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# ON SOME LENGTH PROBLEMS FOR ANALYTIC FUNCTIONS

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**Abstract**

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathbb{D} = \{z \colon |z| < 1\}$. Let $C(r)$ be the closed curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$, $L(r)$ the length of $C(r)$, and let $A(r)$ be the area enclosed by the curve $C(r)$. It was shown in [13] that if $f \in \mathcal{A}$, $f$ is starlike with respect to the origin, and for $0 \leq r < 1$, $A(r) < A$, an absolute constant, then

$$L(r) = \mathcal{O}\left(\log \frac{1}{1-r}\right) \quad \text{as} \quad r \to 1. \tag{0.1}$$

It is the purpose of this work to prove, using a modified methods than that in [13], a strengthened form of (0.1) for Bazilevič functions, strongly starlike functions and for close-to-convex functions.

## 1. Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent in $\mathcal{D}$.

If $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}$$

then $f(z)$ is said to be convex in $\mathbb{D}$ and denoted by $f(z) \in \mathcal{K}$.

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If $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}$$

then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{D}$ and denoted by $f(z) \in \mathcal{S}^*$.

Furthermore, If $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\} > 0, \quad z \in \mathbb{D} \tag{1.2}$$

for some $g(z) \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then $f(z)$ is said to be close-to-convex in $\mathbb{D}$ and denoted by $f(z) \in \mathcal{C}$. An univalent function $f \in \mathcal{S}$ belongs to $\mathcal{C}$ if and only if the complement $E$ of the image-region $F = \{f(z)\colon |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

On the other hand, if $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0, \quad z \in \mathbb{D}$$

for some $g(z) \in \mathcal{S}^*$ and some $\beta \in (0, \infty)$, then $f(z)$ is said to be a Bazilevič function of type $\beta$ and denoted by $f(z) \in \mathcal{B}(\beta)$.

Let $\mathcal{SS}^*(\alpha)$ denote the class of strongly starlike functions of order $\alpha$, $0 < \alpha \le 1$,

$$\mathcal{SS}^*(\alpha) := \left\{f \in \mathcal{A}\colon \left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D}\right\},$$

which was introduced in [12] and [1].

Let $C(r)$ be the closed curve which is the image of $|z| = r < 1$ under the mapping $w = f(z)$. Let $L(r)$ denote the length of $C(r)$ and let $A(r)$ be the area enclosed by $C(r)$.

Let us define $M(r)$ by

$$M(r) = \max_{|z|=r<1} |f(z)|.$$

Then F.R. Keogh [4] has shown that

**Theorem 1.1.** *Suppose that* $f(z) \in \mathcal{S}^*$ *and*

$$|f(z)| \le M < \infty, \quad z \in \mathbb{D}.$$

*Then we have*

$$L(r) = \mathcal{O}\left(\log\frac{1}{1-r}\right) \quad \textit{as} \quad r \to 1,$$

*where $\mathcal{O}$ means Landau's symbol.*

Furthermore, D.K. Thomas in [13] extended this result for bounded close-to-convex functions. Ch. Pommerenke in [9] has shown that

**Theorem 1.2.** *If $f(z) \in \mathcal{C}$, then*

$$L(r) = \mathcal{O}\left\{M(r)\left(\log \frac{1}{1-r}\right)^{5/2}\right\} \quad as \quad r \to 1.$$

Later, D.K. Thomas in [14] has shown that

**Theorem 1.3.** *If $f(z) \in \mathcal{S}^*$, then*

$$L(r) = \mathcal{O}\left\{\sqrt{A(r)} \log \frac{1}{1-r}\right\} \quad as \quad r \to 1.$$

M. Nunokawa in [6, 7] has shown that

**Theorem 1.4.** *If $f(z) \in \mathcal{K}$, then*

$$L(r) = \mathcal{O}\left\{A(r) \log \frac{1}{1-r}\right\}^{1/2} \quad as \quad r \to 1.$$

Moreover, D.K. Thomas in [15] has shown the following two theorems

**Theorem 1.5.** *If $f(z) \in \mathcal{B}(\beta)$ and $|f(z)| < 1$ in $\mathbb{D}$, then we*

$$L(r) = \mathcal{O}\left(\log \frac{1}{1-r}\right) \quad as \quad r \to 1.$$

**Theorem 1.6.** *If $f(z) \in \mathcal{B}(\beta)$ and $0 < \beta \leq 1$, then we*

$$L(r) = \mathcal{O}\left(M(r) \log \frac{1}{1-r}\right) \quad as \quad r \to 1.$$

