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THE ABEL-JACOBI ISOMORPHISM FOR THE CUBIC FIVEFOLD

ALBERTO COLLINO

Let X be a general cubic fivefold, JX the associated intermediate Jacobian, F the Fano surface of the planes contained in X. We prove that the Abel-Jacobi map induces an isomorphism from the Albanese variety of F to JX.

Introduction. It is a standard fact (see [7] Exp. XI (2.9)) that the only smooth hypersurfaces X in $\mathbf{P}^{2d+1}$, $d > 0$, for which the intermediate Jacobian JX is an abelian variety are the quadrics, the cubic and the quartic threefolds in $\mathbf{P}^4$, and the cubic fivefold in $\mathbf{P}^6$. For a quadric $JX = 0$. In [**5**], Clemens and Griffiths proved that the Abel-Jacobi map

$$(+) \qquad a\colon \operatorname{Alb} F \to JX$$

is an isomorphism, where X is the smooth cubic threefold and F is the (smooth) Fano surface of the lines on X. Recently Letizia, [**9**], using a method which he credits Clemens for, [**4**], proved that $(+)$ is an isomorphism also when X is a general smooth quartic threefold and F is the Fano surface of the conics on X.

Here we complete the picture, proving that $(+)$ is an isomorphism also when X is a general smooth cubic fivefold, F being the surface of the planes on X. Our tool is the Clemens-Letizia method coupled with some ideal which originated from [**6**].

We work with varieties defined over the complex numbers field.

(I). Let T be a plane in $\mathbf{P}^6$ and let X be a cubic hypersurface containing it. We choose projective coordinates $(x_0 : x_1 : x_2 : \cdots : x_6)$ in $\mathbf{P}^6$ so that T has equations $x_0 = x_1 = x_2 = x_3 = 0$. The equation of X is then of the form

$$(1.1) \qquad 0 = Q_0x_0 + Q_1x_1 + Q_2x_2 + Q_3x_3 + \sum\left(A_{ijk}x_ix_jx_k\right) + B(x_0, x_1, x_2, x_3)$$

where $4 \le k \le 6$, $0 \le i \le j \le 3$, B is homogeneous cubic and Q_i, $i = 0, \ldots, 3$, are homogeneous polynomials of degree 2 in x_4, x_5, x_6.

Let C_i be the conic on T of equation $Q_i = 0$, X is non-singular along T if and only if $\bigcap C_i = \emptyset$. In the following we shall assume that X is smooth along T, when it is not explicitly otherwise stated. We shall denote $F(X)$, or simply F, the variety of planes contained in X; more precisely we take F to be the Hilbert scheme of the two-dimensional planes of X, [**1**]. In order to study F we need to compute $H^0(T, N(T, X))$, which is the tangent space to F at the point t representing T. When there is no confusion we shall write N for the normal bundle $N(T, X)$ and $h^0(N) = \dim H^0(T, N)$.

(1.2) PROPOSITION. $h^0(N) = 2 \leftrightarrow C_0, C_1, C_2, C_3$, *are linearly independent.*

Proof. We start from the exact sequence of sheaves on T

$$0 \to N \to O(1)^{\oplus 4} \xrightarrow{f} O(3) \to 0, \tag{1.3}$$

where the middle sheaf is $N(T, \mathbf{P}^6)$ and $O(3)$ is $N(X, \mathbf{P}^6)|_T$. If one chooses conveniently the splitting of $N(T, \mathbf{P}^6)$ then f is given by the matrix (Q_0, Q_1, Q_2, Q_3), hence: (a) $h^0(N) = 2 \leftrightarrow$ (b) $f: H^0(T, O(1)^{\oplus 4}) \to H^0(T, O(3))$ is surjective $\leftrightarrow$ (c) given any homogeneous cubic polynomial $K(x_4, x_5, x_6)$ there are linear homogeneous polynomials L_0, L_1, L_2, L_3 such that $(+)$: $K = \sum L_i Q_i$.

