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THE CONNECTED COMPONENT OF THE IDELE CLASS GROUP OF AN ALGEBRAIC NUMBER FIELD

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We shall give another proof of Weil's theorem of the structure of the connected component of the idèle class group of an algebraic number field. Our proof is different from Artin's.

Let Q be the rational number field and k be an algebraic number field of finite degree over Q. We denote by C_k the idèle class group of k and D_k the connected component of unity of C_k. Let T denote the multiplicative group of all complex numbers of absolute value 1 with compact topology, R the additive group of the real numbers with usual topology, and S the Solenoid with compact topology.

Weil ([**3**]) has shown that D_k is isomorphic to $T^{r_2} \times R \times S^r$, by determining the structure of the dual D_k^*. Artin ([**1**]) has exhibited a system of representatives of idèle classes and given a different proof. In this paper we shall give another proof of the above Weil's theorem.

1. Let k be an algebraic number field which has r_1 real infinite primes and r_2 complex infinite primes. As usual we put $r = r_1 + r_2 - 1$. Let I_k be the idèle group of k, C_k the idèle class group of k and D_k the connected component of unity of C_k. An idèle will be denoted by $(a_v) = (a_\mathfrak{p}, a_\lambda)$, where v runs all primes of k, $\mathfrak{p}$ all finite primes and λ all infinite primes of k $(\lambda = 1,\dots,r_1 + r_2)$. We shall agree that λ $(1 \leq \lambda \leq r_1)$ is real and λ $(r_1 + 1 \leq \lambda \leq r_1 + r_2)$ is complex. Let us denote by σ_λ the embedding of k into the complex number field attached to an infinite prime λ. Then σ_λ with $1 \leq \lambda \leq r_1$ is a real embedding and σ_λ with $r_1 + 1 \leq \lambda \leq r_1 + r_2$ a complex one.

For any topological group G, G^* denotes the character group of G. If χ is a character of C_k, i.e., a continuous homomorphism of C_k into T, we can regard it as a character of I_k which is trivial on principal idèles. If we restrict χ to the infinite part $R^{\times r_1} C^{\times r_2}$ of I_k, χ can be written as follows:

$$\chi((a_\lambda)) = \prod_{\lambda=1}^{r_1+r_2} \left(\frac{a_\lambda}{|a_\lambda|} \right)^{f_\lambda} |a_\lambda|^{\sqrt{-1}\,\varphi_\lambda}, \qquad (a_\lambda) \in R^{\times r_1} C^{\times r_2},$$

where $f_\lambda \in Z$ (the rational integers), $\varphi_\lambda \in R$ $(\lambda = 1,\dots,r_1 + r_2)$, and $f_1,\dots,f_{r_1} = 0$ or 1. Such f_λ and φ_λ $(\lambda = 1,\dots,r_1 + r_2)$ are uniquely determined, so we say that χ is of type $(f_\lambda, \varphi_\lambda)$.

It is well known that the following two lemmas hold.

LEMMA 1. *Let* χ *be a character of* C_k, *of type* $(f_\lambda, \varphi_\lambda)$. *Then the conditions* (i), (ii) *and* (iii) *are equivalent to each other*:

(i) χ *is of finite order.*

(ii) $\chi(D_k) = 1$.

(iii) $f_\iota = 0$ $(\iota = r_1 + 1, \ldots, r_1 + r_2)$, $\varphi_\lambda = 0$ $(\lambda = 1, \ldots, r_1 + r_2)$.

LEMMA 2. *Let* $f_\lambda \in Z$ *and* $\varphi_\lambda \in R$ $(\lambda = 1, \ldots, r_1 + r_2)$, *where* $f_1, \ldots, f_{r_1} = 0$ *or* 1. *Then the statements* (i) *and* (ii) *are equivalent to each other*:

(i) *There exists a character of* C_k, *of type* $(f_\lambda, \varphi_\lambda)$.

