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GENERALIZED HORSESHOE MAPS AND INVERSE LIMITS

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The now-classical example due to Smale, the horseshoe map, displays interesting dynamics as well as a topologically complicated attractor. In 1986 Marcy Barge showed that the full attracting sets of horseshoe maps are homeomorphic to inverse limits of the unit interval with a single bonding map. Here we extend Barge's results to a more general class of maps.

1. Introduction. In [**Ba**], Barge describes the attracting sets of horseshoe maps as inverse limits of the unit interval with a single bonding map. Topologically these spaces are chainable continua known as Knaster continua.

In this paper we consider a more general class of maps which we will refer to as generalized horseshoe maps. We will show that the attractors of these maps are homeomorphic to inverse limits of the unit interval with a single bonding map. Both the generalized horseshoe map and the bonding map which defines the inverse limit space described above "follow a pattern" in a sense we will define in the next section. In §3 we will prove two theorems about inverse limit spaces which will be needed in the proof of the main result given in §4. In the final section of the paper we will give some examples, and show that the horseshoe maps which Barge studied in [**Ba**] are special cases of the generalized horseshoes we consider here. For basic information on attractors and inverse limits see [**S**].

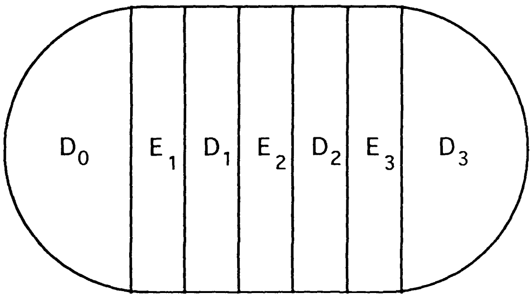
2. Preliminaries. Let I denote the unit interval and $\{f_n\}_{n=1}^{\infty}$ be a sequence of maps of I into I. Let

$$(I, f_n) = \{(x_0, x_1, \dots) : x_n \in I \text{ and } f_n(x_{n+1}) = x_n, n = 1, 2, \dots\}$$

be the inverse limit space with bonding maps f_n and topology induced by the metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

m=3



D

FIGURE 1

For $n = 0, 1, \ldots$, let $\pi_n: (I, f_n) \to I$ be defined by $\pi_n((x_0, x_1, \ldots)) = x_n$. It is often the case that we wish to consider inverse limit spaces with a single bonding map f, i.e., $f_n = f$ for $n = 1, 2, \ldots$. Let (I, f) denote such an inverse limit space.

Next, let $Q: \{0, 1, 2, \ldots, m\} \to \{0, 1, 2, \ldots, m\}$ be a function such that $Q(j) \neq Q(j+1)$, $0 \leq j \leq m-1$, and $\{0, m\} \subset \operatorname{range} Q$. We will use the notation $Q = (Q(0), Q(1), \ldots, Q(m))$ to denote the map $Q: \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\}$. For example, $Q = (0, 2, 1)$ denotes the map $Q: \{0, 1, 2\} \to \{0, 1, 2\}$ where $Q(0) = 0$, $Q(1) = 2$, and $Q(2) = 1$. Let D denote $(I \times I) \cup D_0 \cup D_m$ where D_0 and D_m are half disks attached to the opposites sides $\{0\} \times I$ and $\{1\} \times I$ respectively. Subdivide $I \times I$ as follows: For $1 \leq j \leq m-1$, let $D_j = [\frac{4j-1}{4m}, \frac{4j+1}{4m}] \times I$. Let $E_1 = [0, \frac{3}{4m}] \times I$, $E_m = [1 - \frac{3}{4m}, 1] \times I$, and for $2 \leq j \leq m-1$, let $E_j = [\frac{4j-3}{4m}, \frac{4j-1}{4m}] \times I$. The space D is pictured in Figure 1.