If the C_i are linearly dependent then f cannot be surjective on global sections, because the space of polynomials K in $(+)$ has dimension 9 at most. We assume now that the C_i are linearly independent. Since the four conics have no common point, the general member of the web they span is non singular; without restriction we may assume that C_1 is smooth, so that $C_1 \cap C_2$ is finite and we may take C_3 in such a way that $C_1 \cap C_2 \cap C_3 = \emptyset$. In the ring $R = k[x_4, x_5, x_6]$ we let $I =$ the ideal (Q_1, Q_2, Q_3), $R_3 =$ the vector space of homogeneous cubic polynomials, $I_3 = R_3 \cap I$. By a theorem of Macaulay, [**8**], the pairing $g: (R/I) \times (R/I) \to (R_3/I_3) \simeq \mathbf{C}$ is a perfect duality, where g is given by the product of representative of equivalence classes. It follows:

(i) I_3 has codimension 1 in R_3, (ii) given Q_0, since $Q_0 \notin I$ by hypothesis, there is L_0 such that $g(L_0 Q_0) \neq 0$, i.e. $L_0 Q_0 \notin I_3$. Therefore $I_3 + (L_0 Q_0) = R_3$, hence f is surjective on global sections.

(1.4) COROLLARY. *F is a non-singular surface at the point t representing T if and only if $C_0, \ldots, C_3$ are linearly independent.*

Proof. $h^1(N) = 0 \leftrightarrow h^0(N) = 2 \leftrightarrow (C_0, \ldots, C_3)$ are linearly independent.

We recall that in the preceding corollary we had the tacit assumption that X was smooth along T. For next definition we drop it.

(1.5) DEFINITION. Let T be a plane contained in X; using the notations above we say that T is a *special plane* for X, or that X is *special* with respect to T, if $C_0, \ldots, C_3$ are linearly dependent.

If X is non-singular along T then F is singular at t if and only if T is special; if X has an ordinary node on T we shall see that F is not normal at t, but we shall also see that if T is not special then on the normalization F^+ of F the points t_1 and t_2, which map to t, are non-singular.

Let H be the Hilbert scheme parametrizing the totality of cubic hypersurfaces of $\mathbf{P}^6$, H is naturally isomorphic to $\mathbf{P}^{83}$. We let $H^s =$ the subvariety of H of the cubics with special planes, $D =$ the subvariety of the singular cubics.

(1.6) LEMMA. (a) $\operatorname{cod}(H^s, H) \geq 1$, (b) *$H^s$ is irreducible and $H^s \not\subset D$.*

Proof. Let $H^T \simeq \mathbf{P}^{73}$ be the variety of the cubics containing T. Keeping the notations above (1.1) we note that X is special along T if there are $b_0, \ldots, b_3$ with $(+)$: $\sum b_i Q_i = 0$. Let $\mathbf{A}^{24}$ be the affine space in the variables a^k_{ij}, the coefficients of the Q_k's. Let $(b_0, \ldots, b_3)$ be homogeneous coordinates for $\mathbf{P}^3$. Condition $(+)$ gives six bilinear equations in $\mathbf{P}^3 \times \mathbf{A}^{24}$, let V be the determined variety. It is easy to see that $V \to \mathbf{P}^3$ is an $\mathbf{A}^{18}$ fibration, further the projection $V \to \operatorname{pr}(V) \hookrightarrow \mathbf{A}^{24}$ is birational to the image $\operatorname{pr}(V)$, therefore $\operatorname{pr}(V)$ is irreducible with $\dim = 21$. Let $H^{s,T}$ be the variety of the cubics which are special along T, because of our remark $\operatorname{cod}(H^{s,T}, H^T) = 3$ and $H^{s,T}$ is irreducible. Let $G = G(2,6)$ be the grassmannian of the planes in $\mathbf{P}^6$, then $H^s = \bigcup_{T \in G}(H^{s,T})$, so that H^s is irreducible and $\dim H^s \leq 82$. In order to see that $H^s \not\subset D$ it suffices now to produce one cubic which is non-singular and contains a special plane. The Fermat cubic $\sum x_i^3 = 0$ has this property: let r be a third root of -1, i.e. $r^3 = -1$, change coordinates $z_0 = x_0 - rx_5$, $z_1 = x_1 - rx_6$, $z_2 = x_2$, $z_3 = x_3 - rx_4$, $z_4 = x_4$, $z_5 = x_5$, $z_6 = x_y$, it is easy to see that the plane $z_0 = z_1 = z_2 = z_3$ is special for this cubic.

