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ONE-FIXED-POINT ACTIONS ON SPHERES AND SMITH SETS

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Abstract

Let G be a finite group. The Smith equivalence for real G-modules of finite dimension gives a subset of real representation ring, called the primary Smith set. Since the primary Smith set is not additively closed in general, it is an interesting problem to find a subset which is additively closed in the real representation ring and occupies a large portion of the primary Smith set. In this paper we introduce an additively closed subset of the primary Smith set by means of smooth one-fixed-point G-actions on spheres, and we give evidences that the subset occupies a large portion of the primary Smith set if G is an Oliver group.

1. Introduction

Let G be a finite group. Real G-modules (of finite dimension) V and W are said to be *Smith equivalent* if there exists a homotopy sphere Σ with a smooth G-action such that Σ^G consists of exactly two points, say a and b, and the tangential G-representations at a and b are isomorphic to V and W, respectively. The subset $\mathrm{Sm}(G)$ of the real representation ring $\mathrm{RO}(G)$ consisting of all elements $x = [V] - [W]$ such that V and W are Smith equivalent real G-modules is called the *Smith set* of G. The subset is not additively closed in $\mathrm{RO}(G)$ if G has a quotient G/N, where $N \lhd G$, isomorphic to a cyclic group of order 8, see [2, Theorem II], [3, Theorem A], [22, p. 194, Theorem 0.2], and [15, Theorem 1].

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. For a subset $\mathcal{F}$ of $\mathcal{S}(G)$, a pair (V, W) of real G-modules is called $\mathcal{F}$-matched if $\mathrm{res}^G_H V$ and $\mathrm{res}^G_H W$ are isomorphic for any $H \in \mathcal{F}$. Let A be a subset of $\mathrm{RO}(G)$ and let $\mathcal{F}$ and $\mathcal{G}$ be subsets of $\mathcal{S}(G)$. Then we denote by $A^{\mathcal{G}}_{\mathcal{F}}$ the subset of A consisting of all elements $x \in A$ which can be written in the form $x = [V] - [W]$ such that $V^L = 0 = W^L$ for all $L \in \mathcal{G}$ and (V, W) is $\mathcal{F}$-matched. Thus the equality

$$A^{\mathcal{G}}_{\mathcal{F}} = \mathrm{RO}(G)^{\mathcal{G}}_{\mathcal{F}} \cap A \tag{1.1}$$

holds. We use the abbreviations $A_{\mathcal{F}}$ and $A^{\mathcal{G}}$ for $A^{\emptyset}_{\mathcal{F}}$ and $A^{\mathcal{G}}_{\emptyset}$, respectively.

For a prime p, let $\mathcal{P}_p(G)$ denote the set of all subgroups with p-power order of G and let $G^{\{p\}}$ denote the smallest normal subgroup N of G with p-power index. Let

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$\mathcal{P}(G)$ denote the union of $\mathcal{P}_p(G)$ over all primes p and let $\mathcal{L}(G)$ denote the set of all subgroups H of G such that $H \supset G^{\{p\}}$ for some prime p. The subset $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ of $\mathrm{Sm}(G)$ is called the *primary Smith set* of G. The difference $\mathrm{Sm}(G) \setminus \mathrm{Sm}(G)_{\mathcal{P}(G)}$ is a finite set [15, Theorem 1]. It immediately implies the next fact.

Proposition. *Let A be a subset of* $\mathrm{Sm}(G)$. *If A is additively closed in* $\mathrm{RO}(G)$ *then it is included in* $\mathrm{Sm}(G)_{\mathcal{P}(G)}$.