(ii) *Define the character* X *of the unit group* E *of* k *as follows, then* X *is of finite order*:

$$X(\varepsilon) = \prod_{\lambda=1}^{r_1+r_2} \left(\frac{\varepsilon^{\sigma_\lambda}}{|\varepsilon^{\sigma_\lambda}|} \right)^{f_\lambda} |\varepsilon^{\sigma_\lambda}|^{\sqrt{-1}\varphi_\lambda}, \qquad \varepsilon \in E.$$

2. In order to show that the isomorphism $D_k^* = Z^{r_2} \times R \times Q^r$ as topological groups, consider a homomorphism $\mathcal{H}$ of C_k^* to the additive group $Z^{r_2} \times R^{r_1+r_2}$. For any $\chi \in C_k^*$, if χ is of type $(f_\lambda, \varphi_\lambda)$, we put $\mathcal{H}(\chi) = (f_\iota, \varphi_\lambda)$: the element of $Z^{r_2} \times R^{r_1+r_2}$, where $\iota = r_1 + 1, \ldots, r_1 + r_2$ and $\lambda = 1, \ldots, r_1 + r_2$; as $f_1, \ldots, f_{r_1} = 0$ or 1, they are neglected. It is clear that $\mathcal{H}$ is a homomorphism (algebraically) whose kernel is T_k from Lemma 1. We denote by M the image of $\mathcal{H}$ in $Z^{r_2} \times R^{r_1+r_2}$.

3. Let $\varepsilon_1, \ldots, \varepsilon_r$ be a system of fundamental units of k. Let $f_\iota \in Z$ $(\iota = r_1 + 1, \ldots, r_1 + r_2)$ and $\varphi_\lambda \in R$ $(\lambda = 1, \ldots, r_1 + r_2)$. Assume that $(f_\iota, \varphi_\lambda)$ belongs to M. From Lemma 2, the character X of E,

$$X(\varepsilon) = \prod_{\iota=r_1+1}^{r_1+r_2} \left(\frac{\varepsilon^{\sigma_\iota}}{|\varepsilon^{\sigma_\iota}|} \right)^{f_\iota} \prod_{\lambda=1}^{r_1+r_2} |\varepsilon^{\sigma_\lambda}|^{\sqrt{-1}\varphi_\lambda}, \qquad \varepsilon \in E,$$

is of finite order; for, if λ is a real infinite prime, $\varepsilon^{\sigma_\lambda}/|\varepsilon^{\sigma_\lambda}|$ is always ± 1, so f_λ $(\lambda = 1, \ldots, r_1)$ can be neglected. Then there exists an integer n such that

$$X(\varepsilon_1^n) = X(\varepsilon_2^n) = \cdots = X(\varepsilon_r^n) = 1,$$

which means

$$X(\varepsilon_i) = \prod_{\iota=r_1+1}^{r_1+r_2} \left(\frac{\varepsilon_i^{\sigma_\iota}}{|\varepsilon_i^{\sigma_\iota}|} \right)^{f_\iota} \prod_{\lambda=1}^{r_1+r_2} |\varepsilon_i^{\sigma_\lambda}|^{\sqrt{-1}\varphi_\lambda}$$

is a root of unity ($i = 1,\dots,r$). Put $\varepsilon_i^{\sigma_\iota}/|\varepsilon_i^{\sigma_\iota}| = e^{\sqrt{-1}\theta_{i\iota}}$ ($\iota = r_1 + 1,\dots,r_2$; $i = 1,\dots,r$), then

$$\mathrm{X}(\varepsilon_i) = \prod_{\iota=r_1+1}^{r_1+r_2} e^{\sqrt{-1}\theta_{i\iota}f_\iota} \prod_{\lambda=1}^{r_1+r_2} e^{\sqrt{-1}\varphi_\lambda \log|\varepsilon_i^{\sigma_\lambda}|}$$
$$= e^{\sqrt{-1}(\Sigma_\iota \theta_{i\iota} f_\iota + \Sigma_\lambda \varphi_\lambda \log|\varepsilon_i^{\sigma_\lambda}|)},$$

so we have

$$\sum_{\iota=r_1+1}^{r_1+r_2} \theta_{i\iota} f_\iota + \sum_{\lambda=1}^{r_1+r_2} \varphi_\lambda \log|\varepsilon_i^{\sigma_\lambda}| \in \pi Q, \qquad (i = 1,\dots,r).$$

