsilon B\}$$

for all $f \in \mathscr{H}(U; F)$. Since U is balanced

$$f(k + \varepsilon b) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(k + \varepsilon b)$$

for $k \in K$, $b \in B$. Hence

$$\begin{aligned} &\sup \{\|f(t)\|; t \in K + \varepsilon B\} \\ &\quad \leqq \sum_{n=0}^{\infty} \sup \left\{\left\|\frac{1}{n!} \hat{d}^n f(0)(k + \varepsilon b)\right\|; k \in K, b \in B\right\} \end{aligned}$$

and it follows that

$$p(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(0) \right\|_{K+\varepsilon B} .$$

Conversely let p be a seminorm satisfying (*). Let $\rho > 1$ be such that $\rho K \subset U$. For each $\varepsilon > 0$ such that $\rho K + \varepsilon\rho B \subset U$, we have by Cauchy inequalities

$$\begin{aligned} &\left\| \frac{1}{n!} \hat{d}^n f(0) \right\|_{K+\varepsilon B} \\ &\quad \leqq \frac{1}{\rho^n} \sup \{\|f(\lambda k + \lambda\varepsilon b)\|; k \in K, b \in B, |\lambda| = \rho\} \\ &\quad = \frac{1}{\rho^n} \sup \{\|f(t)\|; t \in \rho\tilde{K} + \rho B\} \end{aligned}$$

where $\tilde{K}$ is the balanced hull of K. Hence

$$p(f) \leqq c(\varepsilon) \frac{\rho}{\rho - 1} \sup \{\|f(t)\|; t \in \rho\tilde{K} + \rho B\}$$

and p is $\rho\tilde{K} - \rho B$ ported.

LEMMA 1.3. *Let* $f = \sum_{n=0}^{\infty} (\hat{d}^n f(0)/n!) \in \mathscr{H}(U; F)$, *where* $U \subset E$ *is balanced. Then for each* $K \subset U$ *balanced compact,* $B \subset E$ *balanced bounded and* $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, *we have*

$$(*) \qquad \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B} < \infty .$$

Proof. Let f, K, B and $(\alpha_n)_{n=0}^{\infty}$ as above. By (4) Proposition 1.8, there is $\varepsilon > 0$ such that $K + \varepsilon B \subset U$ and $\sum_{n=0}^{\infty} \|\hat{d}^n f(0)/n!\|_{K+\varepsilon B} < \infty$. Since $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, let n_0 be a positive integer such that $\alpha_n \leqq \varepsilon$ for $n \geqq n_0$. Then, we get

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} \leqq \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\varepsilon B} \quad \text{for} \quad n \geqq n_0 .$$

Hence

$$\sum_{n=n_0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} < \infty .$$

Since

$$\sum_{n=0}^{n_0-1}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} < \infty ,$$

we have (*).

LEMMA 1.4. *If $f \in \mathscr{H}(U; F)$ and U is balanced, then the Taylor series of f at 0 converges to f in $(\mathscr{H}(U; F), \tau_{\omega s})$.*

PROPOSITION 1.5. *If U is balanced, the topology $\tau_{\omega s}$ in $\mathscr{H}(U; F)$ is generated by all seminorms of the type*

$$(1) \qquad p(f) = \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} ,$$

for all $f \in \mathscr{H}(U; F)$, where $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, $K \subset U$ is balanced compact and $B \subset E$ is bounded and balanced.

Proof. By Lemma 1.3, all seminorms of this type are well defined. Then, it is obviously a seminorm on $\mathscr{H}(U; F)$. Now, we show that p is $\tau_{\omega s}$-continuous. Given $\varepsilon > 0$, choose n_0 a positive integer such that $\alpha_n \leqq \varepsilon$ for all $n \geqq n_0$. As Lemma 1.4, we get

$$\sum_{n=n_0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} \leqq \sum_{n=n_0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\varepsilon B}$$

for all $f \in \mathscr{H}(U; F)$.

For $n = 0, 1, \cdots, n_0 - 1$, there is $\delta > 0$ such that $\delta(K + \alpha_n B) \subset K + \varepsilon B$. So, for all $f \in \mathscr{H}(U; F)$ and $n = 0, 1, \cdots, n_0 - 1$,

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} \leqq \delta^{-n}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\varepsilon B} .$$

Therefore

$$\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\alpha_n B} \leqq \left(\sup_{i \leqq n_0} \delta^{-i}\right)\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\varepsilon B} .$$

Hence p is continuous on $(\mathscr{H}(U; F)\tau_{\omega s})$. Now let p_1 be a continuous seminorm on $(\mathscr{H}(U; F), \tau_{\omega s})$. We show that p_1 is dominated by a norm of the form (1). By Proposition 1.2, for some $K \subset U$ compact and balanced and $B \subset E$ balanced and bounded, p_1 satisfies: for each

$\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that

$$p_1(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\varepsilon B} \quad \text{for all } f \in \mathscr{H}(U;F) .$$

Let $\delta > 1$ such that $\delta K \subset U$. For $P_n \in \mathscr{P}(^nE; F)$, $p_1(P_n) \leqq c(\varepsilon)\|P_n\|_{K+\varepsilon B}$. For each n and $\varepsilon > 0$, let $K_n(\varepsilon)$ be the smallest positive number or zero such that $p_1(P_n) \leqq K_n(\varepsilon)\|P_n\|_{K+\varepsilon B}$ for all $P_n \in \mathscr{P}(^nE; F)$. Since $K_n(\varepsilon) \leqq c(\varepsilon)$ for all n, we get $\lim_{n\to\infty} \sup K_n(\varepsilon)^{1/n} \leqq 1 < \delta$.