Let $\pi: D \to I$ be defined by $\pi(D_0) = 0$, $\pi(D_m) = 1$, and $\pi|_{I \times I}$ be projection onto the first coordinate. Define $p: I \to I$ as follows: $p(0) = 0$, $p(1) = 1$, $p(\pi(D_j)) = \frac{j}{m}$, $1 \leq j \leq m-1$, and p is linear on $\bigcup_{j=1}^{m} \pi(E_j)$. Let $P = p \circ \pi$. Note that $P: D \to I$ and $P(D_j) = \frac{j}{m}$ for $0 \leq j \leq m$.

We say that a map $F_Q: D \to D$ *follows* Q if F_Q is a homeomorphism of D into D which satisfies the following conditions (see Figure 2):

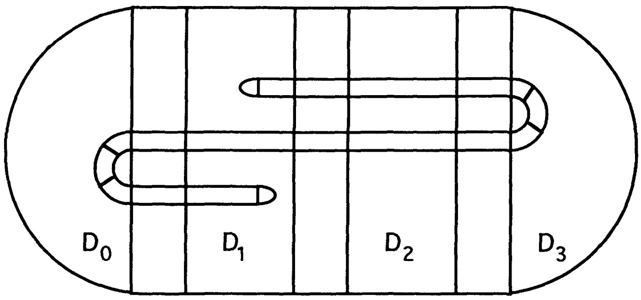
(i) $F_Q(P^{-1}(P(z))) \subset P^{-1}(P(F_Q(z)))$ for each $z \in D$,

(ii) $F_Q(D_j) \subset$ interior $D_{Q(j)}$, $0 \leq j \leq m$,

(iii) $\operatorname{diam} F_Q^k(P^{-1}(P(z)))) \to 0$ uniformly in z as $k \to \infty$.

If $F_Q: D \to D$ follows Q we say that F_Q is a Q-horseshoe map. Let Λ_Q denote the set $\bigcap_{k=0}^{\infty} F_Q^k(D)$ where F_Q is a Q-horseshoe map.

Q=(1,3,0,1)



$F_Q(D)$

FIGURE 2

We say that $g_Q\colon I \to I$ *follows* Q if $g_Q(\frac{j}{m}) = \frac{Q(j)}{m}$, $0 \le j \le m$, and g_Q is linear on $[\frac{j}{m}, \frac{j+1}{m}]$, $0 \le j \le m-1$. Let (I, g_Q) denote the inverse limit space of I with the single bonding map g_Q. The following theorem is our main result, and relates Λ_Q and (I, g_Q).

THEOREM 2.1. *Suppose $Q = (Q(0), Q(1), \dots, Q(m))$ is a function such that $Q(j) \neq Q(j+1)$, $0 \le j \le m-1$, and $\{0, m\} \subset \operatorname{range} Q$. If F_Q is a Q-horseshoe map, and $g_Q\colon I \to I$ follows Q, then Λ_Q is homeomorphic to (I, g_Q).*

We will prove this theorem in §4. To do so, two results about inverse limits of the interval are needed. These results constitute §3.

3. Inverse limits.

THEOREM 3.1. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences of surjective self-maps of $I = [0, 1]$. Suppose that $A = \{0 = a_0 < a_1 < \cdots < a_m = 1\}$ is a finite subset of I such that for each $n \in \mathbb{N}$, f_n and g_n are both strictly increasing or strictly decreasing on $[a_j, a_{j+1}]$, $f_n(a_j) = g_n(a_j)$, $0 \le j \le m$, and f_n and g_n are both invariant on A. Then (I, f_n) is homeomorphic to (I, g_n).*

Proof. Let I_j denote the interval $[a_j, a_{j+1}]$. We will show that if $(x_0, x_1, \dots) \in (I, f_n)$ then there exists a unique point $(y_0, y_1, \dots) \in (I, g_n)$ such that $y_0 = x_0$ and $x_n \in I_j$ if and only if $y_n \in I_j$. Then we can define $\phi\colon (I, f_n) \to (I, g_n)$ by setting $\phi((x_0, x_1, \dots))$ equal to the unique point of (I, g_n) described above. To complete the proof of Theorem 3.1 we will show that ϕ is one-to-one, onto, and continuous.