M. Nunokawa, S. Owa et al. in [8] have shown that

**Theorem 1.7.** *If $f(z) \in \mathcal{B}(\beta)$ and $zf'(z) = f^{1-\beta}(z)g^{\beta}(z)h(z)$, then we*

$$L(r) = \mathcal{O}\left\{\sqrt{A^{1-\beta}(r)G^{\beta}(r)}\left(\log \frac{1}{1-r}\right)^2\right\} \quad as \quad r \to 1,$$

*where*

$$G(r) = \int_0^r \int_0^{2\pi} \varrho |g'(\varrho e^{i\theta})|^2 \,\mathrm{d}\theta \,\mathrm{d}\varrho$$

*or* $G(r)$ *is the area of the image domain of* $|z| \leq r$ *under the starlike mapping* $g$.

Ch. Pommerenke in [9] has also shown that

**Theorem 1.8.** *If* $f(z) \in \mathcal{S}$, *then*

$$M(r) \leq 4\sqrt{\frac{A(r)}{\pi} \log \frac{3}{1-r}} \quad (|z| = r < 1). \tag{1.3}$$

*Therefore, we have*

$$M(r) = \mathcal{O}\left\{A(r) \log \frac{1}{1-r}\right\}^{1/2} \quad as \quad r \to 1.$$

It is the purpose of this work to prove, using a modified methods than that in [13], a strengthened form of (0.1) for Bazilevič functions, strongly starlike functions and for close-to-convex functions.

## 2. Lemmas

**Lemma 2.1.** *If* $h(z)$ *is analytic and* $\mathfrak{Re}\{h(z)\} > 0$ *in* $\mathbb{D}$ *with* $h(0) = 1$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^2 \,\mathrm{d}\theta \leq \frac{1+3r^2}{1-r^2} < \frac{4}{1-r^2}$$

*for* $0 < r < 1$.

Lemma 2.1 can be easily proved using $|h^{(n)}(0)| \leq 2n!$ and the Gutzmer's theorem, see for example [3, p. 31].

**Lemma 2.2.** *If* $f(z) \in \mathcal{S}$, *then we have*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|} < \frac{2}{1-|z|} \quad in \quad \mathbb{D},$$

$$|f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \quad in \quad \mathbb{D}.$$

A proof can be found in [10, p. 21].

**Lemma 2.3** ([2, p. 337]). *If $h(z)$ is analytic and $\mathfrak{Re}\{h(z)\} > 0$ in $\mathbb{D}$ with $h(0) = 1$, then we have*

$$|h'(z)| \leq \frac{2\,\mathfrak{Re}\{h(z)\}}{1-|z|^2} < \frac{2}{1-|z|} \quad \text{in} \quad \mathbb{D}. \tag{2.1}$$

A proof can be found also in [5].

An analytic function $f$ is said to be subordinate to an analytic function $F$, or $F$ is said to be superordinate to $f$, if there exists a function an analytic function $w$ such that

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

and

$$f(z) = F(w(z)) \quad (z \in \mathbb{D}).$$

In this case, we write $f \prec F$ $(z \in \mathbb{D})$ or $f(z) \prec F(z)$ $(z \in \mathbb{D})$. If the function $F$ is univalent in $\mathbb{D}$, then we have

$$[f \prec F \ (z \in \mathbb{D})] \Leftrightarrow [f(0) = F(0) \text{ and } f(\mathbb{D}) \subset F(\mathbb{D})].$$

**Lemma 2.4.** *If $f(z)$ is subordinate to $g(z)$ in $\mathbb{D}$ and if $0 < p$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta$$

*for all $r$, $0 < r < 1$.*

W. Rogosinski has shown Lemma 2.4 in [11].

## 3. Main results

**Theorem 3.1.** *If $f(z) \in \mathcal{S}$ satisfies the condition*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f(z)}\right\} \geq -\,\mathfrak{Re}\left\{\frac{1+z}{1-z}\right\} \quad \text{in} \quad \mathbb{D}, \tag{3.1}$$

*then we have*

$$L(r) = \mathcal{O}\left\{A(r)\log\frac{1}{1-r}\right\}^{1/2} \quad \text{as} \quad r \to 1. \tag{3.2}$$