A finite group G is called an *Oliver group* if it is not a mod-$\mathcal{P}$ hyperelementary group, i.e. if G never admits a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime power order and H/P is cyclic, cf. [6, Section 0, p. 480]. If G is an Oliver group with a normal Sylow 2-subgroup and has a quotient isomorphic to a cyclic group of order pqr for some distinct odd primes p, q and r, then $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is not an additively closed subset of $\mathrm{RO}(G)$ [13, Corollary 1.2.1]. We recall the definition of *gap group* in Section 2. If G is a gap Oliver group then the realization theorem in Pawałowski–Solomon [19, p.850] implies that

$$\mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)},$$

and hence that $\mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ is additively closed. In general, the set $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is larger than $\mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. It is an interesting problem to find another subset of $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ which is additively closed in $\mathrm{RO}(G)$ and occupies a large portion of $\mathrm{Sm}(G)_{\mathcal{P}(G)}$. In this paper, we give such a subset.

Let $\mathfrak{S}^{(1)}_{\mathrm{h,w}}(G)$ denote the family of all smooth G-actions on homotopy spheres having exactly one fixed point and satisfying the $\mathcal{P}(G)$-weak gap condition. The definition of $\mathcal{P}(G)$-*weak gap condition* will be given in Section 2. Let $\mathrm{VO}(G)$ denote the family of all real G-modules obtainable as the tangential representations $T_a(\Sigma)$ of $\Sigma \in \mathfrak{S}^{(1)}_{\mathrm{h,w}}(G)$, where $\{a\} = \Sigma^G$. Now we define

$$\mathrm{DO}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V, W \in \mathrm{VO}(G)\} \cup \{0\}.$$

If G is not an Oliver group then $\mathrm{VO}(G) = \emptyset$ and $\mathrm{DO}(G) = \{0\}$. Let $G^{\cap 2}$ denote the intersection of all subgroups of G with index 1 or 2. By [9, Lemma 2.1], we see

$$\mathrm{VO}(G) = \mathrm{VO}(G)^{\{G^{\cap 2}\}} \quad \text{and} \quad \mathrm{DO}(G) = \mathrm{DO}(G)^{\{G^{\cap 2}\}}. \tag{1.2}$$

Moreover, by [12, Theorem 2.1] and [9, Proposition 2.2], we have the inclusions

$$\mathrm{DO}(G)_{\mathcal{P}(G)} \subset \mathrm{Sm}(G)_{\mathcal{P}(G)} \subset \mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)}. \tag{1.3}$$

If G is a weak gap Oliver group in the sense of [11], then we have

$$\mathrm{DO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}. \tag{1.4}$$

As an answer to the problem raised above, we will prove the next result in this paper.

Theorem 1.1. *The set* $\mathrm{DO}(G)$ *is an additive subgroup of* $\mathrm{RO}(G)$, *and hence so is* $\mathrm{DO}(G)_{\mathcal{P}(G)}$.

The following two theorems indicate that $\mathrm{DO}(G)_{\mathcal{P}(G)}$ occupies a large portion of $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ if G is an Oliver group. Let G^{nil} denote the intersection of $G^{\{p\}}$, where p ranges over all primes dividing $|G|$, cf. [6, Section 2, p. 486].

Theorem 1.2. *Let G be a gap Oliver group. If there exists a pair (V, W) of real G-modules such that $V^{G^{\mathrm{nil}}} = 0 = W^{G^{\mathrm{nil}}}$ and $(\mathbb{R} \oplus V, W)$ is $\mathcal{P}(G)$-matched, then the equalities*

$$\mathrm{DO}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)} \tag{1.5}$$

hold.

We remark that using [10, Lemma 4.6], T. Sumi independently proved the equality $\mathrm{RO}^{\{G^{\cap 2}\}}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)}$ of this theorem.

Theorem 1.3. *Let G be an Oliver group satisfying $G = G^{\{2\}}$, hence $G = G^{\cap 2} = G^{\{2\}}$. If there exists a pair (V, W) of real G-modules such that $V^{G^{\mathrm{nil}}} = 0 = W^{G^{\mathrm{nil}}}$ and $(\mathbb{R} \oplus V, W)$ is $\mathcal{P}_2(G)$-matched, then the equalities*

$$\mathrm{DO}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)} \tag{1.6}$$

hold.