Further we put $\beta_{i\iota} = \theta_{i\iota}/\pi$ and $\alpha_{i\lambda} = (\log|\varepsilon_i^{\sigma_\lambda}|)/\pi$ ($i = 1,\dots,r$; $\iota = r_1 + 1,\dots,r_1 + r_2$; $\lambda = 1,\dots,r_1 + r_2$), then we obtain

$$(\#) \qquad \sum_{\iota=r_1+1}^{r_1+r_2} \beta_{i\iota} f_\iota + \sum_{\lambda=1}^{r_1+r_2} \alpha_{i\lambda}\varphi_\lambda \in Q \qquad (i = 1,\dots,r).$$

The converse is immediate. Hence, for any $(f_\iota, \varphi_\lambda) \in Z^{r_2} \times R^{r_1+r_2}$, $(f_\iota, \varphi_\lambda)$ belongs to the group M if and only if $(f_\iota, \varphi_\lambda)$ satisfies the condition (#).

As $\{\varepsilon_1,\dots,\varepsilon_r\}$ is a system of fundamental units, we should notice that the following hold:

(i) $\alpha_{i1} + \cdots + \alpha_{ir_1} + 2\alpha_{ir_1+1} + \cdots + 2\alpha_{ir_1+r_2} = 0$ ($i = 1,\dots,r$),

(ii) $\det(\alpha_{ij})_{1\le i,j\le r} \ne 0$.

Now we shall show that M is isomorphic to the additive group $Z^{r_2} \times R \times Q^r$ as abstract groups. There uniquely exists $(x_1,\dots,x_r) \in R^r$ satisfying $\beta_{ir_1+1} + \alpha_{i1}x_1 + \cdots + \alpha_{ir}x_r = 0$ ($i = 1,\dots,r$) from (ii). Put

$$N_1 = \left\{ \left(\underbrace{f, 0,\dots,0}_{r_2}; \underbrace{fx_1,\dots,fx_r, 0}_{r_1+r_2} \right) \,\middle|\, f \in Z \right\}$$

and

$$M_1 = \left\{ \left(\underbrace{0, f_{r_1+2},\dots,f_{r_1+r_2}}_{r_2}; \underbrace{\varphi_1,\dots,\varphi_{r_1+r_2}}_{r_1+r_2} \right) \in Z^{r_2} \times R^{r_1+r_2}, \quad \text{satisfying } (\#) \right\},$$

then N_1 and M_1 are subgroups of M and N_1 is isomorphic to Z. We have immediately a direct decomposition $M = N_1 \times M_1 \cong Z \times M_1$, for any element of M $(f_{r_1+1},\dots,f_{r_1+r_2}; \varphi_1,\dots,\varphi_{r_1+r_2})$ is written as $(f_{r_1+1},$

$0,\ldots,0;\ f_{r_1+1}x_1,\ldots,f_rx_r,0) + (0,\ f_{r_1+2},\ldots,f_{r_1+r_2};\ \varphi_1 - f_{r_1+1}x_1,\ldots,\varphi_r - f_rx_r,\varphi_{r_1+r_2}) \in N_1 \times M_1$. Let $(y_1,\ldots,y_r)$ be the element of R^r satisfying $\beta_{ir_1+2} + \alpha_{i1}y_1 + \cdots + \alpha_{ir}y_r = 0\ (i = 1,\ldots,r)$. Put

$$N_2 = \left\{\left(\underbrace{0,f,0,\ldots,0}_{r_2};\underbrace{fy_1,\ldots,fy_r,0}_{r_1+r_2}\right) \mid f \in Z\right\},$$