*Proof.* For the case $0 < r \leq 1/2$, from Lemma 2.2 we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)| \, d\theta \\ &\leq \int_0^{2\pi} \frac{|z|(1+|z|)}{(1-|z|)^3} \, d\theta \\ &< 12\pi. \end{aligned}$$

For the case $1/2 < r < 1$, we have

$$
\begin{aligned}
L(r) &= \int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^r |f'(z) + zf''(z)|\,\mathrm{d}\varrho\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^r \left| f'(z)\left(1 + \frac{zf''(z)}{f(z)}\right)\right| \mathrm{d}\varrho\,\mathrm{d}\theta \\
&\le \left(\int_0^{2\pi}\int_0^r |f'(z)|^2\,\mathrm{d}\varrho\,\mathrm{d}\theta\right)^{1/2} \left(\int_0^{2\pi}\int_0^r \left|1 + \frac{zf''(z)}{f(z)}\right|^2 \mathrm{d}\varrho\,\mathrm{d}\theta\right)^{1/2} \\
&< \left(2\int_0^{2\pi}\int_0^r \varrho|f'(z)|^2\,\mathrm{d}\varrho\,\mathrm{d}\theta\right)^{1/2} \left(\int_0^{2\pi}\int_0^r \left|1 + \frac{zf''(z)}{f(z)}\right|^2 \mathrm{d}\varrho\,\mathrm{d}\theta\right)^{1/2} \\
&= \sqrt{2A(r)}\left(\int_0^{2\pi}\int_0^r \left|1 + \frac{zf''(z)}{f(z)}\right|^2 \mathrm{d}\varrho\,\mathrm{d}\theta\right)^{1/2}.
\end{aligned}
$$

From the hypothesis (3.1), we have

$$
\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0 \quad \text{in} \quad \mathbb{D}
$$

or

$$
\frac{1 + zf''(z)/f(z) + (1+z)/(1-z)}{2} \prec \frac{1+z}{1-z} \quad \text{in} \quad \mathbb{D}. \tag{3.3}
$$

It follows that

$$
1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z} \prec 2\frac{1+z}{1-z} \quad \text{in} \quad \mathbb{D},
$$

where the symbol $\prec$ means the subordination. Then we have

$$
\begin{aligned}
&\int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)}\right|^2 \mathrm{d}\theta\,\mathrm{d}\varrho \\
&= \int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z} - \frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\,\mathrm{d}\varrho \\
&\le \int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\,\mathrm{d}\varrho \\
&\quad + 2\int_0^r\int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z}\right|\left|\frac{1+z}{1-z}\right| \mathrm{d}\theta\,\mathrm{d}\varrho \\
&\quad + \int_0^r\int_0^{2\pi} \frac{|1+z|^2}{|1-z|^2}\,\mathrm{d}\theta\,\mathrm{d}\varrho \\
&= I_1 + 2I_2 + I_3.
\end{aligned}
$$

From Lemma 2.4, (3.3) and Lemma 2.1, we have

$$\begin{aligned}
I_1 &= \int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\, \mathrm{d}\varrho \\
&\le \int_0^r \int_0^{2\pi} 4\left|\frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\, \mathrm{d}\varrho \\
&< 32\pi \int_0^r \frac{1}{1-\varrho^2}\, \mathrm{d}\varrho \\
&= 16\pi \log\frac{1+r}{1-r}.
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
2I_2 &= \left(\int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f(z)} + \frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\, \mathrm{d}\varrho\right)^{1/2} \left(\int_0^r \int_0^{2\pi} \left|\frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\, \mathrm{d}\varrho\right)^{1/2} \\
&\le \left(16\pi \log\frac{1+r}{1-r}\right)^{1/2} \left(8\pi \int_0^r \frac{1}{1-\varrho^2}\, \mathrm{d}\varrho\right)^{1/2} \\
&= \left(16\pi \log\frac{1+r}{1-r}\right)^{1/2} \left(4\pi \log\frac{1+r}{1-r}\right)^{1/2} \\
&= \mathcal{O}\left(\log\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1.
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
I_3 &= \int_0^r \int_0^{2\pi} \left|\frac{1+z}{1-z}\right|^2 \mathrm{d}\theta\, \mathrm{d}\varrho \\
&= 4\pi \log\frac{1+r}{1-r} \\
&= \mathcal{O}\left(\log\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1.
\end{aligned}$$