Now choose a positive integer n_1 such that $(K_n(1))^{1/n} \leqq \delta$ for all $n \geqq n_1$ and by induction take n_k such that $n_k > n_{k-1}$ and $(K_n(1/k))^{1/n} < 2$ for $n \geqq n_k$. Let

$$\alpha_n = \begin{cases} 1 & \text{for } \; n < n_2 \\ 1/k & \text{for } \; n_k \leqq n < n_{k+1} . \end{cases}$$

Then $(\alpha_n)_{n=0} \in C_0^+$ and $K_n(\alpha_n)^{1/n} \leqq \delta$ for $n \geqq n_1$. Hence there is $C > 0$ such that $K_n(\alpha_n) < C.\delta^n$ for all n. Therefore by Lemma 1.4, we get

$$\begin{aligned} p_1(f) &= p_1\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) \leqq \sum_{n=0}^{\infty} p_1\left(\frac{\hat{d}^n f(0)}{n!}\right) \\ &\leqq \sum_{n=0}^{\infty} K_n(\alpha_n) \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B} \\ &\leqq C. \sum_{n=0}^{\infty} \delta^n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B} \\ &\leqq C. \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\delta K+\delta\alpha_n B} . \end{aligned}$$

PROPOSITION 1.6. *Let* $\mathscr{X}$ *be a subset of* $\mathscr{H}(U; F)$. $\mathscr{X}$ *is* $\tau_{\omega s}$-*bounded if, and only if,* $\mathscr{X}$ *is* τ_0-*bounded.*

Proof. It is suffices to show that if $\mathscr{X}$ is τ_0-bounded then $\mathscr{X}$ is $\tau_{\omega s}$-bounded. Suppose $\mathscr{X}$ is τ_0-bounded. By [2], Proposition 4, for all $K \subset U$ compact and $B \subset E$ bounded, there are $C \geqq 0$ and $c \geqq 0$ such that

$$\sup\left\{ \left\| \frac{1}{n!} \hat{d}^n f(x) \right\|_B ; x \in K \text{ and } f \in \mathscr{X} \right\} \leqq Cc^n , \quad \text{for all } n \in N .$$

Therefore, for any p seminorm on $\mathscr{H}(U; F)$, $K - B$ ported, given $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that

$$\begin{aligned} \sup\{p(f); f \in \mathscr{X}\} &\leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \sup\left\{ \left\| \frac{1}{n!} \hat{d}^n f(x) \right\|_B ; x \in K \text{ and } f \in \mathscr{X} \right\} \\ &\leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n C c^n . \end{aligned}$$

Choose $\varepsilon > 0$ so that $\varepsilon c < 1$, we obtain

$$\sup\{p(f); f \in \mathscr{X}\} < \infty ,$$

that is, $\mathscr{X}$ is $\tau_{\omega s}$-bounded.

REMARK 1.7. By [3] Example 3, and above proposition we get $\tau_{\omega s} \neq \tau_{\omega}$ is general.

PROPOSITION 1.8. *Let $\mathscr{X} \subset \mathscr{H}(U; F)$ be τ_0-bounded. Then the uniform structures associated with $\tau_{\omega s}$ and $\tau_{\infty s}$ induce the same uniform structure in $\mathscr{X}$. In particular, $\tau_{\omega s}$ and $\tau_{\infty s}$ induce on $\mathscr{X}$ the same topology.*

Proof. Let us assume first that $0 \in \mathscr{X}$ and prove that a subset of $\mathscr{X}$ is a neighborhood of 0 in the topology on $\mathscr{X}$ induced by $\tau_{\omega s}$ if, and only if, it is a neighborhood of 0 in the topology on $\mathscr{X}$ induced by $\tau_{\infty s}$. One half of this assertion is clear from $\tau_{\infty s} \leqq \tau_{\omega s}$. Conversely, let p be any seminorm on $\mathscr{H}(U; F)$ $K - B$ ported. Then, given $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that

$$p(f) \leqq c(\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \sup\left\{\left\|\frac{1}{n!}\hat{d}^n f(x)\right\|_B; x \in K\right\} .$$

Since $\mathscr{X}$ is τ_0-bounded, there are $C \geqq 0$ and $c \geqq 0$ such that

$$\sup\left\{\left\|\frac{1}{n!}\hat{d}^n f(x)\right\|_B; x \in K \text{ and } f \in \mathscr{X}\right\} \leqq C.c^n ,$$

for all $n \in N$. Next choose $\varepsilon > 0$ so that $\varepsilon c < 1$ and $n_0 \in N$ by

$$C.c(\varepsilon) \sum_{m > n_0} (\varepsilon c)^m < \frac{1}{2} .$$

Define the $\tau_{\infty s}$ continuous seminorm q by

$$q(f) = c(\varepsilon) \sum_{m=0}^{n_0} \varepsilon^m \sup\left\{\left\|\frac{1}{m!}\hat{d}^m f(x)\right\|_B; x \in K\right\} .$$

It is then clear that, if $f \in \mathscr{X}$ and $q(f) \leqq 1/2$, then $p(f) \leqq 1$. This proves that $\tau_{\omega s}|_{\mathscr{X}} = \tau_{\infty s}|_{\mathscr{X}}$.