To define ϕ, let $(x_0, x_1, \dots)$ be an element of (I, f_n). We inductively define a nested sequence $\{Q_n\}_{n=0}^{\infty}$ of closed, nonempty subsets of (I, g_n) with the following properties: if $(y_0, y_1, \ldots) \in Q_n$, then $y_0 = x_0$, and $y_i \in I_j$ if and only if $x_i \in I_j$ for $0 \le i \le n$. Let $Q_0 = \pi_0^{-1}(x_0) \subset (I, g_n)$. Then Q_0 is closed and nonempty. Now suppose $Q_n \subset Q_{n-1} \subset \cdots \subset Q_0$ satisfy the above properties. Define Q_{n+1} as follows: let $(y_0, y_1, \ldots)$ be an element of Q_n, $I_{j(n)}$ an interval which contains y_n and x_n, and $I_{j(n+1)}$ an interval which contains x_{n+1}. Then $f_n(x_{n+1}) = x_n \in I_{j(n)}$ so that $f_n(I_{j(n+1)}) \cap I_{j(n)} \neq \varnothing$. Also, f_n is invariant on A, so $f_n(I_{j(n+1)}) = [a_{k_1}, a_{k_2}]$, where a_{k_1} and a_{k_2} are elements of A. Thus $I_{j(n)} \subset f_n(I_{j(n+1)})$ or $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$. If $I_{j(n)} \subset f_n(I_{j(n+1)}) = g_n(I_{j(n+1)})$ then $y_n \in I_{j(n)} \subset g_n(I_{j(n+1)})$, so there exists $y_{n+1} \in I_{j(n+1)}$ such that $g_n(y_{n+1}) = y_n$. In this case, set $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. If $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$, then $x_n = a_{j(n)}$ or $x_n = a_{j(n+1)}$; so $y_n = x_n$. Let $y_{n+1} = x_{n+1}$ and $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. Obviously Q_{n+1} is closed and nonempty and it is easy to check that $Q_{n+1} \subset Q_n$.

Since each Q_n is closed and nonempty, and the sets $Q_0, Q_1, \ldots$ are nested, it follows that there exists $(y_0, y_1, \ldots) \in \bigcap Q_n$. Suppose that $(y_0', y_1', \ldots)$ is another point of $\bigcap Q_n$. Let i be the first coordinate so that $y_i' \neq y_i$. Then $i > 0$ since $y_0' = y_0 = x_0$. Also, y_i and y_i' are both elements of some interval I_j and $g_{i-1}(y_i) = g_{i-1}(y_i') = y_{i-1}$. But this contradicts the fact that g_{i-1} is one-to-one on I_j. Therefore there is only one point in $\bigcap Q_n$. Thus we define $\phi\colon (I, f_n) \to (I, g_n)$ by setting $\phi((x_0, x_1, \ldots))$ equal to the unique point $(y_0, y_1, \ldots)$ in (I, g_n) such that $x_0 = y_0$ and $x_n \in I_j$ if and only if $y_n \in I_j$.

The same construction shows that given a point $(y_0, y_1, \ldots)$ in (I, g_n) we can find a unique point $(x_0, x_1, \ldots) \in (I, f_n)$ such that $x_0 = y_0$, and $x_n \in I_j$ if and only if $y_n \in I_j$. It follows that ϕ is one-to-one and onto. We now show that ϕ is continuous.

Let $(x_0, x_1, \ldots) \in (I, f_n)$, $(y_0, y_1, \ldots) = \phi((x_0, x_1, \ldots))$, and L be the minimum of the lengths of the intervals $[a_j, a_{j+1}]$. Given $\varepsilon > 0$, choose N so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. For each $n \in \mathbb{N}$ and j between 0 and m, $g_n^{-1}\colon g_n(I_j) \to I_j$ is a homeomorphism since $g_n|_{I_j}$ is one-to-one. From now on, let g_n^j denote $g_n^{-1}\colon g_n(I_j) \to I_j$. Note that if $y_{n+1} \in I_j$, then $y_{n+1} = g_n^j(y_n)$. Next, for each i, $0 \le i \le N+1$, define L_i as follows: if $x_i \in A$, let $L_i = L$. If $x_i \notin A$, then $x_i \in (a_j, a_{j+1})$. In this case, let $L_i = \min\{a_{j+1} - x_i, x_i - a_j\}$. Note that if $|x_i - x_i'| \le L_i$ then x_i and x_i' both lie in some I_j.