(1.7) in $G \times H$ we set $I = \{(T, X)\colon T \subseteq X\}$, I is the incidence correspondence; I is a $\mathbf{P}^{73}$ fibre bundle over G, so that dim $I = 85$. Let p: $I \to G$ and q: $I \to H$ be the projections, the fibre $q^{-1}(x) = F(X)$, the Hilbert scheme of the planes contained in X, the hypersurface represented by x. Fix now a point (T, X) in I such that T is non-special for X and X is smooth along T (it follows from the proof of the lemma that there is such a couple) then $F(X)$ is a smooth surface at the point t representing T. Then the general $F(X)$ is a surface, non-singular because it does not contain special planes, moreover it is irreducible because of

(1.8) PROPOSITION. *For all X, $F(X)$ is connected.*

Proof. Following an idea of Barth and Van de Ven [2] we need only to check that the set $S = \{(T, X)\colon F(X)$ is not a smooth surface at $T\}$ has codimension at least 2 in $p^{-1}(T)$. Now $F(X)$ is not a smooth surface at T only if either T is special for X or if X is singular at some points of T. The first case is covered by the proof of (1.6), indeed $p^{-1}(T) = H^T$ and we proved $\operatorname{cod}(H^{s,T}, H^T) = 3$. On the other hand let $D^T = \{X \in H^T\colon X$ is singular at some point of $T\}$, by a similar argument as for $H^{s,T}$ one can see $\operatorname{cod}(D^T, H^T) = 2$.

(1.9) REMARK. In $G \times H^s$ let $I^s = \{(T, X)\colon T$ is special for $X\}$; it follows from the proof of (1.6) that dim $I^s = 82$, hence either (a) dim $H^s = 82$, so that in a general cubic which is special there is a finite number of special planes, or (b) dim $H^s < 82$, so that given a general pencil of cubics none of the cubics is special.

Collecting the preceding results we see

(1.10) PROPOSITION. *Let $\{X_t\}$, $t \in \mathbf{P}^1$, be a general Lefschetz pencil of cubic fivefolds, let $\{F(X_t)\}$, $t \in \mathbf{P}^1$, be the associated family of Fano varieties, let $t_1, \ldots, t_N$ be such that X_t is smooth for $t \in \mathbf{P}^1 - \{t_1, \ldots, t_N\}$. Then there exists $t_{N+1}, \ldots, t_{N+M}$ in $\mathbf{P}^1$ such that:*

(1) *$F(X_t)$ is a smooth and irreducible surface for $t \in \mathbf{P}^1 - \{t_1, \ldots, t_N, t_{N+1}, \ldots, t_{N+m}\}$.*

(2) *The surface $F(X_{t_{N+J}})$ has only isolated singularities, $1 \leq J \leq M$.*

(3) *The surface $F(X_{t_J})$ has as only singularities the locus of the planes through the ordinary double point of X_{t_J}, $1 \leq J \leq N$.*

In order to complete the program according to the Clemens-Letizia method [**9**] we still need to check that the Abel-Jacobi map $F \to JX$ is not constant, and also to prove the following

(1.11) THEOREM. *Let F_0 be the Fano surface of the planes on X_0, general cubic hypersurface with one single singular point p_0 which is an ordinary node. Then the family D which represents the planes through p_0 is a smooth irreducible curve; $F_0 - D$ is non-singular; along D F_0 is analytically reducible in two smooth components meeting transversally.*

(II). This section is devoted to the proof of Theorem (1.11). We need some preliminary considerations.

Let b: $\mathbf{P}^+ \to \mathbf{P}^6$ be the blow up of $\mathbf{P}^6$ at p_0, let $E = b^{-1}(p_0)$ be the exceptional divisor, let X^+ be the strict transform of X_0, let $Q = X^+ \cap E$ be the exceptional quadric. As in [**5**] the linear projection $X_0 \dashrightarrow \mathbf{P}^5$ of centre p_0 induces a birational morphism λ: $X^+ \to \mathbf{P}^5$, which turns out to be the blow up of $\mathbf{P}^5$ along the threefold Y, the $(2, 3)$ complete intersection of Q with a cubic K. More precisely let $p_0 = (0,\dots,0,1)$, let $\mathbf{P}^5$ be the hyperplane $x_6 = 0$, then the cubic X_0 has equation

$$0 = Q(x_0, x_1,\dots,x_5)x_6 + K(x_0,\dots,x_5) \tag{2.1}$$

and in $\mathbf{P}^5$ the cubic and the quadric have equations $K = 0$ and $Q = 0$. The planes through p_0 are mapped via λ to the lines lying in Y and conversely; hence the family of planes through x_0 is in general a smooth irreducible curve, because such a curve is the family of the lines in a general Y, see [**3**].