If we next consider any subset $\mathscr{X}$ bounded for τ_0, the set $\mathscr{X} - \mathscr{X}$ is bounded for τ_0, and it contains 0. Since the neighborhoods of 0 in the topology on $\mathscr{X} - \mathscr{X}$ induced by $\tau_{\omega s}$ and $\tau_{\infty s}$ are identical, it follows that the uniform structures on $\mathscr{X}$ induced by the uniform structures associated to $\tau_{\omega s}$ and $\tau_{\infty s}$ are identical.

COROLLARY 1.9. *Let $f_\iota \in \mathscr{H}(U; F)$ for all $\iota \in N$ and $f \in \mathscr{H}(U; F)$,* *then* $f_\iota \to f$ *for* $\tau_{\omega s}$ *as* $\iota \to \infty$ *if, and only if,* $f_\iota \to f$ *for* $\tau_{\infty s}$ *as* $\iota \to \infty$.

COROLLARY 1.10. *Let* $\mathscr{X} \subset \mathscr{H}(U; F)$. *Then* $\mathscr{X}$ *is* $\tau_{\omega s}$*-relatively compact if, and only if* $\mathscr{X}$ *is* $\tau_{\infty s}$*-relatively compact.*

PROPOSITION 1.11. *Let* F *be complete and* $\mathscr{X} \subset \mathscr{H}(U; F)$ *lacally bounded. Then* $\mathscr{X}$ *is* $\tau_{\omega s}$*-relatively compact if, and only if,* $\{(1/n!)\hat{d}^n f(t); f \in \mathscr{X}\}$ *is relatively compact in* $\mathscr{P}_s({}^nE; F)$, $\forall n \in N$, $\forall t \in U$.

Proof. First we assume $\mathscr{X}$ is $\tau_{\omega s}$-relatively compact. For each $\xi \in U$ and $n \in N$, the mapping

$$\phi_\xi\colon (\mathscr{H}(U; F), \tau_{\omega s}) \longrightarrow \mathscr{P}_s({}^nE; F)$$
$$f \longrightarrow \frac{1}{n!}\hat{d}^n f(\xi)$$

is continuous. In fact, $q(f) = \|(1/n!)\hat{d}^n f(\xi)\|_B$ is a seminorm $\{\xi\} - B$ ported. Choose p a seminorm on $\mathscr{P}_s({}^nE; F)$ such that $p(P) = \|P\|_B$ for every $P \in \mathscr{P}({}^nE; F)$, we obtain $p(\phi_\xi(f)) = q(f)$. Hence the image of $\mathscr{X}$ is relatively compact in $\mathscr{P}_s({}^nE; F)$ for every $n \in N$, that is, $\{(1/n!)\hat{d}^n f(t); f \in \mathscr{X}\}$ is relatively compact in $\mathscr{P}_s({}^nE; F) \forall n \in N$, $\forall t \in U$.

Conversely, suppose that $\{(1/n!)\hat{d}^n f(t); f \in \mathscr{X}\}$ is relatively compact in $\mathscr{P}_s({}^nE; F) \forall n \in N$, and $t \in U$. Since $\mathscr{X}$ is locally bounded, it is $\tau_{\omega s}$-bounded and $\overline{\mathscr{X}}^{\tau_{\omega s}} = \overline{\mathscr{X}}^{\tau_{\infty s}}$. Hence to prove that $\mathscr{X}$ is relatively compact for $\tau_{\omega s}$ we shall show that $\mathscr{X}$ is relatively compact for $\tau_{\omega s}$ topology. Let

$$\phi\colon \mathscr{H}(U; F) \longrightarrow \prod_{n=0}^{\infty} \mathscr{H}(U; \mathscr{P}_s({}^nE; F))$$
$$f \longrightarrow \left(\frac{1}{n!}\hat{d}^n f\right)_{n \geq 0}.$$

On $\mathscr{H}(U; F)$ consider the topology $\tau_{\infty s}$ and on $\prod_{n=0}^{\infty} \mathscr{H}(U; \mathscr{P}_s({}^nE; F))$ we consider the product of the topologies τ_0 on each factor. By [2], Proposition 2.5, $\phi(\mathscr{H}(U; F)) \subset \prod_{n=0}^{\infty} \mathscr{H}(U; \mathscr{P}_s({}^nE; F))$. ϕ is a continuous linear injection and ϕ^{-1} is also continuous.

To show that $\mathscr{X}$ is $\tau_{\omega s}$-relatively compact it is equivalent to show that $\phi(\mathscr{X})$ is relatively compact for the product topology. It is enough to show that $\prod_n \phi(\mathscr{X}) = \{(1/n!)\hat{d}^n f; f \in \mathscr{X}\}$, that is, the projection in each $(\mathscr{H}(U; \mathscr{P}_s({}^nE; F)), \tau_0)$ is relatively compact.