As we noted above, $g_{N-1}^j\colon g_{N-1}(I_j) \to I_j$ is a homeomorphism. Thus, for each j, $1 \le j \le m$, we may choose δ_{N-1}^j so that if y and

y' are elements of $g_{N-1}(I_j)$ with $|y - y'| < \delta^j_{N-1}$, then $|g^j_{N-1}(y) - g^j_{N-1}(y')| < \frac{\varepsilon}{2N}$. Let $\delta_{N-1} = \min\{\delta^1_{N-1}, \dots, \delta^m_{N-1}\}$. Similarly, for each j, $1 \le j \le m-1$, choose δ^j_{N-2} so that if y and y' are elements of $g_{N-2}(I_j)$ with $|y-y'| < \delta^j_{N-2}$, then $|g^j_{N-2}(y) - g^j_{N-2}(y')| < \min\{\frac{\varepsilon}{2N}, \delta_{N-1}\}$. Let $\delta_{N-2} = \min\{\delta^1_{N-2}, \dots, \delta^m_{N-2}\}$.

Continue in this way to choose $\delta^j_{N-(i+1)}$ so that if y and y' are elements of $g_{N-(i+1)}(I_j)$ with $|y-y'| < \delta^j_{N-(i+1)}$, then $|g^j_{N-(i+1)}(y) - g^j_{N-(i+1)}(y')| < \min\{\frac{\varepsilon}{2N}, \delta_{N-i}\}$. Let

$$\delta_{N-(i+1)} = \min\{\delta^1_{N-(i+1)}, \dots, \delta^m_{N-(i+1)}\}.$$

Thus we obtain $\delta_0, \delta_1, \dots, \delta_{N-1}$ such that if y and y' are elements of $g_i(I_j)$ with $|y-y'| < \delta_i$, then $|g^j_i(y) - g^j_i(y')| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$.

Finally let $\delta = \min\{\delta_0, L_0, L_1/2, \dots L_{N+1}/2^{N+1}, \varepsilon/2N\}$. Now suppose that $(x'_0, x'_1, \dots) \in (I, f_n)$ such that

$$d((x_0, x_1, \dots), (x'_0, x'_1, \dots)) < \delta.$$

Let $(y'_0, y'_1, \dots)$ denote $\phi((x'_0, x'_1, \dots))$. Since

$$d((x_0, x_1, \dots), (x'_0, x'_1, \dots)) < \delta \le L_i/2^i$$

for $0 \le i \le N+1$, it follows that $|x_i - x'_i| < L_i$. Therefore, there exists $I_{j(i)}$ such that x_i and x'_i are both elements of $I_{j(i)}$. This implies that y_i and y'_i are both elements of $I_{j(i)}$.

We now show inductively that $|y_i - y'_i| < \min\{\delta_i, \frac{\varepsilon}{2N}\}$ for $0 \le i \le N$. First, $|y_0 - y'_0| = |x_0 - x'_0| < \delta \le \min\{\delta_0, \frac{\varepsilon}{2N}\}$. Now suppose that $|y_i - y'_i| < \min\{\delta_i, \frac{\varepsilon}{2N}\}$. Let $I_{j(i+1)}$ be an interval which contains y_{i+1} and y'_{i+1}. Then y_i and y'_i are elements of $g_i(I_{j(i+1)})$. Furthermore, $|y_i - y'_i| < \delta_i$ by the induction hypothesis, so $|g^j_i(y_i) - g^j_i(y'_i)| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$. But $g^j_i(y_i) = y_{i+1}$ and $g^j_i(y'_i) = y'_{i+1}$. Therefore $|y_{i+1} - y'_{i+1}| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$. It follows that

$$d((y_0, y_1, \dots), (y'_0, y'_1, \dots)) = \sum_{i=0}^{\infty} \frac{|y_i - y'_i|}{2^i}$$

$$= \sum_{i=0}^{N-1} \frac{|y_i - y'_i|}{2^i} + \sum_{i=N}^{\infty} \frac{|y_i - y'_i|}{2^i} \le \sum_{i=0}^{N-1} \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus ϕ is continuous, and this completes the proof of Theorem 3.1.