2. Gap and weak gap conditions

Let M be a smooth G-manifold and let H and K be subgroups of G such that $H < K$. The H- and K-fixed point sets M^H and M^K are the disjoint unions of connected components M^H_i and M^K_j, respectively. We say that M satisfies the *gap condition* (resp. the *weak gap condition*) for (H, K) if the inequality

$$\dim M^H_i > 2 \dim M^K_j \tag{2.1}$$

$$(\text{resp. } \dim M^H_i \geq 2 \dim M^K_j) \tag{2.2}$$

holds whenever $M^H_i \supset M^K_j$. Let $\mathcal{F}$ be a set of subgroups of G. If M satisfies the gap condition (resp. the weak gap condition) for all (H, K) with $H \in \mathcal{F}$ and $H < K \leq G$ then we say that M satisfies the *$\mathcal{F}$-gap condition* (resp. the *$\mathcal{F}$-weak gap condition*).

For a real G-module V and a set $\mathcal{G}$ of subgroups of G, if the triviality $V^H = 0$ holds for all $H \in \mathcal{G}$ then we say that V is *$\mathcal{G}$-free*. If G has an $\mathcal{L}(G)$-free real G-module satisfying the $\mathcal{P}(G)$-gap condition then G is called a *gap group*.

Proposition 2.1. *Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Let V, W, V_1 and W_1 be $\mathcal{L}(G)$-free real G-modules such that $[V] - [W] = [V_1] - [W_1]$ in* $\mathrm{RO}(G)$. *If V_1 and W_1 satisfy the $\mathcal{P}(G)$-weak gap condition then there exists an $\mathcal{L}(G)$-free real G-module U such that $V \oplus U$ and $W \oplus U$ satisfy the $\mathcal{P}(G)$-weak gap condition.*

Proof. Let l be a natural number such that $l \geq \max(\dim V, \dim W)$, and set $V_2 = V_1 \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus l}$ and $W_2 = W_1 \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus l}$. These V_2 and W_2 are $\mathcal{L}(G)$-free real G-modules satisfying the $\mathcal{P}(G)$-weak gap condition. We can regard $V \subset V_2$ and $W \subset W_2$ up to isomorphisms. We can readily check the equality $([V_2] - [V]) - ([W_2] - [W]) = 0$, and hence the real G-modules $V_2 - V$ and $W_2 - W$ are isomorphic. Set $U = V_2 - V$. Then U is an $\mathcal{L}(G)$-free real G-module, and $V \oplus U = V_2$ and $W \oplus U \cong W_2$ satisfy the $\mathcal{P}(G)$-weak gap condition. □

If for any $\mathcal{P}(G)$-matched pair (V, W), i.e. $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ for all $P \in \mathcal{P}(G)$, of $\mathcal{L}(G)$-free real G-modules, there exists an $\mathcal{L}(G)$-free real G-module U such that $V \oplus U$ and $W \oplus U$ both satisfy the $\mathcal{P}(G)$-weak gap condition, then G is called a *weak gap group*. This definition of weak gap group agrees with that given in [11, p. 627] by Proposition 2.1 under the condition $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. It is readily shown that any gap group G is a weak gap group. Further information of gap groups is found in [16, 28, 29, 30].

Let $\mathcal{G}$ be a set of subgroups of G and V a real G-module. Then we denote by $V^{\mathcal{G}}$ the smallest $\mathbb{R}$-submodule of V such that $V^{\mathcal{G}} \supset V^L$ for all $L \in \mathcal{G}$. With respect to some G-invariant inner product on V, V is decomposed to the direct sum

$$V = V^{\mathcal{G}} \oplus V_{\mathcal{G}}.$$

If all minimal elements of $\mathcal{G}$ are normal in G then $V^{\mathcal{G}}$ and $V_{\mathcal{G}}$ are real G-modules. In the case where $\mathcal{G} = \{H\}$, we use the abbreviations V^H and V_H for $V^{\mathcal{G}}$ and $V_{\mathcal{G}}$, respectively. We can regard V^H and V_H as real $N_G(H)$-modules. By [6, Theorem 2.3], we see that $\mathbb{R}[G]_{\mathcal{L}(G)}$ satisfies the $\mathcal{M}(G)$-weak gap condition, where $\mathcal{M}(G) = \mathcal{S}(G) \setminus \mathcal{L}(G)$.

