PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 1, 1993

THE MODULI OF RATIONAL WEIERSTRASS FIBRATIONS OVER $\mathbf{P}^1$: SINGULARITIES

PABLO LEJARRAGA

The Weierstrass equation $y^2 = x^3 + ax + b$, where a and b are rational functions of one variable, defines a fibration over $\mathbf{P}^1$, which we call a Weierstrass fibration. We consider the moduli space W of rational Weierstrass fibrations over $\mathbf{P}^1$. In this paper we determine the singular locus of W and we compute the general singularities. We work over $\mathbf{C}$, but it seems possible to generalize our methods to characteristic $p \neq 2, 3$.

Introduction. In [**Mi**] Miranda has constructed moduli spaces W_N, $N \geq 0$, for Weierstrass fibrations over $\mathbf{P}^1$ whose zero section has self intersection number $-N$ in the associated elliptic surface. Seiler has generalized and extended this work in [**Sei2**] and [**Sei3**]. For $N = 1$, we have the moduli space of rational fibrations $W = W_1$. The points of W parametrize isomorphism classes of rational Weierstrass fibrations over $\mathbf{P}^1$ with at most rational double point singularities whose associated elliptic surface (= minimal resolution of singularities) has only reduced fibers. By passing to the associated elliptic surface, W can be viewed as parametrizing isomorphism classes of relatively minimal elliptic surfaces over $\mathbf{P}^1$ admitting a section which have only reduced fibers. The basic definitions and constructions are reviewed in §1.

To determine the singular locus of W, we first find the locus S of Weierstrass fibrations that have non-negligible (= nontrivial) automorphisms. By means of the Weierstrass equation, this boils down to finding stable pairs of Weierstrass coefficients whose isotropy group with respect to the action of $G = \mathbf{GL}_2/\pm I$ is nontrivial. This work is the content of §2 and culminates in Theorem 1 where the 7 irreducible components of S are listed.

The general singularities turn out to be cyclic quotient singularities. We compute and classify them with the help of the slice theorem and work of Prill [**Pr**] in Theorem 2, §3.

This work is part of my Ph.D. thesis. I want to thank my advisor M. Artin and Rick Miranda for their help.

1. Generalities. All varieties we consider are defined over the field of complex numbers $\mathbf{C}$. Unless otherwise stated all topological notions refer to the Zariski topology. We refer the reader to **[Mi]**, **[Ka]** and **[M-S]** for proofs of the following facts in this section.

Let S be a variety. Let $p: Y \to S$ be a flat proper morphism of irreducible varieties whose fibers are of one of the following types:

(a) an elliptic curve,

(b) a rational curve with a node,

(c) a rational curve with a cusp.

Let σ be a section of p not touching the nodes and cusps of the fibers. The quadruple (Y, S, p, σ) is called a *Weierstrass fibration over* S. We usually denote Weierstrass fibrations by Y/S when there is no risk of confusion.

A morphism of a Weierstrass fibration (Y, S, p, σ) into a Weierstrass fibration (Y', S', p', σ') is given by a pair of morphisms f: $Y \to Y'$ and $\varphi: S \to S'$ such that $p' \circ f = \varphi \circ p$ and $f \circ \sigma = \sigma' \circ \varphi$.

When $S = C$ is a complete nonsingular connected curve, a Weierstrass fibration with nonsingular general fiber and only rational double point singularities is called a *Weierstrass model*. As is well known, a Weierstrass model Y/C can be described by a *Weierstrass equation* over C, i.e. there exists an invertible sheaf $\mathscr{L}$ over C and sections a of $\mathscr{L}^{\otimes 4}$ and b of $\mathscr{L}^{\otimes 6}$ such that Y is isomorphic to the hypersurface in $\mathbf{P}(O_C \otimes \mathscr{L}^{\otimes(-2)} \oplus \mathscr{L}^{\otimes(-3)})$ given by $y^2 = x^3 + ax + b$. The morphism $J = J(a, b) = 4a^3/(4a^3 + 27b^2)$ of C into $\mathbf{P}^1$ is called the *J-invariant.*

Let $S = \mathbf{P}^1$. Choose coordinates t, s such that $t = 1$, $s = 0$ is the point at infinity. Call V_n the set of homogeneous functions of degree n on $\mathbf{P}^1$ viewed as homogeneous forms of degree n in t, s. Call G the quotient group $\mathbf{GL}_2/(\pm I)$. We use the same notation for a matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ in $\mathbf{GL}_2$ and for its image in G. We also use the notation $\left(\begin{smallmatrix}\alpha & \\ & \delta\end{smallmatrix}\right)$ for diagonal matrices, $\alpha = \left(\begin{smallmatrix}\alpha & \\ & \alpha\end{smallmatrix}\right)$ for scalar matrices and $\left(\begin{smallmatrix} & \beta\\ \gamma & \end{smallmatrix}\right)$ for matrices with zeros in the main diagonal. Let $f(t, s) \in V_n$ and g be an element of G with matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. We define

$$(f \cdot g)(t, s) = f(\alpha t + \beta s, \gamma t + \delta s).$$

This defines a right action of G on V_n. The pair of coefficients (a, b) of a rational Weierstrass model over $\mathbf{P}^1$ can be interpreted as an element of $V_4 \times V_6$.

In this way we get an injection of the set of isomorphism classes of rational Weierstrass models over $\mathbf{P}^1$ into $(V_4 \times V_6)/G$, where G acts by means of its actions on V_4 and V_6. Denote by X the open set of $\mathbf{SL}_2$-*stable* (= finite stabilizer and closed orbit) elements of $V_4 \times V_6$ (to be called just *stable* from now on). The quotient algebraic variety $W = X/G$ is called the *moduli of rational Weierstrass fibrations over* $\mathbf{P}^1$. We denote by $\pi\colon X \to W$ the canonical map. Under the above injection points of W correspond to classes of Weierstrass models whose associated elliptic surface has reduced fibers. For $f \in V_n$ and $\tau \in \mathbf{P}^1$ denote by $v_\tau(f)$ the order of vanishing of f at τ. An element $(a, b) \in V_4 \times V_6$ is stable if and only if the following *numerical criterion* holds:

$$\min(3v_\tau(a), 2v_\tau(b)) < 6$$

for all $\tau \in \mathbf{P}^1$.