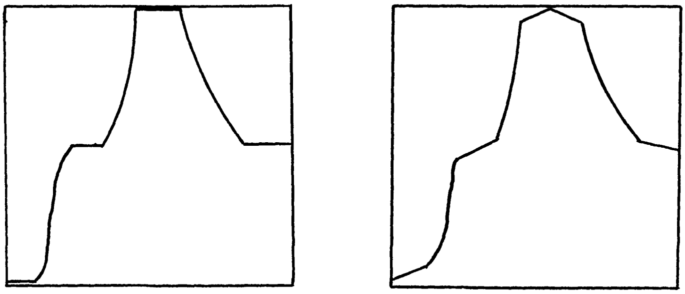


FIGURE 3

THEOREM 3.2. *Let* $f: I \to I$, $A = \{0 = a_0 < a_1 < \cdots < a_m = 1\}$, *and* $B \subset I$ *which has a finite number of nondegenerate components. Suppose that* f, A, *and* B *satisfy the following conditions*:

(i) f *is constant on each component of* B,
(ii) f *is strictly monotone on each component of* $I - B$,
(iii) $f|_{[a_j, a_{j+1}]}$ *is monotone,* $0 \leq j \leq m-1$,
(iv) f *is invariant on* A,
(v) $A \subset B$ *and* A *intersects each component of* B *in at most one point.*

If $g: I \to I$ *satisfies* $g(a_j) = f(a_j)$ *for* $a_j \in A$, *and* g *is linear on* $[a_j, a_{j+1}]$, $1 \leq j \leq m-1$, *then* (I, f) *is homeomorphic to* (I, g).

Proof. We will use the following notation: if B_{i_1} and B_{i_2} are components of B such that $x < y$ for each $x \in B_{i_1}$ and $y \in B_{i_2}$ then write $B_{i_1} < B_{i_2}$. Let $B_1 < B_2 < \cdots < B_r$ be the components of B. Then each B_i is an interval, $[b_i, c_i]$. Note that $0 = b_1$ and $1 = c_r$ since $\{0, 1\} \subset B$. Let $L = \min\{b_{i+1} - c_i : 1 \leq i \leq r-1\}$ and choose N so that if $n \geq N$, then $\frac{1}{n} < \frac{L}{2}$. For each $n \geq N$, let $b_1^n = b_1 = 0$, $b_i^n = b_i - \frac{1}{n}$, $2 \leq i \leq r$, $c_r^n = c_r = 1$, and $c_i^n = c_i + \frac{1}{n}$, $1 \leq i \leq r-1$.

Define f_n as follows: $f_n(x) = f(x)$ if $x \in A \cup \bigcup_{i=1}^{r-1}[c_i^n, b_{i+1}^n]$. If $B_i \cap A = \{a_j\}$, define f_n to be linear on $[b_i^n, a_j]$ and $[a_j, c_i^n]$. If $B_i \cap A = \varnothing$, define f_n to be linear on $[b_i^n, c_i^n]$. The graphs of f and f_n are pictured in Figure 3.

It is a straightforward check that $f_n \to f$ uniformly as $n \to \infty$. Thus it follows from Theorem 3 of **[Br]** that (I, f) is homeomorphic to (I, f_{n_k}) where $\{f_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}_{n=1}^{\infty}$. In order to show that (I, f) is homeomorphic to (I, g) we show that (I, g) is homeomorphic to (I, f_{n_k}). For each $k \in \mathbb{N}$ let $g_k = g$. We will show that $\{f_{n_k}\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$, and A satisfy the conditions of Theorem 1. First note that for each $k \in \mathbb{N}$ and $a_j \in A$, $g_k(a_j) = g(a_j) = f(a_j) = f_{n_k}(a_j)$. Therefore f_{n_k} and g_k agree on A. Furthermore, f is invariant on A and so f_{n_k} and g_k are also invariant on A.