(2.2) Let T be a plane in $\mathbf{P}^6$, if $p_0 \notin T$ then the total transform $b^{-1}(T)$ is isomorphic to T and it is also the proper transform T^b of T. If $p_0 \in T$ then the proper transform T^b of T is the blowing up of T at p_0. In this case T^b is not the correct transform with respect to the behaviour of the Hilbert scheme; in fact in both cases T^b is a complete intersection in $\mathbf{P}^+$, but of different type.

(2.3) DEFINITION. We say that a variety Z in $\mathbf{P}^+$ is a *strict biplane* if $Z = T^b \cup B$, where B is a plane in E, $B \cap T^b = L$, where L is the exceptional line in T^b, proper transform of a plane T through p_0.

It is easy to see that a strict-biplane Z is the same kind of complete intersection in $\mathbf{P}^+$ as it is a 'plane' T^b not meeting E. In the following we denote $G^+ =$ the Hilbert scheme of P^+ of the complete intersections of the same type as a strict biplane. If $a \in G^+$ we let Z_a be the represented scheme; we call Z_a a biplane and remark that there are only two possibilities: either Z_a is a strict biplane or $Z_a = T^b$, where T is a plane not containing p_0. One can see easily that G^+ is non-singular of dimension 12, by explicitly computing the dimension of the tangent space to G^+ at a point representing a given biplane.

There is a more intuitive way to describe G^+. Let S be the Schubert variety of the planes in $\mathbf{P}^6$ through p_0, then S is smooth, being isomorphic to $G(1,5)$. Let $\beta^*\colon G^* \to G$ be the blow up of $G = G(2,6)$ along S. We have $G^* = G^+$. In fact there is a correspondence $\lambda'\colon G \dashrightarrow G(2,5)$ obtained by sending a plane from $\mathbf{P}^6$ to $\mathbf{P}^5$ by means of the linear projection λ of centre p_0. The indeterminacy of λ' at S is solved by blowing up G, so to have $\lambda^*\colon G^* \to G(2,5)$. Via (β^*, λ^*) G^* embeds in $G \times G(2,5)$. Similarly there is a map $\beta^+\colon G^+ \to G$ and a map $\lambda^+\colon G^+ \to G(2,5)$, obtained by setting $\beta^+(a) =$ the point representing $b(Z_a)$, $\lambda^+(a) =$ the point representing $\lambda(Z_a)$. Also G^+ embeds in $G \times G(2,5)$ and it has the same image as G^* has. In this way we get a 1-1 correspondence between G^* and G^+; since they are both non-singular, then they are isomorphic.

We write $F^+ =$ the Fano scheme of the biplanes contained in X^+ and denote $\beta^+\colon F^+ \to F$ the map induced by restriction of $\beta^+\colon G^+ \to G$. Collecting previous remarks we note

(2.4) LEMMA. (1) $\beta^+\colon F^+ - \beta^{-1}(D) \to F - D$ *is* 1-1. (2) $(\beta^+)^{-1}(D) = D_1 \cup D_2$ *where* D_1 *and* D_2 *are isomorphic to* D *via* β^+, $D_1 \cap D_2 = \varnothing$.

Proof. (1) is clear. For (2) let $t \in D$, then $a \in (\beta^+)^{-1}(t)$ means $Z_a \hookrightarrow X$, $b(Z_a) = T_t$ contains p_0, $\lambda(Z_a)$ is a plane B_a contained in the quadric Q and passing through the exceptional line L in the proper transform T_t^b. In other words $Z_a = T_t^b + B_a$. Since Q is a four dimensional smooth quadric then for a fixed L in Q there are only two possible choices for B_a, one for each system of planes. Statement (2) follows easily.

Our next step is

(2.5) THEOREM. *Let* t *represent the plane* T, *if* X_0 *is not special at* T *then* F^+ *is non singular at the points of* $(\beta^+)^{-1}(t)$.

If X is smooth along T the theorem is Corollary (1.4). We assume therefore that $p_0 \in T$ and let $(\beta^+)^{-1}(t) = \{a, b\}$, $a \in D_1$, $b \in D_2$. We denote $N = N(Z_a, X)$ the dual of the conormal bundle of Z_a in X, our program is to prove $h^0(N) = 2$, hence F^+ is a smooth surface at the point a.