3. Preliminaries

In this section we describe two lemmas which have not been stated so far but are readily obtained from known results. These lemmas are useful in our study of Smith equivalence.

Lemma 3.1. *Let G be an Oliver group, let V be an $\mathcal{L}(G)$-free real G-module satisfying the $\mathcal{P}(G)$-weak gap condition, and set $W = V \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 3}$. Then W belongs to*

VO(G), *i.e. there exists a smooth G-action on a homotopy sphere Σ such that $\Sigma^G = \{a\}$, $T_a(\Sigma) \cong W$, and Σ satisfies the $\mathcal{P}(G)$-weak gap condition.*

Proof. First recall that $\mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 3}$ is admissible in the sense of [11, Definition 6.1] and facilitates G-surgery. Let Y be the unit disk of W with respect to some G-invariant inner product on W. Then Y satisfies the $\mathcal{P}(G)$-weak gap condition. By [11, Theorem 5.1], we obtain a G-framed map

$$\boldsymbol{f}' = (f' \colon D \to Y, b' \colon T(D) \oplus \varepsilon_D(\mathbb{R}^l) \to f'^* T(Y) \oplus \varepsilon_D(\mathbb{R}^l))$$

such that $\partial f' \colon \partial D \to \partial Y$ is the identity map on ∂Y, $f' \colon D \to Y$ is a homotopy equivalence, and $D^G = \emptyset$. Then D satisfies the $\mathcal{P}(G)$-weak gap condition because $\dim T_x(D)^H = \dim T_{f'(x)}(Y)^H$ for all $x \in D$ and $H \in \mathcal{S}(G_x)$, where G_x is the isotropy subgroup of G at x in D. Thus, the manifold $\Sigma = Y \cup_\partial D$ obtained by gluing Y and D along the boundaries is a homotopy sphere, Σ^G consists of one point, $T_a(\Sigma)$, where $a \in \Sigma^G$, is isomorphic to W, and Σ satisfies the $\mathcal{P}(G)$-weak gap condition. □

Theorem 3.2 (cf. [11, Theorem 1.8])**.** *If G is a weak gap Oliver group then the equalities*

$$\mathrm{DO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \tag{3.1}$$

hold.

Proof. By (1.3), it suffices to show $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{DO}(G)$. Let $x \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. We can write x in the form $x = [V] - [W]$ with $\mathcal{L}(G)$-free real G-modules V and W. Since G is a weak gap group, there exists an $\mathcal{L}(G)$-free real G-module U such that $V \oplus U$ and $W \oplus U$ satisfy the $\mathcal{P}(G)$-weak gap condition. By Lemma 3.1, $V_1 = V \oplus U \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 3}$ and $W_1 = W \oplus U \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 3}$ belong to VO(G). Clearly we have $x = [V_1] - [W_1]$. Thus x belongs to DO(G). □

We need the next for further study of DO(G).