Next we check that g is strictly monotone on $[a_j, a_{j+1}]$ for each j between 0 and $m-1$. Since g is linear on $[a_j, a_{j+1}]$, it suffices to show that $g(a_j) \neq g(a_{j+1})$. Suppose that $g(a_j) = g(a_{j+1})$. Then $f(a_j) = f(a_{j+1})$. Condition (iii) in the hypothesis of the theorem says that $f|_{[a_j, a_{j+1}]}$ is monotone, so $f(a_j) = f(a_{j+1})$ implies that f is constant on $[a_j, a_{j+1}]$. Also, $a_j \in B_{i_1} = [b_{i_1}, c_{i_1}]$ and $a_{j+1} \in B_{i_2} = [b_{i_2}, c_{i_2}]$, where $i_1 < i_2$. Thus, $[c_{i_1}, b_{i_2}] \subset [a_j, a_{j+1}]$, which implies that f is constant on $[c_{i_1}, b_{i_2}]$. But $[c_{i_1}, b_{i_2}]$ must contain at least one component of $I - B$, and f is strictly monotone on each component of $I - B$. We have reached a contradiction and so it must be the case that $g(a_j) \neq g(a_{j+1})$.

Finally, we check that each f_{n_k} is strictly increasing (decreasing) on $[a_j, a_{j+1}]$ if g is strictly increasing (decreasing) on $[a_j, a_{j+1}]$. Suppose that g is strictly increasing on $[a_j, a_{j+1}]$. Then f is increasing on $[a_j, a_{j+1}]$ and

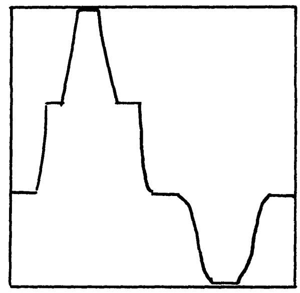
$$[a_j, a_{j+1}] = [a_j, c_i^{n_k}] \cup [c_i^{n_k}, b_{i+1}^{n_k}] \cup \cdots \cup [c_{i+s-1}^{n_k}, b_{i+s}^{n_k}] \cup [b_{i+s}^{n_k}, a_{j+1}].$$

It is a straightforward check that f_{n_k} is strictly increasing on each of these subintervals of $[a_j, a_{j+1}]$. The case where g is strictly decreasing is proved similarly.

Since Theorem 3.1 applies, it follows that (I, f_{n_k}) and (I, g_k) are homeomorphic. Furthermore, $(I, g_k) = (I, g)$ and (I, f_{n_k}) is homeomorphic to (I, f). Thus (I, g) is homeomorphic to (I, f).

4. Proof of Theorem 2.1. We are now ready for the proof of our main result, Theorem 2.1. Suppose $Q = (Q(0), Q(1), \ldots, Q(m))$ is a function such that $Q(j) \neq Q(j+1)$, $0 \leq j \leq m-1$, and $\{0, m\} \subset \operatorname{range} Q$ and let F_Q be a Q-horseshoe. Define $f_Q\colon I \to I$ by $f_Q(x) = P(F_Q(P^{-1}(x))$. The graph of f_Q for $Q = (1, 3, 0, 1)$ is pictured in Figure 4 (see next page) (F_Q is pictured in Figure 2).

It is easy to check that f_Q is well defined, continuous, and that $P \circ F_Q = f_Q \circ P$. Thus we may define $\widehat{P}\colon \Lambda_Q \to (I, f_Q)$ by $\widehat{P}(z) = (P(z), P(F_Q^{-1}(z)), P(F_Q^{-2}(z)), \ldots)$. It follows from the proof of Theorem 1 in **[Ba]** that $\widehat{P}$ is a homeomorphism. Thus Λ_Q is homeomorphic to (I, f_Q).