Let $x = (a, b) \in X$. Denote by Y_x the Weierstrass fibration with equation $\eta^2 = \xi^3 + a\xi + b$. Denote by $\operatorname{Stab} x$ the *isotropy group* (= stabilizer) of x with respect to the action of G. Denote by $\operatorname{Aut}_{WF}(Y_x/\mathbf{P}^1)$ the automorphism group of the Weierstrass fibration $Y_x/\mathbf{P}^1$ and by N the normal subgroup of *negligible* automorphisms, i.e., those of the form

$$\eta = \pm\eta', \quad \xi = \xi', \quad t = t', \quad s = s'.$$

Define $\operatorname{Aut}_{RWF}(Y_x/\mathbf{P}^1) = (\operatorname{Aut}_{WF}(Y_x/\mathbf{P}^1))/N$, the *reduced automorphism group of* $Y_x/\mathbf{P}^1$. Given $g \in \operatorname{Stab} x$ with matrix $\lambda\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, $\lambda \neq 0$, $\alpha\delta - \beta\gamma = 1$, the formulas

$$\eta = \lambda^{-3}\eta', \quad \xi = \lambda^{-2}\xi', \quad t = \alpha t' + \beta s', \quad s = \gamma t' + \delta s'$$

define an element of $\operatorname{Aut}_{RWF}(Y_x/\mathbf{P}^1)$ denoted by $\operatorname{Aut} g$. The following proposition follows from well known facts.

PROPOSITION 1. *The canonical group homomorphism* $\operatorname{Stab} x \to \operatorname{Aut}_{RWF}(Y_x/\mathbf{P}^1)$, $g \mapsto \operatorname{Aut} g$ *is bijective.*

We view the J-invariant $J(x) = J(a, b) = 4a^3/(4a^3 + 27b^2)$ as a morphism of $\mathbf{P}^1$ into $\mathbf{P}^1$. We denote by $\operatorname{Aut} J(x)$ the group of deck transformations of $J(x)\colon \mathbf{P}^1 \to \mathbf{P}^1$. For g an element of G with matrix $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ we denote by Pg the linear fractional transformation $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$, viewed as an element of $\mathbf{PGL}_2 = \operatorname{Aut} \mathbf{P}^1$. The proof of the following easy corollary is left to the reader.

COROLLARY. *Suppose that* $x \in X$ *has nonconstant* J*-invariant. The canonical group homomorphism* $\operatorname{Stab} x \to \operatorname{Aut} J(x)$, $g \mapsto Pg$ *is injective.*

REMARK. In fact the homomorphism of the above corollary is *bijective*, but the proof is more involved.

2. Components of S. Recall that $\pi: X \to W$ is the canonical morphism. Define

$$S = \pi\{x \in X \mid \operatorname{Stab} x \neq 1\}.$$

By the corollary to Proposition 1, this set is the locus in moduli of Weierstrass fibrations with nontrivial automorphisms. In this section we determine the irreducible components of the closed set S.

Let the group Γ operate on the set E. Let H be a subgroup of Γ. We denote

$$E^H = \{x \in E \mid xg = x \text{ for all } g \in H\}.$$

For $g \in \Gamma$, define $E^g = E^{(g)}$, where (g) is the group generated by g; we remark that E^g is the set of x in E such that $xg = x$. When $E = X$, $\Gamma = G$, H subgroup of G, $g \in G$, we use the notations

$$\operatorname{Inv} H = X^H, \qquad \operatorname{Inv} g = X^g.$$

It is clear that

$$S = \bigcup\{\pi(\operatorname{Inv} g) \mid g \in G,\ g \neq 1,\ g \text{ of finite order}\}.$$

LEMMA 1. *Let $g \in G$ be of finite order. The sets* $\operatorname{Inv} g$ *and* $\pi(\operatorname{Inv} g)$ *are irreducible closed in X and W respectively.*

REMARK. It follows from Lemma 1 that the maximal elements among the $\pi(\operatorname{Inv} g)$ are the irreducible components of S. Since a Noetherian topological space has a finite number of irreducible components, the set S is closed.

Proof of Lemma 1. We have

$$\operatorname{Inv} g = (V_4 \times V_6)^g \cap X$$

where $(V_4 \times V_6)^g$ is a sub-vector space of $V_4 \times V_6$ and X is open in $V_4 \times V_6$. It follows that $\operatorname{Inv} g$ is irreducible and closed. Consequently $\pi(\operatorname{Inv} g)$ is irreducible. We have not used the fact that g is of finite order up to here.

Now let C be the conjugacy class of g. Since g is of finite order it follows from [**Bo**, pp. 227–228] that C is closed. Moreover G acts properly on X by [**GIT**, p. 41, Converse 1.13] and the fact that $\pi: X \to W = X/G$ is affine. Hence the morphism

$$X \times G \xrightarrow{\psi} X \times X,$$
$$(x, h) \mapsto (xh, x)$$

is proper.