Lemma 3.3. *Let G be an Oliver group and M a compact smooth G-manifold fulfilling the following conditions.*
(1) *For each $L \in \mathcal{L}(G)$, M^L is a closed manifold.*
(2) *M satisfies the $\mathcal{P}(G)$-weak gap condition.*
(3) *$[\mathrm{res}^G_{\{e\}} T(M)] = 0$ in $\widetilde{\mathrm{KO}}(M)$.*
(4) *For any prime p and $P \in \mathcal{P}_p(G)$, $[\mathrm{res}^G_P T(M)] = 0$ in $\widetilde{\mathrm{KO}}_P(M)_{(p)}$.*
Then there exist a natural number N such that for an arbitrary integer $n \geq N$, there exist smooth G-actions on a disk D and a sphere S possessing the following properties.
(i) *$M \subset D \subset S$.*

(ii) $D^G = M^G = S^G$.
(iii) *For any* $x \in M^G$, $T_x(D) \cong T_x(M) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus n} \cong T_x(S)$.
(iv) D *and* S *satisfy the* $\mathcal{P}(G)$*-weak gap condition.*
If M *satisfies the* $\mathcal{P}(G)$*-gap condition then* (iv) *can be improved to*
(iv)′ D *and* S *satisfy the* $\mathcal{P}(G)$*-gap condition.*

This lemma follows from [12, Section 4, Proof of Theorem 2.1].

4. Proof of Theorem 1.1

We can assume that G is an Oliver group without any loss of generality. Let x, $y \in \mathrm{DO}(G)$. We can write x and y in the form $x = [V_1] - [W_1]$ and $y = [V_2] - [W_2]$ with V_1, V_2, W_1, $W_2 \in \mathrm{VO}(G)$. Next take homotopy spheres Σ_i and Ξ_i with smooth G-actions such that $\Sigma_i^G = \{a_i\}$ and $\Xi_i^G = \{b_i\}$, $T_{a_i}(\Sigma_i) = V_i$ and $T_{b_i}(\Xi_i) = W_i$, and Σ_i and Ξ_i satisfy the $\mathcal{P}(G)$-weak gap condition, for $i = 1, 2$. Let M_i and N_i be the G-regular neighborhoods of

$$\bigcup_{L \in \mathcal{L}(G)} \Sigma_i^L$$

and

$$\bigcup_{L \in \mathcal{L}(G)} \Xi_i^L$$

in Σ_i and Ξ_i, respectively. Then M_i and N_i satisfy the conditions (1) and (2) in Lemma 3.3, and hence so do $M_1 \times M_2$ and $N_1 \times N_2$. By [12, Lemma 3.1 (7)], we can readily check that M_i and N_i, and furthermore $M_1 \times M_2$ and $N_1 \times N_2$ also, satisfy the conditions (3) and (4) in Lemma 3.3. By Lemma 3.3, there exist smooth G-actions on spheres S_1 and S_2 such that $S_1^G = \{a\}$ and $S_2^G = \{b\}$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, $T_a(S_1) \cong V_1 \oplus V_2$ and $T_b(S_2) \cong W_1 \oplus W_2$, and S_1 and S_2 satisfy the $\mathcal{P}(G)$-weak gap condition. Thus $x + y = [V_1 \oplus V_2] - [W_1 \oplus W_2]$ belongs to $\mathrm{DO}(G)$. $\square$

5. Proof of Theorem 1.2

It suffices to show that $\mathrm{RO}(G)_{\mathcal{P}(G)}^{\{G^{\cap 2}\}} \subset \mathrm{DO}(G)$. Let $x \in \mathrm{RO}(G)_{\mathcal{P}(G)}^{\{G^{\cap 2}\}}$. We can write x in the from $x = [M_1] - [M_3]$ with $\{G^{\cap 2}\}$-free real G-modules M_1 and M_3. In this proof, let K denote G^{nil}. For $i = 1, 3$, we set

$$\begin{aligned} N_i &= V \oplus ((\mathbb{R} \oplus V) \otimes M_i^K), \\ U_i &= \mathbb{R} \oplus N_i, \\ U_{i+1} &= W \oplus (W \otimes M_i^K). \end{aligned}$$