f_Q

FIGURE 4

Next, we use Theorem 3.2 to show that (I, f_Q) is homeomorphic to (I, g_Q). To this end, let $A = \{\frac{j}{m} : 0 \leq j \leq m\}$ and let $B = \bigcup_{j=0}^{m} P(F_Q^{-1}(D_j \cap F_Q(D)))$. Then f_Q, A, and B satisfy the conditions of Theorem 3.2. Therefore (I, f_Q) is homeomorphic to (I, g) where $g(a_j) = f_Q(a_j)$ for $a_j \in A$ and g is linear on $[a_j, a_{j+1}]$. But

$$f_Q(a_j) = P(F_Q(P^{-1}(a_j))) = P(F_Q(D_j)) \subset P(D_{Q(j)}) = \frac{Q(j)}{m}.$$

Therefore $g(a_j) = f_Q(a_j) = \frac{Q(j)}{m}$ and g is linear on $[a_j, a_{j+1}]$. Thus g follows Q, and the theorem is proved.

5. Examples. For our first example, we show that the horseshoe maps studied in **[Ba]** are special cases of the generalized horseshoes considered here. Consider

$$Q = \begin{cases} (0, m, 0, m, \dots, m, 0) : m \text{ even}, \\ (0, m, 0, m, \dots, 0, m) : m \text{ odd}. \end{cases}$$

Then any Q-horseshoe map, F_Q, is an m-fold horseshoe map described in **[Ba]**. Its attracting set is a Knaster continuum. Next consider $Q = (0, 2, 1)$. Then F_Q and g_Q are pictured in Figure 5. It is well known that (I, g_Q) is homeomorphic to the $\sin(\frac{1}{x})$ continuum, and thus the attracting set of F_Q is homeomorphic to this continuum.

Finally consider $Q = (1, 2, 0)$. Then F_Q and g_Q are pictured in Figure 6. It is well known that (I, g_Q) is homeomorphic to the three point indecomposable continuum described in **[HY]**, pages 141–142. Thus the attracting set of F_Q is homeomorphic to this continuum.

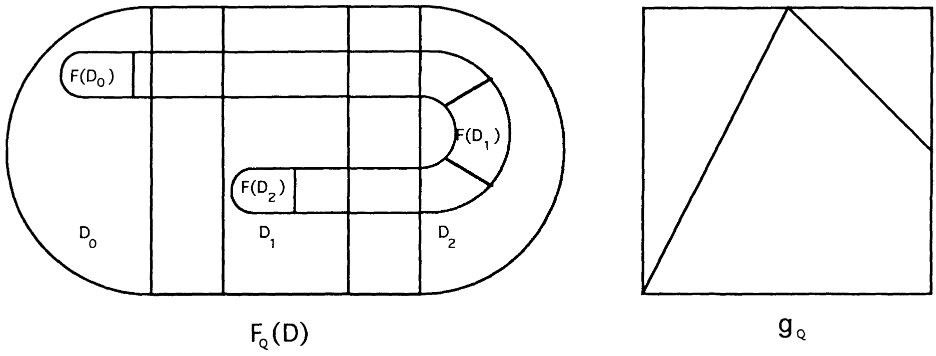


FIGURE 5

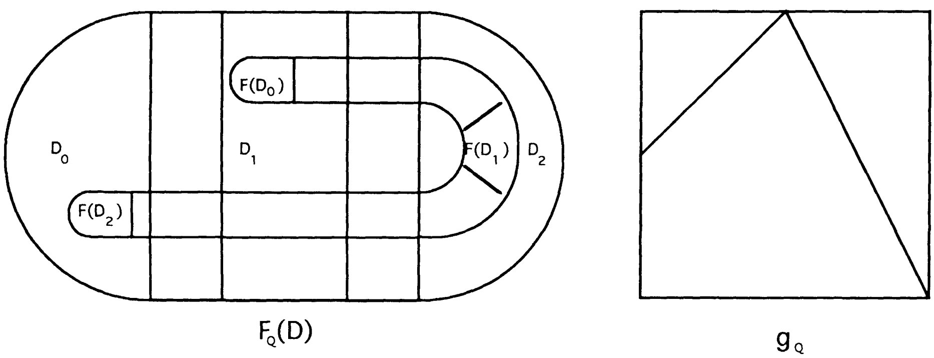


FIGURE 6

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