Denote by Δ_X the diagonal morphism of X into $X \times X$. It follows that the set $\Delta_X^{-1}(\psi(X \times C))$ is closed. Since

$$\Delta_X^{-1}(\psi(X \times C)) = \{x \in X | xg = x \text{ for some } g \in C\}$$

is G-saturated, it is clear that

$$\pi(\operatorname{Inv} g) = \pi(\Delta_X^{-1}(\psi(X \times C)))$$

is closed. □

For any prime number p, let R_p be a system of representatives of the equivalence classes of elements of $\mathbf{F}_p^* - \{1\} = (\mathbf{Z}/p\mathbf{Z}) - \{0, 1\}$ with respect to the equivalence relation between elements u, v of $\mathbf{F}_p^* - \{1\}$ defined by the condition "$u = v$ or $u = v^{-1}$". Moreover we define $\zeta_n = e^{2\pi i/n}$.

LEMMA 2. *We have*

$$S = \bigcup \pi(\operatorname{Inv} g)$$

where g *runs over the following list*:

$$\begin{pmatrix} i & \\ & i \end{pmatrix}, \quad \begin{pmatrix} \zeta_3 & \\ & \zeta_3 \end{pmatrix},$$
$$\begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}, \qquad l \in R_p, \; p = 3, 5, 7, 11.$$
$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} -i & \\ & i \end{pmatrix},$$

The inclusion

$$\bigcup \pi(\operatorname{Inv} g) \subset S$$

is obvious. Now let $u \in S$. There are two cases:

(i) $J(x) = 0$ (resp. $J(x) = 1$) for all $x \in \pi^{-1}(u)$.
(ii) $J(x)$ is nonconstant for all $x \in \pi^{-1}(u)$.

Case (i). The conditions $J(x) = 0$ and $J(x) = 1$ are equivalent to $x \in \operatorname{Inv} \zeta_3$ and $x \in \operatorname{Inv} i$ respectively. We conclude in this case that

$$u \in \bigcup \pi(\operatorname{Inv} g)$$

where $g = i, \zeta_3$.

Case (ii). Since $J(x)$ is nonconstant, it follows from the rationality of the Weierstrass model determined by x, that $\deg J(x) \leq 12$, where $\deg J(x)$ denotes the degree of the cover $J(x)\colon \mathbf{P}^1 \to \mathbf{P}^1$. The following argument shows that every element φ of $\operatorname{Aut} J(x)$ has order $\leq d = \deg J(x) \leq 12$. Take a classical nonempty open set U in $\mathbf{P}^1$ such that $(J(x))^{-1}(U)$ is a disjoint union of d copies of U. Suppose that V is one of such copies. Then the sequence $V = \varphi^0(V), \varphi^1(V), \dots, \varphi^d(V)$ has a repetition, say

$$\varphi^i(V) = \varphi^j(V) \quad \text{for } 0 \leq i < j \leq d.$$

Thus

$$V = \varphi^{j-i}(V)$$

which implies, since φ is an analytic function, that φ has order $\leq j - i \leq d$. We conclude by the corollary to Proposition 1 that every element of $\operatorname{Stab} x$ has order ≤ 12. Now we notice the following facts.

(α) If $u \in \pi(\operatorname{Inv} g)$, there exists $x \in \pi^{-1}(u)$ such that $x \in \operatorname{Inv} g$. Thus $\operatorname{Stab} x \supset (g)$. It follows that g has order ≤ 12 by the above considerations.

(β) If $u \in \pi(\operatorname{Inv} g)$, there exists $x \in \pi^{-1}(u)$ such that $x \in \operatorname{Inv} g$. Since $J(x)$ is nonconstant, $x = (a, b)$ with $a \neq 0$, $b \neq 0$. Suppose g were scalar with matrix $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$. It follows that

$$ag = \lambda^4 a = a, \qquad bg = \lambda^6 b = b$$

which implies $\lambda^4 = \lambda^6 = 1$. Thus $\lambda^2 = 1$, which contradicts the fact that $g \neq 1$ in G. Consequently g is nonscalar.

(γ) Given g of finite order there exists $g' \in (g)$ of prime order such that

$$\pi(\operatorname{Inv} g) \subset \pi(\operatorname{Inv} g').$$

(δ) Given g of finite order there exists a diagonal element g' conjugate to g such that

$$\pi(\operatorname{Inv} g) = \pi(\operatorname{Inv} g').$$

(ε) If (g) is conjugate to (g'), then

$$\pi(\operatorname{Inv} g) = \pi(\operatorname{Inv} g').$$

We conclude from (α) to (ε) that

$$u \in \bigcup \pi(\operatorname{Inv} g),$$

where g runs through a system of representatives of the equivalence classes of nonscalar diagonal elements of G of prime order ≤ 12 with respect to the equivalence relation between elements g, g' of

G defined as follows. We say that g is *equivalent* to g' if (g) is conjugate to (g').

Let g be of prime order $p \leq 12$ with matrix $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, $\lambda_1 \neq \lambda_2$. In case $p = 2$, we have either $\lambda_1^2 = \lambda_2^2 = 1$ or $\lambda_1^2 = \lambda_2^2 = -1$. Thus g is equivalent to one of $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$, $\begin{pmatrix} -i & \\ & i \end{pmatrix}$. In case p is odd, suppose first that $\lambda_1^p = \lambda_2^p = 1$. If $\lambda_2 = 1$, then $\lambda_1 \neq 1$. There exists an integer μ such that

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}.$$

Thus g is equivalent to $\begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}$. The case $\lambda_1 = 1$ reduces to the previous one by conjugation with the matrix

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

If $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, there exists an integer μ such that

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \lambda_1^\mu & \\ & \zeta_p \end{pmatrix}.$$

For some integer $l \neq 0, 1$

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}.$$

Thus g is equivalent to

$$\begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}, \qquad l \neq 0, 1.$$

When $\lambda_1^p = \lambda_2^p = -1$, set $\lambda_i' = -\lambda_i$, $i = 1, 2$ and reduce to the previous case.