For simplicity we write $T_t^b = A$, so that $A \cup B = Z_a$ and further we set $Z = Z_a$. The standard exact sequences of "normal" bundles for the triple $(Z, X^+, \mathbf{P}^+)$ is

$$(2.6) \qquad 0 \to N(Z, X^+) \to N(Z, \mathbf{P}^+) \xrightarrow{f} N(X^+, \mathbf{P})|_Z \to 0.$$

(2.7) Let D_0, D_1, D_2, D_3 be four hypersurfaces in $\mathbf{P}^+$ which intersect completely in Z, then

$$N(Z, \mathbf{P}^+) = O(D_0)|_Z \oplus O(D_1)|_Z \oplus O(D_2)|_Z \oplus O(D_3)|_Z.$$

In the following (2.9) we fix such a splitting so that the restrictions of sequence (2.6) to A, B, L are respectively

$$\begin{array}{llllllllll} (s_A) & 0 & \to & N_A & \to & \oplus^3 O_A(H-L) \oplus O_A(H) & \xrightarrow{f_A} & O_A(3H-2L) & \to & 0 \\ (s_B) & 0 & \to & N_B & \to & \oplus^3 O_B(1) \oplus O_B & \xrightarrow{f_B} & O_B(2) & \to & 0 \\ (s_L) & 0 & \to & N_L & \to & \oplus^3 O_L(1) \oplus O_L & \xrightarrow{f_L} & O_L(2) & \to & 0. \end{array}$$

Here H is the divisor on A which is the total transform of the line in T, L is the exceptional line in A. N_A, N_B and N_L are the restrictions of N to A, B, L.

(2.8) There is a standard Mayer-Vietoris sequence

$$0 \to H^0(Z, N) \to H^0(A, N_A) \oplus H^0(B, N_B) \to H^0(L, N_L);$$

in order to prove $h^0(Z, N) = 2$ we shall prove: (i) $H^0(B, N_B) \xrightarrow{\sim} H^0(L, N_L)$, (ii) $h^0(A, N_A) = 2$. We need first to compute f_A, f_B, f_L.

(2.9) Looking at the sequences above we remark that f_A is a global section in $H^0(A, \oplus^3 O_A(2H - L) \oplus O_A(2H - 2L))$, f_B is a global section in $H^0(B, \oplus^3 O_B(1) \oplus O_B(2))$, and f_L is in $H^0(L, \oplus^3 O_L(1) \oplus O_L(2))$. To compute f_A, f_B, f_L means to identify them as sections of the indicated sheaves, in particular both f_A and f_B are determined if we find their restrictions to $A - L$ and $B - L$ respectively.

We let $(x_0, \ldots, x_6; y_0, \ldots, y_5)$ be the bihomogeneous coordinates of $\mathbf{P}^6 \times \mathbf{P}^5$, then $\mathbf{P}^+$ is the subvariety determined by the equations $x_i y_j = x_j y_i$, $0 \le i, j \le 5$. The biplane Z is the complete intersection in $\mathbf{P}^+$ of equations $y_0 = y_1 = y_2 = x_3 = 0$. We let $D_i = \text{locus}(y_i = 0)$ $i = 0,1,2$; $D_3 = \text{locus}(x_3 = 0)$, cf. (2.7). If f_{A-L} denotes the restriction of f_A to $A - L = T - p_0$, then the same proof of (1.2) gives

$$(2.10) \qquad f_{A-L} = (Q_0, Q_1, Q_2, Q_3)|_{A-L}.$$

Let C_i^- be the proper transform in A of the conic C_i and let $f_A = (f_A^0, f_A^1, f_A^2, f_A^3)$. We shall see below that one can choose the coordinates $x_0, \ldots, x_3$ so that both C_0 and C_3 have a double point in p_0, while C_1 and C_2 are smooth there. From this and (2.10) it follows

(2.11) PROPOSITION. *$(C_0^- + L)$ is the divisor of the zeros of f_A^0, C_1^- of f_A^1, C_2^- of f_A^2, C_3^- of f_A^3.*

With the notations above, the equations of the plane B in the exceptional divisor E of $\mathbf{P}^+$ are $y_0 = y_1 = y_2 = 0$, while the equations of T in $\mathbf{P}^6$ are as before $x_0 = x_1 = x_2 = x_3 = 0$. Since p_0 is a node the equations of the conics C_i are of type

$$(2.12) \qquad Q_i = L_i(x_4, x_5)x_6 + Q_i^0(x_4, x_5)$$

cf. (1.1). In E the exceptional quadric Q of X^+ is therefore

$$(2.13) \quad y_0L_0(y_4, y_5) + y_1L_1(y_4, y_5) + y_2L_2(y_4, y_5) + y_3L_3(y_4, y_5) + \sum(A_{ij6}y_iy_j) = 0, \qquad 0 \le i, j \le 3.$$