By the assumption that $\mathscr{X}$ is locally bounded, we have $\mathscr{X}$ equicontinuous by [1], Proposition 3.4, and then $\prod_n \phi(\mathscr{X})$ is equicontinuous. So we have by assumption that $\prod_n \phi(\mathscr{X})$ is pointwise relatively compact. Hence by the Ascoli's theorem (Bourbaki-cap. X)

$\prod_n \phi(\mathscr{X})$ is τ_0-relatively compact.

Since F is complete the closure of $\phi(\mathscr{X})$ for the product topology is contained in $\phi(\mathscr{H}(U; F))$, so we have $\bar{\mathscr{X}}^{\tau_{\infty s}}$ is compact for $\tau_{\infty s}$ topology.

The next proposition belongs to J. A. Barroso, [1], where the proof contains some small mistakes and here they are corrected.

PROPOSITION 1.12. *Suppose $F \neq \{0\}$ and E be a locally convex space such that corresponding to every bounded subset $B \subset E$ there is a compact subset $K \subset E$ such that B is contained in the closure of the absolutely convex hull of K, $\Gamma(K)$. Then $\tau_0 = \tau_{\infty s}$ on $\mathscr{H}(U; F)$. Conversely, if $\tau_0 = \tau_{\infty s}$ on $\mathscr{H}(U) = \mathscr{H}(U; C)$, then corresponding to every bounded subset $B \subset E$, there is a compact subset $K \subset E$ such that B is contained in $\Gamma(K)$.*

Proof. We prove the first part. Thus let E be a locally convex space such that corresponding to every bounded subset $B \subset E$ there is a compact subset $K \subset E$ such that B is contained in $\Gamma(K)$. Since $\tau_0 \leqq \tau_{\infty s}$, it is enough to show that $\tau_{\infty s} \leqq \tau_0$ on $\mathscr{H}(U; F)$. Let p be a $\tau_{\infty s}$ continuous seminorm, $K_0 \subset U$ compact, B, such that

$$p(f) = \sup_{\substack{n \in J \\ t \in K_0}} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!} \right\|_B \right\},$$

for all $f \in \mathscr{H}(U; F)$, where J is a finite subset of N.

For each $n \in N$, we have that

$$\begin{aligned} \sup_{t \in K_0} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!} \right\|_B \right\} &\leqq \sup_{t \in K_0} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!} \right\|_{\Gamma(K)} \right\} \\ &\leqq \sup_{\substack{t \in K_0 \\ x \in \Gamma(K)}} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!}(x, \cdots, x) \right\| \right\} \\ &\leqq \sup_{\substack{t \in K_0 \\ x_i \in K; i=1, \cdots, n}} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!}(x_1, \cdots, x_n) \right\| \right\}. \end{aligned}$$

By the polarization formula, we have for $\tilde{K}$ ($\tilde{K}$ be the balanced hull of K), that

$$\sup_{\substack{t \in K_0 \\ x_i \in K; i=1, \cdots, n}} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!}(x_1, \cdots, x_n) \right\| \right\} \leqq \frac{n^n}{n!} \sup_{\substack{t \in K_0 \\ x \in \tilde{K} + \cdots + \tilde{K}}} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!}(x) \right\| \right\}.$$

By the Cauchy formula, we obtain

$$\sup_{t \in K_0} \left\{ \left\| \frac{\hat{d}^n f(t)}{n!} \right\|_B \right\} = \frac{n^n}{n!} \frac{1}{\rho^n} \sup_{y \in K_0 + \rho(\tilde{K} + \cdots + \tilde{K})} \|f(y)\| .$$

If we take $\rho > 0$ such that $K_0 + \rho(\tilde{K} + \cdots + \tilde{K}) = L$ is contained in U, we obtain a compact subset of U and

$$p(f) \leqq \sup_{n \in J} \frac{n^n}{n!\rho^n} \|f\|_L$$

therefore $\tau_{\infty s} \leqq \tau_0$ on $\mathscr{H}(U; F)$.

Conversely, if $\tau_0 = \tau_{\infty s}$ on $\mathscr{H}(U)$, then $\tau_0/E' = \tau_{\infty s}/E'$. Therefore, the τ_{β_0}-topology of uniform convergence on bounded subsets of E is induced by $\tau_{\infty s}$ in E' and $\tau_0/E' = \tau_{\beta_0}$. Hence if $B \subset E$ is a subset bounded of E, B^0 is the polar of B, there is a compact subset K of E and $\varepsilon > 0$, such that if

$$V = \{T \in E'; \|T\|_K = \sup\{|T(x)|; x \in K\} \leqq \varepsilon\},$$

then $V \subseteqq B^0$. Therefore $B \subset B^{00} \subset V^0$. But if $x \in V^0$, we claim $|T(x)| \leqq \varepsilon^{-1}\|T\|_K$ for all $T \in E'$. In fact if $\delta > 0$ and $T \in E'$, then for $G = \varepsilon T/(\|T\|_K + \delta)$, we have $\|G\|_K \leqq \varepsilon$ and so $G \in V$ and so $|G(x)| \leqq 1$. This gives $|T(x)| \leqq \varepsilon^{-1}(\|T\|_K + \delta)$ and as δ is arbitrary, it follows that $|T(x)| \leqq \varepsilon^{-1}\|T\|_K$ for all $T \in E'$. So if L is $\Gamma(K)$ then $|T(\varepsilon x)| \leqq \|T\|_K \leqq \|T\|_L$, if $x \in V^0$ and $T \in E'$. This implies, via the separation theorem, that $\varepsilon x \in L$ for $x \in V^0$. Thus $V^0 \subset \varepsilon^{-1}L$. So $B \subset V^0 \subset \varepsilon^{-1}L$. This completes the proof.