This shows (3.2) which completes the proof of Theorem 3.1. □

**Theorem 3.2.** *If $f(z) \in \mathcal{B}(\beta)$ is a Bazilevič function of type $\beta$, $0 < \beta \le 1$, then we have*

$$L(r) = \mathcal{O}\left\{A(r)\left(\log\frac{1}{1-r}\right)^{3/2}\right\} \quad \text{as} \quad r \to 1. \tag{3.4}$$

Proof. Because $f(z) \in \mathcal{B}(\beta)$, there exists $g(z) \in \mathcal{S}^*$ and there exists an analytic function $h(z)$, $h(0) = 1$, $\mathfrak{Re}\{h(z)\} > 0$ in $\mathbb{D}$, such that

$$zf'(z) = f^{1-\beta}(z) g^{\beta}(z) h(z). \tag{3.5}$$

Therefore we have

$$\begin{aligned}
L(r) &= \int_0^{2\pi} |zf'(z)|\, \mathrm{d}\theta \\
&= \int_0^{2\pi} |f^{1-\beta}(z)g^{\beta}(z)h(z)|\, \mathrm{d}\theta \\
&\le M^{1-\beta}(r)\int_0^{2\pi} |g^{\beta}(z)h(z)|\, \mathrm{d}\theta \\
&\le M^{1-\beta}(r)\left\{\int_0^r\int_0^{2\pi} \beta|g^{\beta-1}(z)g'(z)h(z)|\, \mathrm{d}\theta\, \mathrm{d}\varrho + \int_0^r\int_0^{2\pi} |g^{\beta}(z)h'(z)|\, \mathrm{d}\theta\, \mathrm{d}\varrho\right\} \\
&\le M^{1-\beta}(r)(I_1(r)+I_2(r)).
\end{aligned}$$

Applying Ch. Pommerenke's result (1.3), we have

$$L(r) \le \left(\frac{16}{\pi}A(r)\log\frac{3}{1-r}\right)^{(1-\beta)/2}(I_1(r)+I_2(r)).$$

D.K. Thomas in [15] has shown that if $f(z)$ is a Bazilevič function of type $\beta$, $0<\beta$, then

$$\begin{aligned}
I_1(r) &\le 2\sqrt{2\pi}\,\beta K(\beta)\left(\frac{1}{r}\log\frac{1+r}{1-r}\right)^{1/2} \\
&= \mathcal{O}\left\{\left(\log\frac{1}{1-r}\right)^{1/2}\right\} \quad \text{as} \quad r\to 1,
\end{aligned} \tag{3.6}$$

where

$$K(\beta) = \max\{1, (4/r)^{1-\beta}\} \tag{3.7}$$

is a bounded constant not necessarily the same each time. On the other hand

$$I_2(r) = \int_0^r\int_0^{2\pi} |g^{\beta}(z)h'(z)|\, \mathrm{d}\theta\, \mathrm{d}\varrho.$$

Using (2.1) we obtain

$$\begin{aligned}
I_2(r) &\le \int_0^r\int_0^{2\pi} |g(z)|^{\beta}\, \mathfrak{Re}\{h(z)\}\frac{2}{1-\varrho^2}\, \mathrm{d}\theta\, \mathrm{d}\varrho \\
&\le 2\,\mathfrak{Re}\left\{\int_0^r\int_0^{2\pi} \frac{|g^{\beta}(z)|}{g^{\beta}(z)}g^{\beta}(z)h(z)\frac{1}{1-\varrho^2}\, \mathrm{d}\theta\, \mathrm{d}\varrho\right\}.
\end{aligned}$$

Using (3.5) we can write

$$I_2(r) \le 2\,\mathfrak{Re}\left\{\int_0^r\int_0^{2\pi} zf'(z)f^{\beta-1}(z)\frac{e^{-i\beta\arg g(z)}}{1-\varrho^2}\, \mathrm{d}\theta\, \mathrm{d}\varrho\right\}.$$