The proof of Lemma 2 is finished by the observation that whenever $m \cdot l = 1 \pmod p$,

$$\begin{pmatrix} \zeta_p^m & \\ & \zeta_p \end{pmatrix} \text{ is equivalent to } \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}. \qquad \square$$

For $g \in G$, we have

$$\operatorname{Inv} g = (V_4 \times V_6)^g \cap X = (V_4^g \times V_6^g) \cap X.$$

Let g be diagonal. The g-invariant monomials of V_n form a vector basis of V_n^g. Thus a general element of V_n^g is given by a linear combination with general coefficients of elements from such a basis. A general element of $V_4^g \times V_6^g$ is just a pair of general elements of V_4^g and V_6^g. Such a general element is also a general element of $\operatorname{Inv} g$ since X is open. It is stable if some specialization is stable.

Now we choose the following R_p for $p = 3, 5, 7, 11$:

$$\begin{aligned} R_3 &= \{2\}, \\ R_5 &= \{2, 4\}, \\ R_7 &= \{2, 3, 6\}, \\ R_{11} &= \{2, 3, 5, 7, 10\}. \end{aligned}$$

In Table 1 we give bases of g-invariant monomials of V_4^g and V_6^g for the different values of g that appear in Lemma 2 subject to the above choice of R_p's, except for the cases $g = i$, ζ_3 which are trivial. We also indicate for which values of g the set $\operatorname{Inv} g$ is nonempty.

TABLE 1. *Invariant Monomials.* We list all g-invariant monomials of degrees 4 and 6

p	g	degree 4	degree 6	g-invariant pairs
2	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	t^4, t^2s^2, s^4	t^6, t^4s^2, t^2s^4, s^6	some stable
	$\begin{pmatrix} -i & \\ & i \end{pmatrix}$	t^4, t^2s^2, s^4	t^5s, t^3s^3, ts^5	some stable
3	$\begin{pmatrix} \zeta_3 & \\ & 1 \end{pmatrix}$	t^3s, s^4	t^6, t^3s^3, s^6	some stable
	$\begin{pmatrix} \zeta_3^2 & \\ & \zeta_3 \end{pmatrix}$	t^2s^2	t^6, t^3s^3, s^6	some stable
5	$\begin{pmatrix} \zeta_5 & \\ & 1 \end{pmatrix}$	s^4	t^5s, s^6	some stable
	$\begin{pmatrix} \zeta_5^2 & \\ & \zeta_5 \end{pmatrix}$	ts^3	t^4s^2	some stable
	$\begin{pmatrix} \zeta_5^4 & \\ & \zeta_5 \end{pmatrix}$	s^4	s^6	all unstable
7	$\begin{pmatrix} \zeta_7 & \\ & 1 \end{pmatrix}$	s^4	s^6	all unstable
	$\begin{pmatrix} \zeta_7^2 & \\ & \zeta_7 \end{pmatrix}$	t^3s	ts^5	some stable
	$\begin{pmatrix} \zeta_7^3 & \\ & \zeta_7 \end{pmatrix}$	No solutions	No solutions	
	$\begin{pmatrix} \zeta_7^6 & \\ & \zeta_7 \end{pmatrix}$	t^2s^2	t^3s^3	all semistable
11	$\begin{pmatrix} \zeta_{11}^{10} & \\ & \zeta_{11} \end{pmatrix}$	t^2s^2	t^3s^3	all semistable
	No other solutions for $p = 11$.			

TABLE 2. *Components of S. Here Γ is the component $\pi(\operatorname{Inv} g)$, $x=(a,b)$ is an element of the general orbit over Γ*

Γ	g	$x=(a,b)$	Stab x	$\deg J(x)$	$\dim\Gamma$
A	$\begin{pmatrix}-1 & \\ & 1\end{pmatrix}$	$\begin{cases} a=(t^2-s^2)(t^2-k^2s^2) \\ b=c(t^2-m^2s^2)(t^2-n^2s^2)(t^2-p^2s^2)\end{cases}$	C_2	12	5
B	$\begin{pmatrix}-i & \\ & i\end{pmatrix}$	$\begin{cases} a=(t^2-s^2)(t^2-k^2s^2) \\ b=cts(t^2-m^2s^2)(t^2-n^2s^2)\end{cases}$	C_2	12	4
C	$\begin{pmatrix}\zeta_3 & \\ & 1\end{pmatrix}$	$\begin{cases} a=(t^3-s^3)s \\ b=c(t^3-m^3s^3)(t^3-n^3s^3)\end{cases}$	C_3	12	3
D	$\begin{pmatrix}\zeta_5 & \\ & 1\end{pmatrix}$	$\begin{cases} a=s^4 \\ b=c(t^5-s^5)s\end{cases}$	C_5	10	1
E	$\begin{pmatrix}\zeta_5^2 & \\ & \zeta_5\end{pmatrix}$	$\begin{cases} a=ts^3 \\ b=t^4s^2\end{cases}$	C_5	5	0
F	$\begin{pmatrix}\zeta_7^2 & \\ & \zeta_7\end{pmatrix}$	$\begin{cases} a=t^3s \\ b=ts^5\end{cases}$	C_7	7	0
G	$\begin{pmatrix}\zeta_3 & \\ & \zeta_3\end{pmatrix}$	$\begin{cases} a=0 \\ b=t(t-s)s(t-ms)(t-ns)(t-ps)\end{cases}$	C_3	$J\equiv 0$	3

THEOREM 1. *The irreducible components of S are listed in Table 2. Suppose g and x are entries in a row of Table 2 with x an element of the general orbit over $\Gamma=\pi(\operatorname{Inv} g)$. Then* $\operatorname{Stab} x=(g)$.

In Table 2 the element $x=(a,b)$ is obtained by taking a general element of $\operatorname{Inv} g$ constructed from Table 1 and eliminating parameters redundant with respect to the action of G. The resulting parameters are chosen in such a way as to make explicit the zeros of a and b.