Since B is contained in Q one has

$$(2.14) \qquad A_{336} = 0, \qquad L_3(y_4, y_5) = 0$$

so that C_3 has a node in p_0. Next we use the hypothesis of the linear independence of the C_i to remark that, up to a linear change in x_0, x_1, x_2, one may assume that also C_0 has a node in p_0, i.e. $L_0(x_4, x_5) = 0$. Now we recall that the exceptional line L has equations $y_0 = y_1 = y_2 = y_3 = 0$ in E and that we have $Z_a = A + B$ with $B = B_a$ and also $Z_b = A + B_b$, where $A \cap B_b = L = A \cap B$. Without restriction we may require that B_b has equations $y_1 = y_2 = y_3 = 0$. The equation of Q in E is then

$$(2.15) \qquad y_1L_1(y_4, y_5) + y_2L_2(y_4, y_5) + \sum A_{ij6}y_iy_j, \quad \text{with } 0 \le i \le j \le 3, \text{ and } A_{006} = A_{336} = 0.$$

By hypothesis Q is of maximal rank, then by a linear change of coordinates we may assume $L_1(y_4, y_5) = y_4$, $L_2(y_4, y_5) = y_5$ and also note that $A_{036} \neq 0$.

(2.16) COROLLARY.

$$f_B = \left(A_{036}y_3,\, y_4 + A_{136}y_3,\, y_5 + A_{236}y_3,\, Q_3^0(y_4, y_5) + (\cdots)y_3\right)$$

(2.17) COROLLARY. $f_L = (0, y_4, y_5, Q_3^0(y_4, y_5))$.

Proof (2.16). We just outline the computation. It suffices to compute the restriction of f_B to the affine plane $B - L$ (i.e. the locus $y_3 \neq 0$), so we restrict everything to the affine space V which is the locus in $\mathbf{P}^+$ where $y_3 \neq 0$, $x_6 \neq 0$. There $B - L$ is the complete intersection $y_0 = y_1 = y_2 = x_3 = 0$. Now in V $y_0, y_1, y_2, x_3, y_4, y_5$ induce natural linear parameters which we write $y_0^0, y_1^0, y_2^0, x_3^0, y_4^0, y_5^0$. The equation of the restriction of X^+ to V is then of the form $M_0(\,)y_0^0 + M_1(\,)y_1^0 + M_2(\,)y_2^0 + M_3(\,)x_3^0 = 0$. It follows that the restriction of f_B to $B - L$ is equal to the restriction of (M_0, M_1, M_2, M_3). An explicit computation of the M_i's yields the result.

(2.18) COROLLARY. $h^0(B, N_B) = h^0(L, N_L) = 4$.

Proof. Obvious since $A_{036} \neq 0$.

Let $g\colon H^0(B, N_B) \to H^0(L, N_L)$ be the restriction map; using the short exact sequences of global sections associated with s_B and s_L and the snake lemma one finds

$$\operatorname{Ker}(g) = \operatorname{Ker}\left(h\colon \oplus^3 H^0(B,0) \oplus H^0(B,0(-1)) \to H^0(B,0(1))\right)$$

where the matrix of h is just the matrix of f_B. So h is surjective and g and h are both isomorphisms.

(2.19) COROLLARY. $g\colon H^0(B, N_B) \xrightarrow{\sim} H^0(L, N_L)$.

We have proved part (i) in (2.8); next we show

(2.20) PROPOSITION. *If Q_0, Q_1, Q_2, Q_3 are linearly independent then $h^0(A, N_A) = 2$.*

Proof. Looking at the long exact sequence of cohomology associated with s_A we see $h^0(A, N_A) = 2 \leftrightarrow h^1(A, N_A) = 0 \leftrightarrow f_A$ is surjective on global sections. Let $P(x_4, x_5, x_6) = W(x_4, x_5)x_6 + V(x_4, x_5)$ be a cubic polynomial, to prove that f_A is surjective amounts to produce $L_0(x_4, x_5)$,

$L_1(x_4, x_5)$, $L_2(x_4, x_5)$, $L_3(x_4, x_5, x_6)$ linear polynomials in the indicated variables so that $\sum L_i Q_i = P$ where Q_i are the quadrics in (2.12). For later use we note that we shall in fact produce $L_3(x_4, x_5)$.