Because $g(z)$ is a starlike function, then $\arg g(\varrho e^{i\theta})$ is an increasing function of $\theta$ and maps the interval $[0, 2\pi]$ onto oneself. Applying D. K. Thomas method [15, p. 357], after a suitable substitution and integrating by parts, we obtain

$$\begin{aligned}
I_2(r) &\le \frac{2}{\beta}\,\mathfrak{Re}\left\{\int_0^r\int_{|z|=\varrho} z\left(\frac{\mathrm{d}f^\beta(z)}{\mathrm{d}z}\right)\frac{e^{-i\beta\arg g(z)}}{1-\varrho^2}\frac{\mathrm{d}z}{iz}\,\mathrm{d}\varrho\right\}\\
&= 2\,\mathfrak{Re}\left\{\int_0^r\int_{|z|=\varrho}\frac{1}{i\beta}\frac{e^{-i\beta\arg g(z)}}{1-\varrho^2}\left(\frac{\mathrm{d}f^\beta(z)}{\mathrm{d}_\theta\arg g(z)}\right)\mathrm{d}_\theta\arg g(z)\,\mathrm{d}\varrho\right\}\\
&= 2\,\mathfrak{Re}\left\{\int_0^r\frac{\mathrm{d}\varrho}{i\beta(1-\varrho^2)}\int_{|z|=\varrho} e^{-i\beta\arg g(z)}\left(\frac{\mathrm{d}f^\beta(z)}{\mathrm{d}_\theta\arg g(z)}\right)\mathrm{d}_\theta\arg g(z)\right\}\\
&= 2\,\mathfrak{Re}\left\{\int_0^r\frac{\mathrm{d}\varrho}{i\beta(1-\varrho^2)}\left\{[f^\beta(z)e^{-i\beta\arg g(z)}]_{\arg g(z)=0}^{\arg g(z)=2\pi}\right.\right.\\
&\qquad\left.\left.+\int_0^r\int_{|z|=\varrho} i\beta f^\beta(z)e^{-i\beta\arg g(z)}\,\mathrm{d}_\theta\arg g(z)\right\}\right\}\\
&= 2\,\mathfrak{Re}\left\{\int_0^r\int_{|z|=\varrho} f^\beta(z)e^{-i\beta\arg g(z)}\frac{1}{1-\varrho^2}\,\mathrm{d}_\theta\arg g(z)\,\mathrm{d}\varrho\right\}\\
&\le 4\pi\int_0^r M^\beta(\varrho)/(1-\varrho^2)\,\mathrm{d}\varrho.
\end{aligned}$$

Applying Ch. Pommerenke's result (1.3), we have

$$\begin{aligned}
I_2(r) &\le 16\sqrt{\pi}\int_0^r\left(A(\varrho)\log\frac{3}{1-\varrho}\right)^{\beta/2}/(1-\varrho^2)\,\mathrm{d}\varrho\\
&\le 16\sqrt{\pi}A^{\beta/2}(r)\int_0^r\left(\log\frac{3}{1-\varrho}\right)^{\beta/2}\frac{1}{1-\varrho}\,\mathrm{d}\varrho\\
&= 16\sqrt{\pi}A^{\beta/2}(r)\frac{2}{\beta+2}\int_0^r\left\{\left(\log\frac{3}{1-\varrho}\right)^{(\beta+2)/2}\right\}'\mathrm{d}\varrho\\
&= \mathcal{O}\left\{A^{\beta/2}(r)\left(\log\frac{1}{1-r}\right)^{(\beta+2)/2}\right\}\quad\text{as}\quad r\to 1.
\end{aligned}$$

Applying it together with (3.6) we obtain (3.4). □

**Theorem 3.3.** *If $f(z)\in\mathcal{B}(\beta)$ is a Bazilevič function of type $\beta$, $1<\beta$, then we have*

$$L(r)=\mathcal{O}\left\{A^\beta(r)\left(\log\frac{1}{1-r}\right)^{\beta+2}\right\}^{1/2}\quad \textit{as}\quad r\to 1. \tag{3.8}$$

Proof. For the case $0 < r \le 1/2$, because $\mathcal{B}(\beta) \subset \mathcal{S}$, by Lemma 2.2 we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta \\ &\le \int_0^{2\pi} \frac{r(1+r)}{(1-r)^3}\,\mathrm{d}\theta \\ &< 12\pi, \end{aligned}$$

where $r = |z|$. Assume that

$$h(z) = \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}, \quad \mathfrak{Re}\{h(z)\} > 0,\ z \in \mathbb{D},\ g \in \mathcal{S}^*.$$