Keeping in mind the remark after Lemma 1, we first prove that the sets $\pi(\operatorname{Inv} g)$ for g in Table 2 are an irredundant decomposition of S. Among the $\pi(\operatorname{Inv} g)$ in Lemma 2 the following inclusions hold:

$$(\alpha)\qquad \pi\operatorname{Inv}\begin{pmatrix}i & \\ & i\end{pmatrix}\subset\pi\operatorname{Inv}\begin{pmatrix}-1 & \\ & 1\end{pmatrix},$$

$$(\beta)\qquad \pi\operatorname{Inv}\begin{pmatrix}\zeta_3^2 & \\ & \zeta_3\end{pmatrix}\subset\pi\operatorname{Inv}\begin{pmatrix}-1 & \\ & 1\end{pmatrix}.$$

(α) By putting $c=0$ in the element of the general orbit over $\pi\operatorname{Inv}\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)$ we get

$$\begin{cases} a=(t^2-s^2)(t^2-k^2s^2), \\ b=0.\end{cases}$$

By dimension considerations we get that the locus $J\equiv 1$ which equals $\pi\operatorname{Inv}\left(\begin{smallmatrix}i & \\ & i\end{smallmatrix}\right)$ is contained in $\pi\operatorname{Inv}\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)$.

(β) Since $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$ we have

$$\pi \operatorname{Inv}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \pi \operatorname{Inv}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

But the general element of $\operatorname{Inv}\left(\begin{smallmatrix} \zeta_3^2 & \\ & \zeta_3 \end{smallmatrix}\right)$ is invariant under $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.

To check that there are no other inclusions we use systems of eigenvalues. More precisely, let R be the equivalence relation on $\mathbf{C}^2$ generated by the relations "$(x', y') = (y, x)$" and "$(x', y') = (-x, -y)$" both between the elements (x, y) and (x', y') of $\mathbf{C}^2$. Thus the class of (x, y) consists of the elements (x, y), (y, x), $(-x, -y)$ and $(-y, -x)$. The system of eigenvalues of an element of finite order $g \in G$ will be considered as an element of $\mathbf{C}^2/R$.

For a subgroup of finite order H of G we denote by $\operatorname{Eigenval}(H)$ the set of systems of eigenvalues of elements of H. Clearly $\operatorname{Eigenval}(H)$ depends only on the conjugacy class of H.

Let g_1, g_2 appear in Table 2 and suppose that $\pi(\operatorname{Inv} g_1) \subset \pi(\operatorname{Inv} g_2)$. Suppose x_1 is the element of the general orbit over $\pi(\operatorname{Inv} g_1)$, given in Table 2. Then $\pi(x_1) \in \pi(\operatorname{Inv} g_2)$, which implies that $x_1 \in \operatorname{Inv} h^{-1} g_2 h$ for some h. By (ii) it follows that

$$(g_1) = \operatorname{Stab} x_1 \supset (h^{-1} g_2 h).$$

Thus $\operatorname{Eigenval}(g_1) \supset \operatorname{Eigenval}(g_2)$. The reader can easily check case by case that this can only happen when $g_1 = g_2$.

Now it remains to prove that $\operatorname{Stab} x \subset (g)$, for g, x satisfying the conditions of the theorem. The inclusion $(g) \subset \operatorname{Stab} x$ is obvious. In cases D, E, F, suppose $h = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \operatorname{Stab} x$, $x = (a, b)$. By comparing coefficients in the equations $ah = a$ and $bh = b$, one concludes that $h \in (g)$. The remaining cases depend on a series of lemmas.

As usual, we identify $\mathbf{P}^1$ with $\mathbf{C} \cup \{\infty\}$ and automorphisms of $\mathbf{P}^1$ with linear fractional transformations. Moreover, given a set E of $n \geq 3$ distinct points of $\mathbf{P}^1$, every automorphism of $\mathbf{P}^1$ stabilizing the set E is determined by the induced permutation of E. We indicate such automorphisms by giving only the induced permutation. We omit the proof of the following well-known lemmas.

LEMMA 3. *Every automorphism of $\mathbf{P}^1$ that permutes the points 0, 1, ∞, m, n, p, where m, n, p are in general position, is the identity.*

LEMMA 4. *The group of automorphisms of* $\mathbf{P}^1$ *that permute the points* 0, 1, ∞, k, *for general* k *is Klein's four-group consisting of* $(0\ 1)(\infty\ k)$, $(0\ \infty)(1\ k)$, $(1\ \infty)(0\ k)$ *and the identity* e.

LEMMA 5. *The group of automorphisms of* $\mathbf{P}^1$ *that permute the points* 1, ζ_3, ζ_3^2, ∞ *is the tetrahedral group.*

LEMMA 6. *The group of automorphisms of* $\mathbf{P}^1$ *that permute the points* 1, -1, k, $-k$, *for general* k *is Klein's four-group consisting of* $(1\ -1)(k\ -k)$, $(1\ k)(-1\ -k)$, $(1\ -k)(-1\ k)$ *and the identity* e.

Now we return to the proof of Theorem 1. We omit Case B because it is similar to case A. We treat case G first.

Case G. Suppose $bh = b$ for $b = t(t-s)s(t-ms)(t-ns)(t-ps)$, m, n, p in general position and $h \in G$. By Lemma 3, we infer that h has the matrix $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$ since the automorphism of $\mathbf{P}^1$ induced by h permutes the zeros of b. Thus $\lambda^6 = 1$.

Case A. Suppose $ah = a$ for $a = (t^2 - s^2)(t^2 - k^2 s^2)$, k general, $h \in G$. By Lemma 5 the linear fractional transformation Ph is one of $(1\ -1)(k\ -k)$, $(1\ k)(-1\ -k)$, $(1\ -k)(-1\ k)$ or the identity. But $(1\ k)(-1\ -k)$ and $(1\ -k)(-1\ k)$ cannot stabilize the set $\{m, -m, n, -n, p, -p\}$ for m, n, p in general position.