$$M_2 = \left\{\left(\underbrace{0,0,f_{r_1+3},\ldots,f_{r_1+r_2}}_{r_2};\underbrace{\varphi_1,\ldots,\varphi_{r_1+r_2}}_{r_1+r_2}\right) \in Z^{r_2} \times R^{r_1+r_2},\right.$$

$$\left.\text{satisfying } (\#)\right\},$$

then we have immediately that $M_1 = N_2 \times M_2 \cong Z \times M_2$ in the same way, which implies $M \cong Z^2 \times M_2$. By induction we obtain that $M \cong Z^{r_2} \times M_{r_2}$, where

$$M_{r_2} = \left\{\left(\underbrace{0,\ldots,0}_{r_2};\varphi_1,\ldots,\varphi_{r_1+r_2}\right) \mid \alpha_{i1}\varphi_1 + \cdots + \alpha_{ir_1+r_2}\varphi_{r_1+r_2} \in Q\right.$$

$$\left.(i = 1,\ldots,r)\right\}.$$

Put

$$N' = \left\{\left(\underbrace{0,\ldots,0}_{r_2};\underbrace{\varphi/2,\ldots,\varphi/2}_{r_1},\underbrace{\varphi,\ldots,\varphi}_{r_2}\right) \mid \varphi \in R\right\},$$

then, from (i), N' is a subgroup of M_{r_2}; isomorphic to R. And put

$$N'' = \left\{\left(\underbrace{0,\ldots,0}_{r_2};\underbrace{\varphi_1,\ldots,\varphi_r,0}_{r_1+r_2}\right) \mid \alpha_{i1}\varphi_1 + \cdots + \alpha_{ir}\varphi_r \in Q\ (i = 1,\ldots,r)\right\},$$

then N'' is a subgroup of M_{r_2}, and it is clear that the map $(0,\ldots,0;\ \varphi_1,\ldots,\varphi_r,0) \to (b_1,\ldots,b_r)$, satisfying

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix},$$

is an isomorphism of N'' onto Q^r from (ii): $N'' \cong Q^r$. It immediately follows that

$$M_{r_2} = N' \times N'' \cong R \times Q^r.$$

Thus we obtain the isomorphism as abstract groups

$$M \cong Z^{r_2} \times R \times Q^r.$$

4. Identifying M with $Z^{r_2} \times R \times Q^r$ through the algebraic isomorphism, the map $\mathcal{H}$: $C_k^* \to M$, defined in 2, is as follows:

$$C_k^* \to Z^{r_2} \times R \times Q^r$$

$$(*) \qquad \chi \,(\text{of type } (f_\lambda, \varphi_\lambda)) \to \Big(\underbrace{*, \ldots, *}_{r_2};\ \varphi_{r_1+r_2},\ \underbrace{*, \ldots, *}_{r} \Big).$$

We shall agree that R has the usual topology and that Z and Q have discrete topology. If we show the map $\mathcal{H}$ is open and continuous, we have the isomorphism as topological groups $C_k^*/T_k = Z^{r_2} \times R \times Q^r$ because the kernel of $\mathcal{H}$ is T_k as mentioned in 2. By Lemma 1 $C_k^*/T_k \cong D_k^*$, which shows $D_k^* \cong Z^{r_2} \times R \times Q^r$, and hence we have from the duality theorem

$$D_k \cong T^{r_2} \times R \times S^r.$$

We shall show that the map $\mathcal{H}$ is open and continuous. Let C_k^0 be the set of all idèle classes with volume 1, and A the set of all idèle classes of (a_v) which has component $a_{\lambda_{r_1+r_2}}$ at the infinite prime $\lambda_{r_1+r_2}$ and all other components equal to 1. As is well-known it holds that $C_k = A \times C_k^0$. We denote by A^* and $(C_k^0)^*$ the group of all characters of C_k which is trivial on C_k^0 and A, respectively, and then we have $C_k^* = A^* \times (C_k^0)^*$.