Then $N_1^K \cong M_1^K$, $U_2^K = 0$, $N_3^K \cong M_3^K$, $U_4^K = 0$, and (U_1, U_2) and (U_3, U_4) are $\mathcal{P}(G)$-matched. Then we can rewrite x in the form

$$x = [N_1 \oplus W_1] - [N_3 \oplus W_3]$$

for some $\{K\}$-free real G-modules W_1 and W_3. Set

$$V_1 = N_1 \oplus W_1$$

and

$$V_3 = N_3 \oplus W_3.$$

Then we have $V_1 = (U_1 - \mathbb{R}) \oplus W_1$ and $V_3 = (U_3 - \mathbb{R}) \oplus W_3$. Thus we get the situation required in [10, Proof of Lemma 4.6]. Let A be an $\mathcal{L}(G)$-free real G-module satisfying the $\mathcal{P}(G)$-gap condition. In [10, Proof of Lemma 4.6], for each $i = 1, 3$, and for sufficiently large (arbitrary) natural numbers a and b, we have obtained a disk D_i and a sphere Σ_i with smooth G-actions such that $D_i \subset \Sigma_i$, $D_i^G = \{x_i\} = \Sigma_i^G$, and

$$T_{x_i}(\Sigma_i) = V_i \oplus A^{\oplus a} \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus b}.$$

The G-manifold Σ_i is obtained by means of [10, Proof of Lemma 4.5] and hence Σ_i satisfies the $\mathcal{P}(G)$-gap condition. Thus $x = [T_{x_1}(\Sigma_1)] - [T_{x_3}(\Sigma_3)]$ belongs to $\mathrm{DO}(G)$. $\square$

6. Proof of Theorem 1.3

We prove Theorem 1.3 in a slightly generalized form, that is, we prove the next theorem.

Theorem 6.1. *Let G be an Oliver group and set $K = G^{\mathrm{nil}}$ and $N = G^{\{2\}}$. If there exists a pair (V, W) of $\{K\}$-free real N-modules such that $(\mathbb{R} \oplus V, W)$ is $\mathcal{P}_2(N)$-matched, then the inclusions*

$$\mathrm{ind}_N^G(\mathrm{RO}(N)^{\{N\}}_{\mathcal{P}(N)}) \subset \mathrm{DO}(G)^{\{N\}}_{\mathcal{P}(G)} \subset \mathrm{Sm}(G)^{\{N\}}_{\mathcal{P}(G)} \tag{6.1}$$

hold.

Proof. By the existence of the pair (V, W) stated in the theorem, K possesses a subquotient isomorphic to a dihedral group of order $2p$ with odd prime p. Thus any subgroup H containing K is not of prime power order.

It suffices to show that $\mathrm{ind}_N^G(\mathrm{RO}(N)^{\{N\}}_{\mathcal{P}(N)}) \subset \mathrm{DO}(G)$. Let $x \in \mathrm{RO}(N)^{\{N\}}_{\mathcal{P}(N)}$. We can write x in the from $x = [M_1] - [M_3]$ with $\{N\}$-free real N-modules M_1 and M_3. Since N/K is supersolvable, by [26, 8.5, Theorem 16] we can decompose M_i^K to the direct sum

$$M_i^K = \bigoplus_j \mathrm{ind}_{H_{i,j}}^N M_{i,j}$$

for some subgroups $H_{i,j} \triangleright K_{i,j} \supset K$ of N and faithful real $H_{i,j}/K_{i,j}$-modules $M_{i,j}$ of dimension 2, where $H_{i,j}/K_{i,j}$ are cyclic groups of odd order. For $i = 1, 3$, we set

$$\begin{aligned} N_{i,j} &= \operatorname{res}^N_{H_{i,j}} V \oplus ((\mathbb{R} \oplus \operatorname{res}^N_{H_{i,j}} V) \otimes M_{i,j}), \\ U_{i,j} &= \mathbb{R} \oplus N_{i,j}, \\ U_{i+1,j} &= \operatorname{res}^N_{H_{i,j}} W \oplus (\operatorname{res}^N_{H_{i,j}} W \otimes M_{i,j}). \end{aligned}$$

Then $N_{i,j}^{H_{i,j}} = 0$ and $(U_{i,j}, U_{i+1,j})$ is $\mathcal{P}_2(H_{i,j})$-matched for each $i = 1, 3$.