Case C. Suppose $ah = a$ for $h \in G$. By Lemma 6, Ph belongs to the tetrahedral group permuting the points 1, ζ_3, ζ_3^2, ∞, which is isomorphic to the alternating group of the set $\{1, \zeta_3, \zeta_3^2, \infty\}$. Taking into account the form of the elements of this group ([**Se**], p. 41), for m, n in general position the subgroup stabilizing the set of zeros of b is $((1\ \zeta_3\ \zeta_3^2))$. □

3. Singularities. In this section we prove that S is the singular locus of W and we determine the general singularities.

All the representations we consider in the following are finite dimensional linear representations over $\mathbf{C}$ of finite groups.

We need the notion of isomorphism of two representations $\rho\colon H \to \mathbf{GL}(V)$ and $\rho'\colon H' \to \mathbf{GL}(V')$ of not necessarily identical groups H, H'. The definition is obvious. The representation $\rho\colon H \to \mathbf{GL}(V)$ is called *small* if no element in the image of ρ has 1 as eigenvalue of multiplicity $\dim V - 1$. We gather in the following proposition the results we need from [**Pr**].

PROPOSITION 2. *Two small faithful representations $\rho\colon H \to \mathbf{GL}(V)$ and $\rho'\colon H' \to \mathbf{GL}(V')$ are isomorphic if and only if the germs of analytic space $(V/H, 0)$ and $(V'/H', 0)$ are isomorphic.*

A small faithful representation $\rho\colon H \to \mathbf{GL}(V)$ is identically equal to the identity if and only if $(V/H, 0)$ is nonsingular.

Now let u be a point of W and x an element of X such that $u = \pi(x)$. Put $H = \operatorname{Stab} x$ and let N be an H-invariant complement to $T_x(xG)$ in $T_x(X)$. By the slice theorem ([**Sch**], p. 56) and the fact that π is affine, there exists an isomorphism of germs of analytic space $(W, u) \xrightarrow{\sim} (N/H, 0)$. Call $\rho = \rho_{x,N}$ the representation of H defined by its action on N and $\rho^f = \rho^f_{x,N}$ the faithful representation of $H/\operatorname{Ker}\rho$ induced by ρ. The isomorphism class of the germ (W, u) depends only on the isomorphism class of the representation ρ. We say that the representation $\rho = \rho_{x,N}$ is *associated* to the point $u = \pi(x)$.

THEOREM 2. *The set S is the singular locus of W. Representations associated to the general singularities, which are given in Table 3, are faithful and small.*

The following corollary is immediate by Proposition 2.

COROLLARY. *The isomorphism classes of the associated representations classify the general singularities up to isomorphism.*

TABLE 3. *Associated Representations.* Here Γ is the component of $\pi(\operatorname{Inv} g)$, x is the element of the general orbit over Γ such that $\operatorname{Stab} x = (g)$, $\rho = \rho_{x,N}$ is a representation of $\operatorname{Stab} x$ associated to $u = \pi(x)$

Γ	$\operatorname{Stab} x$	Eigenvalues of $\rho(g)$							
A	C_2	$1,$	$1,$	$1,$	$1,$	$1,$	$-1,$	$-1,$	-1
B	C_2	$1,$	$1,$	$1,$	$1,$	$-1,$	$-1,$	$-1,$	-1
C	C_3	$1,$	$1,$	$1,$	$\zeta_3,$	$\zeta_3,$	$\zeta_3,$	$\zeta_3^2,$	ζ_3^2
D	C_5	$\zeta_5,$	$\zeta_5,$	$\zeta_5^2,$	$\zeta_5^2,$	$\zeta_5^2,$	$\zeta_5^3,$	$\zeta_5^3,$	ζ_5^4
E	C_5	$1,$	$\zeta_5,$	$\zeta_5,$	$\zeta_5^2,$	$\zeta_5^2,$	$\zeta_5^3,$	$\zeta_5^3,$	ζ_5^4
F	C_7	$\zeta_7,$	$\zeta_7^2,$	$\zeta_7^2,$	$\zeta_7^3,$	$\zeta_7^4,$	$\zeta_7^5,$	$\zeta_7^6,$	ζ_7^6
G	C_3	$1,$	$1,$	$1,$	$\zeta_3,$	$\zeta_3,$	$\zeta_3,$	$\zeta_3,$	ζ_3

Proof of Theorem 2. It is clear that the representations in Table 3 are faithful and small. We have to prove that they are associated to the general points of the components of S.

First of all we recall some generalities on infinitesimals of first order. Let X be an analytic space, $x \in X$. Let $\mathbf{C}[\varepsilon]$ be the algebra of dual numbers. Let $\operatorname{Specan} \mathbf{C}[\varepsilon]$ be the analytic space with only one point o and with local ring $\mathbf{C}[\varepsilon]$ at that point. We use the notation

$$\begin{aligned} X(\mathbf{C}[\varepsilon])_x &= \operatorname{Hom}((\operatorname{Specan} \mathbf{C}[\varepsilon], o), (X, x)) \\ &= \operatorname{Hom}_{\mathbf{C}-\mathrm{alg\,loc}}(\mathscr{O}_{X,x}, \mathbf{C}[\varepsilon]) \end{aligned}$$

for the set of $\mathbf{C}[\varepsilon]$-valued points of X at x.

The map

$$\begin{gathered} \operatorname{Der}_{\mathbf{C}}(\mathscr{O}_{X,x}, \mathbf{C}) \to \operatorname{Hom}_{\mathbf{C}-\mathrm{alg\,loc}}(\mathscr{O}_{X,x}, \mathbf{C}[\varepsilon]), \\ t \to u = x + \varepsilon t \end{gathered}$$

establishes a bijection of $T_x(X)$ onto $X(\mathbf{C}[\varepsilon])_x$.