REMARK 1.13. In the second part of this proof it is enough to suppose that $\tau_0 = \tau_{1s}$ where τ_{1s} is the locally convex topology generated by all seminorms of the type:

$$p(f) = \sup_{t \in K} \|\hat{d}^1 f(t)\|_B,$$

for all $f \in \mathscr{H}(U; F)$, where $K \subset U$ is compact and $B \subset E$ is bounded.

COROLLARY 1.14. *Let E be a locally convex space so that the closure of the absolutely convex hull of every compact is compact. Then $\tau_0 = \tau_{\omega s}$ on $\mathscr{H}(E)$ if, and only if, E is a semi-Montel space.*

One part of the proof of the Corollary 1.14 is given more directly by considering seminorms of the type

$$f \longrightarrow \sup\{\|\hat{d}^1 f(0)x\|; x \in B\},$$

for all $f \in \mathscr{H}(E)$, where $B \subset E$ is bounded. This seminorm is τ_{1s}-continuous, thus $\tau_{\omega s}$-continuous and, by hypothesis, τ_0-continuous. Therefore for all $T \in E'$ and $x \in B$, $|T(x)| \leqq c\|T\|_K \leqq c\|T\|_{\Gamma(K)}$, for some $K \subset U$ compact and $c > 0$. Then, via the separation theorem, we obtain that $B \subset c.\Gamma(K)$. Sence $\Gamma(K)$ is compact by hypothesis, this implies that E is a semi-Montel space.

2. **The completion of $(\mathscr{H}(E), \tau_{\omega s})$.** In here, the completions are considered as subspaces of the space of the G-holomorphic functions in E. We denote by $\mathscr{H}_c(E)$ the set of all functions $f: E \to \mathbf{C}$ such that there is P_n in the completion of $\mathscr{P}_s({}^nE)$, for $n = 0, 1, \cdots$, so that for each $K \subset E$ compact, $B \subset E$ bounded there is $\alpha = \alpha(B) > 0$ with $f = \sum P_n$ uniformly on $K + \alpha B$. We use the notation $\widehat{\mathscr{H}_s(E)}$ for the completion of $(\mathscr{H}(E), \tau_{\omega s})$ and $\widehat{\mathscr{P}_s({}^nE)}$ for the completion of $\mathscr{P}_s({}^nE)$. Here we prove that $\widehat{\mathscr{H}_s(E)} = \mathscr{H}_c(E)$. For this we need the following lemma.

LEMMA 2.1. *If $f \in \widehat{\mathscr{H}_s(E)}$, there is $P_n \in \widehat{\mathscr{P}_s({}^nE)}$, for $n = 0, 1, \cdots$, such that $f(t) = \sum_{n=0}^{\infty} P_n(t)$ for each $t \in E$. Furthermore, if p is a $\tau_{\omega s}$-continuous seminorm on $\mathscr{H}(E)$, such that*

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B}$$

for all $f \in \mathscr{H}(E)$, for some $K \subset U$ balanced compact, $B \subset E$ bounded balanced and $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, then the extension $\hat{p}$ of p on $\widehat{\mathscr{H}_s(E)}$ is given by:

$$\hat{p}(f) = \sum_{n=0}^{\infty} \|P_n\|_{K+\alpha_n B} \text{ for all } f \in \widehat{\mathscr{H}_s(E)} .$$

Proof. Let $f \in \widehat{\mathscr{H}_s(E)}$ and $(f_\alpha)_{\alpha \in A}$ a Cauchy net in $(\mathscr{H}(E), \tau_{\omega s})$ such that $\lim_{\alpha \in A} f_\alpha = f$. Note that in particular, for each $t \in E$, $\lim_{\alpha \in A} f_\alpha(t) = f(t)$. Then, if p is a $\tau_{\omega s}$-continuous seminorm on $(\mathscr{H}(E), \tau_{\omega s})$ given $K \subset E$ compact balanced, $B \subset E$ bounded balanced, $(\alpha_n)_{n=0}^{\infty} \in C_0^+$ and $\varepsilon > 0$, there is $\lambda \in A$ such that, for $\alpha, \beta > \lambda$,

$$p(f_\alpha - f_\beta) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f_\alpha(0)}{n!} - \frac{\hat{d}^n f_\beta(0)}{n!} \right\|_{K+\alpha_n B} < \varepsilon .$$

Hence for any positive integer m and $\alpha, \beta \geqq \lambda$, we have

$$(1) \qquad \sum_{n=0}^{m} \left\| \frac{\hat{d}^n f_\alpha(0)}{n!} - \frac{\hat{d}^n f_\beta(0)}{n!} \right\|_{K+\alpha_n B} < \varepsilon .$$

Since $(f_\beta)_{\beta \in A}$ is a Cauchy net in $(\mathscr{H}(E), \tau_{\omega s})$, for $n = 0, 1, \cdots$, $(\hat{d}^n f_\beta / n!)_{\beta \in A}$ is a Cauchy net in $\mathscr{P}_s({}^nE)$. For each $n = 0, 1, \cdots$, let $P_n = \lim_{\beta \in A} \hat{d}^n f_\beta(0)/n!$. If we take $\alpha \geqq \lambda$ and the limit in (1) for $\beta \in A$, we get

$$(2) \qquad \sum_{n=0}^{m} \left\| \frac{\hat{d}^n f_\alpha(0)}{n!} - P_n \right\|_{K+\alpha_n B} < \varepsilon ,$$

for any positive integer m.