Let $P_{i,j} = P(U_{i,j}^K)$ be the real projective space associated with $U_{i,j}^K$, and $\gamma_{i,j}$ the canonical line bundle of $P_{i,j}$, where $i = 1, 3$. Let $\varepsilon_B(F)$ denote the product bundle over B with fiber F. Set

$$\begin{aligned} \tau_{i,j} &= \gamma_{i,j} \otimes U_{i,j}^K, \\ \nu_{i,j} &= (\gamma_{i,j} \otimes U_{i,j\,K}) \oplus (\gamma_{i,j}^{\perp} \otimes U_{i+1,j}), \\ \xi_{i,j} &= \tau_{i,j} \oplus \nu_{i,j} \quad (= (\gamma_{i,j} \otimes U_{i,j}) \oplus (\gamma_{i,j}^{\perp} \otimes U_{i+1,j})), \end{aligned}$$

where $\gamma_{i,j} \oplus \gamma_{i,j}^{\perp} = \varepsilon_{P_{i,j}}(U_{i,j}^K)$ and $i = 1, 3$. By [23, Theorem 2], or alternatively by [24, Theorem 1], we have $\gamma_{i,j}{}^{\oplus 4} \cong \varepsilon_{P_{i,j}}(\mathbb{R}^{\oplus 4})$ as real $H_{i,j}$-vector bundles. This implies $4[\gamma_{i,j}] = 0$, $4[\gamma_{i,j}^{\perp}] = 0$ and $4[\xi_{i,j}] = 0$ in $\widetilde{\mathrm{KO}}_{H_{i,j}}(P_{i,j})$. If P is a 2-subgroup of $H_{i,j}$ then

$$\begin{aligned} \operatorname{res}^{H_{i,j}}_P \xi_{i,j} &\cong \operatorname{res}^{H_{i,j}}_P (\gamma_{i,j} \oplus \gamma_{i,j}^{\perp}) \otimes \operatorname{res}^{H_{i,j}}_P U_{i,j} \\ &\cong \operatorname{res}^{H_{i,j}}_P \varepsilon_{P_{i,j}}(U_{i,j}^K \otimes U_{i,j}), \end{aligned}$$

and $[\xi_{i,j}] = 0$ in $\widetilde{\mathrm{KO}}_P(\operatorname{res}^{H_{i,j}}_P P_{i,j})$. Thus we get

$$[\xi_{i,j}] = 0 \quad \text{in} \quad \widetilde{\mathrm{KO}}(\operatorname{res}^{H_{i,j}}_{\{e\}} P_{i,j})$$

and

$$[\xi_{i,j}] = 0 \quad \text{in} \quad \widetilde{\mathrm{KO}}_P(\operatorname{res}^{H_{i,j}}_P P_{i,j})_{(p)} \quad \text{for all primes} \quad p \quad \text{and} \quad P \in \mathcal{P}_p(H_{i,j}).$$

Let $E_{i,j}$ be the total space of the disk bundle $D(\nu'_{i,j})$ associated with the real $H_{i,j}$-vector bundle

$$\nu'_{i,j} = \nu_{i,j} \oplus \varepsilon_{P_{i,j}}(\mathbb{R}[H_{i,j}]_K{}^{\oplus a_{i,j}}),$$

where $a_{i,j}$ is a natural number. For any $P \in \mathcal{P}(H_{i,j})$, since $K \not\subset P$, $\operatorname{res}^{H_{i,j}}_P \mathbb{R}[H_{i,j}]_K$ has a direct summand isomorphic to $\mathbb{R}[P]$. Thus for a sufficiently large natural number $a_{i,j}$, $E_{i,j}$ satisfies the following conditions.

(1) $E_{i,j}^K = P_{i,j}^K$.