Now denote by X the set of stable elements of $V_4 \times V_6$ and by G the group $\mathbf{GL}_2/(\pm I)$ as before. Let $x \in X$. The orbital map $\rho \colon G \to X$, $g \mapsto xg$ is étale because $\operatorname{Stab} x$ is a finite set. We get the following commutative diagram:

$$\begin{array}{ccc} G(\mathbf{C}[\varepsilon])_1 & \xrightarrow{\ \rho\ } & X(\mathbf{C}[\varepsilon])_x \\ {\scriptstyle\varphi}\big\uparrow & & \big\uparrow{\scriptstyle\psi} \\ \mathbf{M}_2(\mathbf{C}) & \xrightarrow[d\rho_1]{} & V_4 \times V_6 \end{array}$$

where we identify $T_1(G) = T_I(\mathbf{GL}_2) = \mathbf{M}_2(\mathbf{C}) = 2 \times 2$ matrices, $T_x(X) = T_x(V_4 \times V_6) = V_4 \times V_6$, $T_x(G) = \operatorname{Im} d\rho_1$ and φ and ψ are the bijections described above. Note that ρ and $d\rho_1$ are injective. We have a canonical basis $t_1, \dots, t_4$ of $\operatorname{Im} d\rho_1$ namely the image of the canonical basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathbf{M}_2(\mathbf{C})$. It can be computed explicitly from the equation

$$x \cdot (I + \varepsilon E_i) = x + \varepsilon t_i$$

which follows from

$$(\rho \circ \varphi)(e_i) = x \cdot (I + \varepsilon E_i)$$

and

$$(\psi \circ d\rho_1)(e_i) = x + \varepsilon t_i .$$

Now let us explain how to choose N. Let $\theta \colon V_4 \times V_6 \xrightarrow{\sim} \mathbf{C}^{12}$ be the isomorphism defined by the canonical basis, $(t^4, 0), (t^3 s, 0), \ldots, (s^4, 0), (0, t^6), \ldots,$ of $V_4 \times V_6$. Let $t_i = (\sum \alpha^i_{kl} t^k s^l), (\sum \beta^i_{mn} t^m s^n)$. Let A be the matrix

$$\left(\begin{array}{c|c} \alpha^1_{4,0} \cdots \alpha^1_{0,4} & \beta^1_{6,0} \cdots \beta^1_{0,6} \\ \alpha^4_{0,4} \cdots \alpha^4_{0,4} & \beta^4_{6,0} \cdots \beta^4_{0,6} \end{array}\right).$$

The row space of A is $\operatorname{Im} d\rho_1$. We choose a square submatrix B of A such that $\det B \neq 0$. The submatrix B is gotten from A by deleting a row $(\alpha^i_{k,l})$ (resp. a row $(\beta^i_{m,n})$) if and only if $(k, l) \in D$ (resp. $(m, n) \in E$) for well determined sets D, E. The subspace $N = N_B$ generated by $(t^k s^l, 0)$, $(k, l) \in D$ and $(0, t^m s^n)$, $(m, n) \in E$ is a complement of $\operatorname{Im} d\rho_1$. This is obvious by considering their images under θ.

To calculate the matrix A we use the following formulas, where $f = \sum a f_{ij} t^i s^j$ is an element of V_n.

$$\begin{aligned} f \cdot (I + \varepsilon E_1) &= f + \varepsilon \sum f_{ij} i t^i s^j , \\ f \cdot (I + \varepsilon E_2) &= f + \varepsilon \sum f_{ij} i t^{i-1} s^{j+1} , \\ f \cdot (I + \varepsilon E_3) &= f + \varepsilon \sum f_{ij} j t^{i+1} s^{j-1} , \\ f \cdot (I + \varepsilon E_4) &= f + \varepsilon \sum f_{ij} j t^i s^j . \end{aligned}$$

We indicate the explicit choice of the square submatrix B in each case by underlining the corresponding columns of A. The reader should keep in mind Table 2.

Case A. By setting

$$\begin{aligned} \kappa &= -(1 + k^2), \\ \lambda &= k^2, \\ \mu &= -(m^2 + n^2 + p^2), \\ \nu &= m^2 n^2 + m^2 p^2 + n^2 p^2, \\ \pi &= -m^2 n^2 p^2, \end{aligned}$$

the general element x can be written

$$\begin{cases} a = t^4 + \kappa t^2 s^2 + \lambda s^4 , \\ b = c(t^6 + \mu t^4 s^2 + \nu t^2 s^4 + \pi s^6) . \end{cases}$$

The matrix A has the form

$$\left[\begin{array}{ccccc|ccccccc} 4 & 0 & 2\kappa & 0 & 0 & 6c & 0 & 4\mu c & 0 & 2\nu c & 0 & 0 \\ 0 & 4 & 0 & 2\kappa & 0 & 0 & 6c & 0 & 4\mu c & 0 & 4\nu c & 0 \\ 0 & 2\kappa & 0 & 4\lambda & 0 & 0 & 2\mu c & 0 & 4\nu c & 0 & 6\pi c & 0 \\ \underline{0} & \underline{0} & \underline{2\kappa} & \underline{0} & 4\lambda & 0 & 0 & 2\mu c & 0 & 4\nu c & 0 & 6\pi c \end{array}\right]$$

and the submatrix B consists of the underlined columns. It is clear that $\det B \neq 0$ for k, m, n, p general enough.

The action of $g = \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$ on N_B is given by

$$\begin{array}{cccccccc} s^4; & t^6, & t^5 s, & t^4 s^2, & t^3 s^3, & t^2 s^4, & ts^5, & s^6 \\ +1, & +1, & -1, & +1, & -1, & +1, & -1, & +1 \end{array}$$

where the first line is the canonical basis of N_B which consists of eigenvectors of g and the second line are the corresponding eigenvalues.