In particular we get

$$\sum_{n=0}^{m} \|P_n\|_{K+\alpha_n B} \leqq \sum_{n=0}^{\infty} \left\|\frac{\hat{d}^n f_\lambda(0)}{n!}\right\|_{K+\alpha_n B} + \varepsilon .$$

Thus

$$(3) \qquad \sum_{n=0}^{\infty} \|P_n\|_{K+\alpha_n B} \leqq \sum_{n=0}^{\infty} \left\|\frac{\hat{d}^n f_\lambda(0)}{n!}\right\|_{K+\alpha_n B} + \varepsilon < \infty .$$

By (3) we have in particular that $\sum_{n=0}^{\infty} P_n(t)$ is finite for each $t \in E$. If we take the limit in (2) for $m \to \infty$, we get in particular for each $t \in E$;

$$\left|\sum_{n=0}^{\infty} \frac{\hat{d}^n f_\alpha(0)(t)}{n!} - P_n(t)\right| < \varepsilon \text{ for } \alpha \geqq \lambda \text{ and } \varepsilon > 0 ,$$

that is,

$$\left|f_\alpha(t) - \sum_{n=0}^{\infty} P_n(t)\right| < \varepsilon .$$

Therefore, for each $t \in E$, $\lim_{\alpha \in \Lambda} f_\alpha(t) = \sum_{n=0}^{\infty} P_n(t)$, that is, $f(t) = \sum_{n=0}^{\infty} P_n(t)$. This proves the first part of the lemma.

By (3), wehave $\hat{p}(f)$ is finite for each $f \in \widehat{\mathscr{H}_s(E)}$. Now, to prove that $\hat{p}(f) = \sum_{n=0}^{\infty} \|P_n\|_{K+\alpha_n B}$, it is enough to prove that, for $\varepsilon > 0$ there is $n_0 \in N$ and $\lambda \in \Lambda$, such that for $m \geqq n_0$ and $\alpha \geqq \lambda$,

$$\left|\sum_{n=0}^{m} \|P_n\|_{K+\alpha_n B} - \sum_{n=0}^{m} \left\|\frac{\hat{d}^n f_\alpha(0)}{n!}\right\|_{K+\alpha_n B}\right| < \varepsilon .$$

But by (2), for $\alpha \geqq \lambda$, $m \in N$ and $\varepsilon > 0$,

$$\begin{aligned}
&\left|\sum_{n=0}^{m} \|P_n\|_{K+\alpha_n B} - \sum_{n=0}^{m} \left\|\frac{\hat{d}^n f_\alpha(0)}{n!}\right\|_{K+\alpha_n B}\right| \\
&\qquad \leqq \sum_{n=0}^{m} \left|\|P_n\|_{K+\alpha_n B} - \left\|\frac{\hat{d}^n f_\alpha(0)}{n!}\right\|_{K+\alpha_n B}\right| \\
&\qquad \leqq \sum_{n=0}^{m} \left\|P_n - \frac{\hat{d}^n f_\alpha(0)}{n!}\right\|_{K+\alpha_n B} < \varepsilon .
\end{aligned}$$

REMARK 2.2. On $\mathscr{H}_c(E)$ we may define the $\tau_{\omega s}$-topology and show in the same way as Proposition 1.5 that this topology is equivalent to the topology defined by the seminorms of the type used there. Furthermore as in the Lemma 1.4, we obtain that if $f \in \mathscr{H}_c(E)$, there is P_n in $\widehat{\mathscr{P}_s(^nE)}$ for each $n = 0, 1, \cdots$, so that $\sum_{n=0}^{\infty} P_n$ converges to f in $(\mathscr{H}_c(E), \tau_{\omega s})$.

PROPOSITION 2.3. $\mathscr{H}_c(E) = \widehat{\mathscr{H}_s(E)}$.

Proof. By definition, we have $\widehat{\mathscr{H}_s(E)} \subset \mathscr{H}_c(E)$. Let now $f \in \mathscr{H}_c(E)$. Then by the remark above, there is $P_n \in \widehat{\mathscr{P}_s(^nE)}$ for each $n = 0, 1, \cdots$, such that $\sum_{n=0}^{\infty} P_n$ converges to f in $(\mathscr{H}_c(E), \tau_{\omega s})$. For each $k=0, 1, \cdots, n$, $P_k \in \widehat{\mathscr{P}_s(^nE)} \subset \widehat{\mathscr{H}_s(E)}$. Therefore $\sum_{k=0}^{n} P_k \in \widehat{\mathscr{H}_s(E)}$. Since $(\sum_{k=0}^{n} P_k)_{n=0}^{\infty}$ is a Cauchy sequence in $(\mathscr{H}_c(E), \tau_{\omega s})$ we have by the remark above and the previous lemma, that $(\sum_{k=0}^{n} P_k)_{n=0}^{\infty}$ is a Cauchy sequence in $\widehat{\mathscr{H}_s(E)}$. Therefore, $f = \sum_{n=0}^{\infty} P_n$ belongs to $\widehat{\mathscr{H}_s(E)}$.