(2) $T(E_{i,j})|_{P_{i,j}} \oplus \varepsilon_{P_{i,j}}(\mathbb{R}) \cong \xi_{i,j} \oplus \varepsilon_{P_{i,j}}(\mathbb{R}[H_{i,j}]_K{}^{\oplus a_{i,j}})$.

(3) $\mathrm{res}^{H_{i,j}}_{\{e\}} T(E_{i,j})$ is isomorphic to a product bundle.

(4) $\mathrm{res}^{H_{i,j}}_{P} T(E_{i,j})^{\oplus q_{i,j}(P)}$ are (P-equivariantly) isomorphic to product bundles for all primes p and $P \in \mathcal{P}_p(H_{i,j})$, where $q_{i,j}(P)$ is a natural number prime to p.

We consider the N-manifold

$$Z_i^{(1)} = \prod_j \mathfrak{ind}^N_{H_{i,j}} E_{i,j},$$

where $\mathfrak{ind}^N_{H_{i,j}} E_{i,j} = \mathrm{Map}_{H_{i,j}}(N, E_{i,j})$ is the multiplicative induction of $E_{i,j}$, i.e.

$$\mathrm{Map}_{H_{i,j}}(N, E_{i,j}) = \{ f : N \to E_{i,j} \mid f(ga^{-1}) = af(g) \text{ for all } g \in N,\ a \in H_{i,j}\}$$

with the N-action

$$N \times \mathrm{Map}_{H_{i,j}}(N, E_{i,j}) \to \mathrm{Map}_{H_{i,j}}(N, E_{i,j})$$

given by

$$(b, f) \mapsto bf; \quad (bf)(g) = f(b^{-1}g)$$

for $b \in N$, $f \in \mathrm{Map}_{H_{i,j}}(N, E_{i,j})$, and $g \in N$. The N-fixed point set of $Z_i^{(1)}$ consists of one point, x_i say. Then we can rewrite x in the form

$$x = [T_{x_1}(Z_1^{(1)}) \oplus W_1] - [T_{x_3}(Z_3^{(1)}) \oplus W_3]$$

with $\{K\}$-free real N-modules W_1 and W_3. Set $Z_i^{(2)} = Z_i^{(1)} \times D(W_i)$. It is clear that

(i) $\mathrm{res}^N_{\{e\}} T(Z_i^{(2)}) \cong \mathrm{res}^N_{\{e\}} \varepsilon_{Z_i^{(2)}}(\mathbb{R}^n)$ for some integer n,

(ii) $[\mathrm{res}^N_P T(Z_i^{(2)})] = 0$ in $\widetilde{\mathrm{KO}}_P(Z_i^{(2)})_{(p)}$ for all primes p and $P \in \mathcal{P}_p(N)$.

For sufficiently large (arbitrary) integer l,

$$Z_i^{(3)} = Z_i^{(2)} \times D(\mathbb{R}[N]_{\mathcal{L}(N,G)}{}^{\oplus l})$$

satisfies the $\mathcal{P}(N)$-gap condition, where

$$\mathcal{L}(N, G) = \{H \in \mathcal{S}(N) \mid H \supset N \cap G^{\{q\}} \text{ for some prime } q\}.$$

Set

$$Z_i = \mathfrak{ind}^G_N Z_i^{(3)}.$$

Thus by Lemma 3.3, for each sufficiently large (arbitrary) integer n, there exists a smooth G-action on a sphere S_i such that

(a) $S_i^G = Z_i^G$ $(= \{(x_i, 0)\})$,

(b) $T_{(x_i,0)}(S_i) \cong T_{x_i}(Z_i) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus n}$,

(c) S_i satisfies the $\mathcal{P}(G)$-weak gap condition.

Thus $\operatorname{ind}_N^G x = [T_{(x_1,0)}(S_1)] - [T_{(x_3,0)}(S_3)]$ belongs to $\mathrm{DO}(G)$. $\square$

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