In the following cases we just indicate the matrices A, B.

Case B. Here

$$\kappa = -(1+k^2), \quad \lambda = k^2, \quad \mu = -(m^2+n^2), \quad \nu = m^2 n^2.$$

$$\left[\begin{array}{ccccc|ccccccc} 4 & 0 & 2\kappa & 0 & 0 & 0 & 6c & 0 & 3\mu c & 0 & \nu c & 0 \\ 0 & 4 & 0 & 2\kappa & 0 & 0 & 0 & 6c & 0 & 3\mu c & 0 & \nu c \\ 0 & 2\kappa & 0 & 4 & 0 & c & 0 & 3\mu c & 0 & 5\nu c & 0 & 0 \\ \underline{0} & 0 & 2\kappa & 0 & \underline{4} & \underline{0} & c & 0 & 3\mu c & 0 & 5\nu c & \underline{0} \end{array}\right].$$

Case C. Here

$$\mu = (m^3+n^3), \qquad \nu = m^3 n^3.$$

$$\left[\begin{array}{ccccc|ccccccc} 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 & 3\mu c & 0 & 0 & 6\nu c \\ 0 & 3 & 0 & 0 & 0 & 6c & 0 & 0 & 3\mu c & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 6c & 0 & 0 & 3\mu c & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & \underline{0} & \underline{0} & \underline{3\mu c} & 0 & 0 & 6\nu c & \underline{0} \end{array}\right].$$

Case D.

$$\left[\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 5c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & -6c & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \underline{c} & \underline{0} & 0 & 0 & \underline{0} & -\underline{6c} \end{array}\right].$$

Case E.

$$\left[\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{0} & \underline{3} & \underline{0} & 0 & 0 & \underline{2} & 0 & 0 & 0 & 0 \end{array}\right].$$

Case F.

$$\left[\begin{array}{ccccc|ccccccc} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{5} & 0 \end{array}\right].$$

Case G. Here

$$\begin{aligned} \mu &= -1 - m - n - p, \\ \nu &= m + n + p + mn + mp + np, \\ \pi &= pmn - mp - np - mnp, \\ \rho &= mnp, \end{aligned}$$

$$\left[\begin{array}{c|ccccccc} & 0 & 5 & 4\mu & 3\nu & 2\pi & \rho & 0 \\ 0 & 0 & 0 & 5 & 4\mu & 3\nu & 2\pi & \rho \\ & 0 & \mu & 2\nu & 3\pi & 4\rho & 0 & 0 \\ & 0 & \underline{0} & \mu & 2\nu & \underline{3\mu} & \underline{4\rho} & \underline{0} \end{array}\right].$$

The reader can check by specialization that the underlined matrix B has $\det B \neq 0$ for m, n, p general enough.

References

[Bo] A. Borel, *Linear Algebraic Groups*, Benjamin, 1969.

[Br] E. Brieskorn, *Rationale Singularitäten komplexer Flächen*, Invent. Math., **4** (1968), 336–358.

[Fu] W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. of Math., **90** (1969), 542–575.

[H-M] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67** (1982), 23–86.

[Ho] H. Holman, *Quotienten komplexer Räumen*, Math. Ann., **142** (1961), 407–440.

[Hu] A. Hurwitz, *Über Riemansche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann., **39** (1981), 1–61.

[Ka] A. Kas, *Weierstrass normal forms and invariants of elliptic surfaces*, Trans. Amer. Math. Soc., **225** (1977), 259–266.

[Kau] B. Kaup, *Äquivalenzrelationen auf allgemeinen komplexen Räumen*, Schriftenreihe Math. Institut Univ. Münster, Heft **39** (1968).

[Ko1] K. Kodaira, *On compact analytic surfaces*, II Ann. of Math., **77** (1963), 563–626.

[Ko2] ——, *On compact analytic surfaces*, III Ann. of Math., **78** (1963), 1–4.

[Le1] P. Lejarraga, *The moduli of elliptic surfaces*, thesis, Brandeis, 1985.

[Le2] ——, *The moduli of rational Weierstrass fibrations over* $\mathbf{P}^1$: *stratification*, Comm. Algebra, **20** (6), (1992), 1557–1590.

[Le3] ——, *The moduli of Weierstrass fibrations over* $\mathbf{P}^1$: *rationality*, to appear in The Rocky Mountain J. Math.

[Mi] R. Miranda, *The moduli of Weierstrass fibrations over* $\mathbf{P}^1$, Math. Ann., **255** (1981), 379–394.

[GIT] D. Mumford, *Geometric Invariant Theory*, Springer, 1963.

[M-S] D. Mumford and K. Suominen, *Introduction to the theory of moduli*, 5th Nordic Summer School in Mathematics, Oslo, 1970.

[Ne] A. Néron, *Modèles minimaux de variétés abéliennes sur les corps locaux et globaux*, Pub. Math. IHES, 1964.

[Pr] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J., **34** (1967), 375–386.

[Sch] G. Schwartz, *Lifting Smooth Homotopies of Orbit Spaces*, Publ. Math. IHES, No. 51, (1980), 37–135.

[Sei1] W. K. Seiler, *Moduli elliptischer Flächen mit Schnitt*, thesis, Universität Karlsruhe, 1982.

[Sei2] ——, *Global moduli for elliptic surfaces with a section*, Compositio Math., **62** (1987), 169–185.

[Sei3] ——, *Global moduli for polarized elliptic surfaces*, Compositio Math., **62** (1987), 187–213.

[Sei4] ——, *Deformations of Weierstrass elliptic surfaces*, Math. Ann., **281** (1988), 263–278.

[Se] J. P. Serre, *Linear Representations of Finite Groups*, Springer, 1977.

Received February 12, 1991.

St. John's College
Annapolis, MD 21404
E-mail address: pablo@math2.sma.usna.navy.mil