3. The τ_{0b} topology on $H(E)$. We denote by $H(E)$ the set of all functions $f: E \to C$ such that there is P_n in $\mathscr{P}(^nE)$ for $n = 0, 1, \cdots$, so that for each $K \subset E$ compact, $B \subset E$ bounded, there is $\alpha = \alpha(B) > 0$ with $f = \sum_{n=0}^{\infty} P_n$ uniformly on $K + \alpha B$. $\hat{\delta}^n f(0)/n!$ denotes the nth coefficient of the Taylor series of f at 0, $n = 0, 1, \cdots$, for each $f \in H(E)$. We may define the $\tau_{\omega s}$-topology on $H(E)$ as in 1, and obtain similar results to Propositions 1.2 and 1.5. τ_{0b} denotes the bornological topology on $H(E)$ associated with τ_0.

PROPOSITION 3.1. *Let $f = \sum_{n=0}^{\infty} P_n$ pointwise, with $P_n \in \mathscr{P}(^nE)$, $n \in N$. Then the following three conditions are equivalent:*

(1) $f \in H(E)$.

(2) *For each $K \subset E$ compact balanced, $B \subset E$ bounded balanced and $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, we have*

$$\sum_{n=0}^{\infty} \|P_n\|_{K+\alpha_n B} < \infty$$

(3) *For each $K \subset E$ compact balanced, $B \subset E$ bounded balanced and $(\alpha_n)_{n=0}^{\infty} \in C_0^+$, we have*

$$\lim_{n\to\infty} \|P_n\|_{K+\alpha_n B}^{1/n} = 0 .$$

LEMMA 3.2. *$\mathscr{X} \subset H(E)$ is τ_0-bounded if, and only if, is $\tau_{\omega s}$-bounded.*

LEMMA 3.3. *Let $(f_n)_{n=0}^{\infty}$ be a bounded subset of $(H(E), \tau_{\omega s})$, then*

$$g = \sum_{n=0}^{\infty} \frac{\hat{\delta}^n f_n(0)}{n!} \in H(E) .$$

LEMMA 3.4. *Let $\mathscr{X}$ be a bounded subset of $(H(E), \tau_{\omega s})$ then the set $(n^2 \hat{\delta}^n f(0)/n!)_{n=0, f \in \mathscr{X}}^{\infty}$ is bounded in $(H(E), \tau_0)$.*

The proof of the Lemma 3.2 is the same as Proposition 1.6.

Proposition 3.1 and Lemma 3.3 are proved in the same way Dineen proves Proposition 2 and Lemma 5 [4] with minor modifications. Lemma 3.4 is proved in a similar way as Lemma 1.2 [5].

PROPOSITION 3.5. *Let p be a seminorm on $H(E)$ with the following properties*:

(1) *For each $n = 0, 1, \cdots, p$ induces on $\mathscr{P}(^nE)$ a topology weaker than or equal to the τ_{0b}-topology.*

(2) $\sum_{n=0}^{\infty} p(\hat{\delta}^n f(0)/n!) < \infty$ *for every* $f = \sum_{n=0}^{\infty} \hat{\delta}^n f(0)/n! \in H(E)$. *Then,* $p_1(f) = \sum_{n=0}^{\infty} p(\hat{\delta}^n f(0)/n!)$ *is a continuous seminorm on* $(H(E), \tau_{0b})$.

Proof. Since τ_{0b} is a bornological topology it is suffices, to show that for each bounded set $\mathscr{X}$ on $(H(E), \tau_0)$, we have $\sup_{f\in\mathscr{X}} p_1(f) < \infty$. By condition (1), we get for each n that

$$(*) \qquad \sup_{f\in\mathscr{X}} p\left(\frac{\hat{\delta}^n f(0)}{n!}\right) < \infty .$$

Now suppose $\sup_{f\in\mathscr{X}} p_1(f) = \infty$. By (*) and the definition of p_1, we thus have for each positive integer n_0

$$\sup_{f\in\mathscr{X}} \sum_{n=n_0}^{\infty} p\left(\frac{\hat{\delta}^n f(0)}{n!}\right) = \infty .$$

Choose f_1 such that $\sum_{n=0}^{\infty} p(\hat{\delta}^n f_1(0)/n!) \geqq 2$ and take n_1 such that

$$\sum_{n=0}^{n_1} p\left(\frac{\hat{\delta}^n f_1(0)}{n!}\right) \geqq 1 .$$

By induction, choose for each k, f_k such that

$$\sum_{n=n_{k-1}+1}^{\infty} p\left(\frac{\hat{\delta} f_k(0)}{n!}\right) \geqq 2$$

and take n_k such that

$$\sum_{n=n_{k-1}+1}^{n_k} p\left(\frac{\hat{\delta}^n f_{n_k}(0)}{n!}\right) \geqq 1 \qquad (n_k \geqq k) .$$

Let

$$g_n = \begin{cases} f_1 & \text{for} \quad 0 \leqq n \leqq n_1 , \\ f_k & \text{for} \quad n_{k-1} < n \leqq n_k \qquad (k \geqq 2) . \end{cases}$$

By Lemma 3.3,

$$g = \sum_{n=0}^{\infty} \frac{\hat{\delta}^n g_n(0)}{n!} \in H(E) .$$

But

$$p_1(g) = \sum_{n=0}^{\infty} p\Big(\frac{\hat{\delta}^n g_n(0)}{n!}\Big) = \infty ,$$

which contradicts (2). Then $\sup_{f \in \mathscr{B}} p_1(f) < \infty$ and p_1 is a continuous seminorm on $(H(E), \tau_{0b})$.