Now for any real number φ and any ideal $\mathfrak{a}$ of k, we define

$$\psi(\mathfrak{a}) = \operatorname{Norm}(\mathfrak{a})^{-\sqrt{-1}\varphi/2},$$

then ψ is a Grössencharakter mod 1. For any principal ideal (γ), $\gamma \in k^\times$, we can see

$$\psi((\gamma)) = \prod_{\lambda=1}^{r_1} |\gamma^{\sigma_\lambda}|^{-\sqrt{-1}\varphi/2} \prod_{\lambda=r_1+1}^{r_1+r_2} |\gamma^{\sigma_\lambda}|^{-\sqrt{-1}\varphi},$$

so ψ is of type

$$\Big(\underbrace{0, \ldots, 0}_{r_2};\ \underbrace{-\varphi/2, \ldots, -\varphi/2}_{r_1},\ \underbrace{-\varphi, \ldots, -\varphi}_{r_2} \Big).$$

We denote by χ_ψ the character of C_k associated with ψ, i.e., for any idèle $a = (a_v)$

$$\chi_\psi(a) = \psi(\operatorname{id}(a)) \prod_{\lambda=1}^{r_1} |a_\lambda|^{\sqrt{-1}\varphi/2} \prod_{\lambda=r_1+1}^{r_1+r_2} |a_\lambda|^{\sqrt{-1}\varphi},$$

it is of type

$$\Big(\underbrace{0,\dots,0}_{r_2};\underbrace{\varphi/2,\dots,\varphi/2}_{r_1},\underbrace{\varphi,\dots,\varphi}_{r_2}\Big).$$

Each idèle $a = (a_v)$ determines in an obvious manner an ideal of k, and so denote it by $\mathrm{id}(a)$.

Let Y be the set of all such characters χ_ψ: $Y = \{\chi_\psi \mid \varphi \in R\}$. As is well-known, a character χ_ψ defined as mentioned above is trivial on C_k^0, and the converse is also valid, that is $Y = A^*$. For any character $\chi_\psi \in Y$ of type $(0,\dots,0;\ \varphi/2,\dots,\varphi/2,\ \varphi,\dots,\varphi)$ and any idèle $a = (a_v) \in A$, we can see

$$\chi_\psi(a) = \begin{cases} a_{\lambda_{r_1+r_2}}^{\sqrt{-1}\varphi/2} & \big(\text{if } \lambda_{r_1+r_2} \text{ is a real prime, i.e., } r_2 = 0\big), \\ a_{\lambda_{r_1+r_2}}^{\sqrt{-1}\varphi} & \big(\text{if } \lambda_{r_1+r_2} \text{ is a complex prime, i.e., } r_2 \neq 0\big). \end{cases}$$

A with the relative topology of C_k is isomorphic to $R_+^\times$ (the multiplicative group of positive real numbers). Therefore the map $\chi_\psi \to \varphi$ is an isomorphism of $A^* = Y$ with the relative topology of C_k^* to the additive group R as topological groups: $Y \cong R$.

We restrict the map $\mathcal{H}$ on Y, then we have (cf. (*))

$$Y \to Z^{r_2} \times R \times Q^r$$
$$\chi_\psi \to (\mathbf{0}, \varphi, \mathbf{0}).$$

It is open and continuous because $\chi_\psi \to \varphi$ is a topological isomorphism as mentioned above. Since $(C_k^0)^*$, Z and Q are discrete, the map

$$\mathcal{H}\colon C_k^* = Y \times (C_k^0)^* \to Z^{r_2} \times R \times Q^r$$

is clearly open and continuous. This completes the proof.

References

1. E. Artin, *Representatives of the connected component of idèle class group*, Proceedings of International Symposium on Algebraic Number Theory at Tokyo-Nikko (1955), 51–54.
2. A. Weil, *On a certain type of characters of the idèle-class group of an algebraic number field*, ibid., 1–7.
3. A. Weil, *Sur la thèorie du corps de classes*, J. Math. Soc. Japan, **3** (1951), 1–35.

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