Using the simplifications established above we have $Q_0 = AB$, $Q_1 = x_4 x_6 + FG$, $Q_2 = x_5 x_6 + DE$, $Q_3 = CH$ with $A, B, \ldots, H$ linear homogeneous polynomials in x_4, x_5. Using the hypothesis of the linear independence of $Q_0, \ldots, Q_3$ we can assume $FG = 0$, up to a linear change of coordinates x_4, x_5. We notice first that there are L_1 and L_2 such that $L_1 x_4 + L_2 x_5 = W(x_4, x_5)$ and that also $L_1' = L_1 - \alpha x_5$, $L_2' = L_2 + \alpha x_4$ satisfy the equation. So we need to find $L_0(x_4, x_5)$, $L_3(x_4, x_5)$ and a constant α such that

$$(+) \qquad L_0 AB + (L_2 + \alpha x_4) DE + L_3 CH = V(x_4, x_5).$$

Equivalently for any $U(x_4, x_5)$ we look for L_0, L_3, α such that

$$(++) \qquad L_0 AB + \alpha x_4 DE + L_3 CH = U.$$

in other words we want to show that the dimension of the vector space of polynomials in $(++)$ is 4. If AB and CH have no common factor a solution for $(++)$ exists with $\alpha = 0$, because of the theorem of Macaulay. If AB and CH have a common factor then it can be only a linear factor, because $AB = Q_0$ and $CH = Q_3$ are linearly independent by hypothesis. We assume then that $B = H$ and also that A and C are not proportional. In this case the linear system $\{(L_0 A + L_3 C)B\}$ has dimension 3 and the system $\{L_0 AB + L_3 CH + \alpha x_4 DE\}$ has dimension 4 if $x_4 DE \notin \{(L_0 A + L_3 C)B\}$; we need therefore to exclude that either (i) x_4 or (ii) D or (iii) E is proportional to B. In case (iii) or (ii) the point $x_6 = B = 0$ on T is a point in the intersection of the conics C_i, hence it is a second singular point on X_0, which is a contradiction. In case (i) similarly the set $x_4 = Q_2 = 0$ contains another singular point on X_0, again a contradiction.

(2.21) In order the complete the proof of (1.11) we show below that the differential of β^+ at the point a is injective and next that if a and b are the two points in the fibre $(\beta^+)^{-1}(t)$ then

$$\dim\left(d\beta^+(T_a(F^+)) \cap d\beta^+(T_b(F^+))\right) = 1.$$

PROPOSITION. *$d\beta_a^+ : T_a(F^+) \to T_{\beta^+(a)}(G)$ is injective.*

Proof. Recall $T_a(F^+) = H^0(Z_a, N) = H^0(A, N_a)$. From the long sequence of cohomology of the sequence (s_A) we get the upper exact row in the following diagram.

(D_a):

$$\begin{array}{c} 0 \to T_a(F^+) \overset{i_a}{\to} H^0(A, O(H-E) \oplus O(H-E) \oplus O(H-E) \oplus O(H)) \overset{f_A}{\to} H^0(A, O(3H-2E)) \to 0 \\ \downarrow j_a \\ H^0(T, O(H) \oplus O(H) \oplus O(H) \oplus O(H)) \to H^0(T, O(3H)) \to 0 \\ \| \\ T_t(G) \end{array}$$

Now $d\beta_a^+ = j_a i_a$, hence $d\beta_a^+$ is injective.

In order to compute the intersection of $T_a(F^+)$ and $T_b(F^+)$ in $T_t(G)$ one has to recall that the given splitting of $T_t(G)$ depends on the ordered choice of x_0, x_1, x_2, x_3, the equations of the plane T. If in the analogous diagram (D_b) we want to give the map j_b by means of the natural inclusion of the summands then the diagram is

(D_b):

$$\begin{array}{c} 0 \to T_b(F^+) \overset{i_b}{\to} H^0(A, O(H) \oplus O(H-E) \oplus O(H-E) \oplus O(H-E)) \to H^0(A, O(3H-2E)) \to 0 \\ \downarrow j_b \\ H^0(T, O(H) \oplus O(H) \oplus O(H) \oplus O(H)) \to H^0(T, O(3H)) \to 0. \end{array}$$

Next we note that the foldlowing sequence is exact.

$$0 \to T_a(F^+) \cap T_b(F^+) \overset{i}{\to} H^0\big(A, O(H-E)^{\oplus 4}\big)$$

$$\overset{(C_0^-, C_1^-, C_2^-, C_3^-)}{\to} H^0(A, O(3H-2E)) \to 0.$$

The exactness follows from the proof of (2.20) and more precisely from the remark that one can produce a $L_3(x_4, x_5)$. The statement about the dimension of the intersection is then obvious. We remark that on the other hand if C_0, C_1, C_2, C_3 were not linearly independent then the above sequence could not be exact to the right, for simple reasons of rank, so that the analytical branches of F would not be transversal.