PROPOSITION 3.6. *Let p be a continuous seminorm on $(H(E), \tau_{0b})$. Then:*

(i) *For each $n = 0, 1, \cdots, p$ induces on $\mathscr{P}(^nE)$ a topology weaker than or equal to the τ_{0b} topology.*

(ii) *If $f = \sum_{n=0}^{\infty} \hat{\delta}^n f(0)/n! \in H(E)$, then $\sum_{n=0}^{\infty} p(\hat{\delta}^n f(0)/n!) < \infty$.*

LEMMA 3.7. *If $f \in H(E)$, then the Taylor series of f at 0 converges to f in $(H(E), \tau_{0b})$.*

Lemma 3.7 is proved in the same way Dineen proves Proposition 7, [4] with minor modifications.

PROPOSITION 3.8. *The topology τ_{0b} on $H(E)$ is generated by all seminorms which satisfy the following conditions:*

(1) *$p(f) = \sum_{n=0}^{\infty} p(\hat{\delta}^n f(0)/n!)$ for all $f \in H(E)$.*

(2) *For each $n = 0, 1, \cdots, p$ induces on $\mathscr{P}(^nE)$ a topology weaker than or equal to the τ_{0b}-topology.*

Proof. By Proposition 3.5, if p satisfy (1) and (2), then p is τ_{0b}-continuous on $H(E)$.

Let q be a τ_{0b}-continuous seminorm on $H(E)$. Proposition 3.6 gives that

$$\sum_{n=0}^{\infty} q\Big(\frac{\hat{\delta}^n f(0)}{n!}\Big) < \infty \quad \text{for each} \quad f = \sum_{n=0}^{\infty} \frac{\hat{\delta}^n f(0)}{n!} \in H(E) .$$

By Proposition 3.6,

$$p_1(f) = \sum_{n=0}^{\infty} q\Big(\frac{\hat{\delta}^n f(0)}{n!}\Big)$$

is a continuous seminorm on $(H(E), \tau_{0b})$. Lemma 3.7 gives

$$\begin{aligned} q(f) &= \lim_{m\to\infty} q\Big(\sum_{n=0}^{m} \frac{\hat{\delta}^n f(0)}{n!}\Big) \\ &\leqq \lim_{m\to\infty} \sum_{n=0}^{m} q\Big(\frac{\hat{\delta}^n f(0)}{n!}\Big) = p_1(f) . \end{aligned}$$

Hence every continuous seminorm on $(H(E), \tau_{0b})$ is dominated by a continuous seminorm with satisfies the required conditions. This proves the proposition.

REMARK 3.9. If U is an open balanced subset in E, we obtain the same results on $H(U)$.

PROPOSITION 3.10. *The topology* τ_{0b} *on* $\mathscr{H}(E)$ *is generated by all seminorms which satisfy the following conditions*:

(1) $p(f) = \sum_{n=0}^{\infty} p(\hat{\delta}^n f(0)/n!)$ *for all* $f \in H(E)$.

(2) *For each* $n = 0, 1, \cdots, p$ *induces on* $\mathscr{P}(^nE)$ *a topology weaker than or equal to the* τ_{0b}*-topology.*

Proof. If p is a seminorm on $\mathscr{H}(E)$ satisfying (1) and (2) then it can be defined in $H(E)$ and, by Proposition 3.8, p is τ_{0b}-continuous. Hence p is bounded on the τ_0-bounded subsets of $H(E)$, thus on the τ_0-bounded subsets of $\mathscr{H}(E)$. This implies that p is τ_{0b}-continuous on $\mathscr{H}(E)$. Now, if p is τ_{0b}-continuous seminorm on $\mathscr{H}(E)$, then it is clear that (2) holds. If $f \in H(E)$, $\mathscr{X} = \{f\}$ is τ_0-bounded in $H(E)$. By Lemma 3.4, $(n^2(\hat{\delta}^n f(0)/n!))_{n=0}^{\infty}$ is τ_0-bounded in $H(E)$, hence in $\mathscr{H}(E)$. Thus

$$\sup_n p\left(\frac{\hat{\delta}^n f(0)}{n!}\right) \leqq \frac{M}{n^2}$$

and

$$\sum_{n=0}^{\infty} p\left(\frac{\hat{\delta}^n f(0)}{n!}\right) \leqq M \sum_{n=0}^{\infty} \frac{1}{n^2} < +\infty .$$

Therefore

$$p_1(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{\delta}^n f(0)}{n!}\right)$$

defines a seminorm on $\mathscr{H}(E)$ satisfying (1) and (2), hence τ_{0b}-continuous (by first part of the proof). Thus, since $p \leqq p_1$, τ_{0b} can be defined by seminorms satisfying (1) and (2).

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