(III). In this section we complete the Clemens-Letizia program by proving that the Abel-Jacobi map is not constant on the Fano surface. Let X be a smooth cubic fivefold, we show that the Abel-Jacobi map a: $F \to JX$ is an immersion at a point t which represents a plane T if T is non-special in the sense of (1.5). Our method follows **[10]** p. 24.

The cotangent space of F at t is

$$\big(H^0(T, N(T, X))\big)^* \simeq H^2\big(T, N^*(-3)\big);$$

the cotangent space to JX is $H^2(X, \Omega_X^3)$; the codifferential a^* turns out to be the map k in the following commutative diagram, which we explain in a moment:

$$\begin{array}{ccc} H^0(X, O_X(6) \otimes K_X) & \xrightarrow{f} & H^2(X, \Omega^3_X) \\ \downarrow h & & \downarrow k \\ H^0(T, O_T(2)) & \xrightarrow{g} & H^2(T, N^*(-3)). \end{array}$$

We shall show that h and g are surjective, hence k is also surjective so that a is an immersion at t.

The top row is obtained as follows. Start from

$$0 \to O(-3)_X \to \Omega^1_{\mathbf{P}^6|X} \to \Omega^1_X \to 0 \tag{3.1}$$

take then Λ^5 and tensor with $O(3)$ so to have

$$0 \to \Omega^4_X \to \Omega^5_{\mathbf{P}^6|X}(3) \to \Omega^5_X(3) \to 0. \tag{3.2}$$

Taking instead Λ^4 one has

$$0 \to \Omega^3_X(-3) \to \Omega^4_{\mathbf{P}^6|X} \to \Omega^4_X \to 0. \tag{3.3}$$

Putting (3.2) and (3.3) together and tensoring with $O(3)$

$$0 \to \Omega^3_X \to \Omega^4_{\mathbf{P}^6|X}(3) \to \Omega^5_{\mathbf{P}^6|X}(6) \to \Omega^5_X(6) \to 0. \tag{3.4}$$

The top row comes from the (hyper)cohomology sequence associated with (3.4).

To find the bottom row in the diagram one starts from the usual sequence of conormal bundles for $(T, X, \mathbf{P}^6)$:

$$0 \to O(-3)_T \to O_T(-1)^{\oplus 4} \to N^* \to 0 \tag{3.5}$$

taking Λ^3 and tensoring with $O(3)$ it follows

$$0 \to \Lambda^2 N^* \to O_T^{\oplus 4} \to \Lambda^3 N^* \otimes O(3) \to 0. \tag{3.6}$$

Since $c_1(N^*) = \Lambda^3 N^*$, then $\Lambda^3 N^* \otimes O_T(3) = O_T(2)$.

Taking Λ^2 instead one has

$$0 \to N^*(-3) \to O_T(-2)^{\oplus 6} \to \Lambda^2 N^* \to 0. \tag{3.7}$$

Putting the sequences together we have the exact sequence

$$0 \to N^*(-3) \to O_T(-2)^{\oplus 6} \to O_T^{\oplus 4} \to O_T(2) \to 0. \tag{3.8}$$

The bottom map g in the diagram is obtained by looking at the associated (hyper)cohomology sequence. To check the commutativity of the diagram is now a standard exercise, the main point is to provide a map from the restriction of (3.4) to T to sequence (3.8). We leave the details to the reader, everything is based on the commutativity of

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & O_T(-3) & \to & O_T(-3) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_T(-1)^{\oplus 4} & \to & \Omega^1_{\mathbf{P}^6|T} & \to & \Omega^1_T & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N^* & \to & \Omega^1_{X|T} & \to & \Omega^1_T & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The map h in the diagram is obviously surjective, being the restriction map. The surjectivity of g is a simple consequence of the vanishing $h^2(O_T(-2)) = 0 = h^1(O_T)$.

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UNIVERSITA DI TORINO
VIA PRINCIPE AMEDO 8
10123 TORINO, ITALY