For the case $1/2 < r < 1$, we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta \\ &= \int_0^{2\pi} |f^{1-\beta}(z)g^{\beta}(z)h(z)|\,\mathrm{d}\theta \\ &\le \int_0^{2\pi} \left|\frac{(1+r)^2}{r}\right|^{\beta-1} |g^{\beta}(z)h(z)|\,\mathrm{d}\theta \\ &\le \left(\frac{9}{2}\right)^{\beta-1} \int_0^{2\pi} |g^{\beta}(z)h(z)|\,\mathrm{d}\theta \\ &\le \left(\frac{9}{2}\right)^{\beta-1} \left\{\int_0^{2\pi}\int_0^{r} \beta|g'(z)g^{\beta-1}(z)h(z)|\,\mathrm{d}\varrho\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{r} |g^{\beta}(z)h'(z)|\,\mathrm{d}\varrho\,\mathrm{d}\theta\right\} \\ &= \left(\frac{9}{2}\right)^{\beta-1} (I_1(r) + I_2(r)). \end{aligned}$$

Using the result (3.7) for $1/2 < r < 1$, we have

$$I_1(r) \le 2\sqrt{2\pi}\,\beta K_1(\beta)\left(2\log\frac{1}{1-r}\right)^{1/2},$$

where $K_1(\beta) \le \max\left\{1, 8^{1-\beta}\right\}$. Furthermore, in the same way as in the previous proof, we obtain

$$\begin{aligned} I_2(r) &= \int_0^{2\pi}\int_0^{r} |g^{\beta}(z)h'(z)|\,\mathrm{d}\varrho\,\mathrm{d}\theta \\ &= \mathcal{O}\left\{(A(r))^{\beta/2}\left(\log\frac{1}{1-r}\right)^{(\beta+2)/2}\right\} \quad \text{as} \quad r \to 1, \end{aligned}$$

where $K_2(r)$ is a bounded function of $\beta$. This completes the proof. $\square$

REMARK 3.4. D.K. Thomas in [15] has shown that if $f(z)$ is a Bazilevič function of type $\beta$, $0 < \beta \le 1$, then

$$L(r) \le K(\beta) M(r) \log \frac{1}{1-r},$$

where $K(\beta)$ is a bounded function of $\beta$. On the other hand, from Ch. Pommerenke's result [9], we have

$$L(r) \le K(\beta) \sqrt{A(r)} \left( \log \frac{1}{1-r} \right)^{3/2}.$$

From Theorems 3.2 and 3.3 we have that if $f(z)$ is a Bazilevič function of type $\beta$, $0 < \beta \le 1$, then

$$L(r) = \begin{cases} \mathcal{O}\left\{ A^{\beta/2}(r) \left( \log \frac{1}{1-r} \right)^{\beta+2/2} \right\} & \text{for} \quad 1 < \beta, \\ \mathcal{O}\left\{ A^{1/2}(r) \left( \log \frac{1}{1-r} \right)^{3/2} \right\} & \text{for} \quad 0 < \beta \le 1, \end{cases} \quad \text{as} \quad r \to 1.$$

**Theorem 3.5.** *Let $f \in \mathcal{SS}^*(\alpha)$ be strongly starlike function of order $\alpha$, $0 < \alpha < 1$. Then we have*

$$L(r) = \mathcal{O}\left\{ A(r) \left( \log \frac{1}{1-r} \right)^{1/2} \right\} \quad \textit{as} \quad r \to 1. \tag{3.9}$$

Proof. From the hypothesis of the Theorem and applying Ch. Pommerenke's [9] and Rogosinski's [11] results, we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |z f'(z)| \, \mathrm{d}\theta \\ &= \int_0^{2\pi} |f(z)| \left| \frac{z f'(z)}{f(z)} \right| \mathrm{d}\theta \\ &\le M(r) \int_0^{2\pi} \left| \frac{z f'(z)}{f(z)} \right| \mathrm{d}\theta \\ &\le \sqrt{-K A(r) \log(1-r)} \int_0^{2\pi} \left| \frac{1+z}{1-z} \right|^{\alpha} \mathrm{d}\theta \\ &\le \sqrt{-K A(r) \log(1-r)} \int_0^{2\pi} \frac{2}{|1-z|^{\alpha}} \, \mathrm{d}\theta \\ &= \mathcal{O}\left\{ A(r) \left( \log \frac{1}{1-r} \right)^{1/2} \right\} \quad \text{as} \quad r \to 1, \end{aligned}$$

where $K$ is a bounded constant and because we have

$$\int_0^{2\pi} \frac{2}{|1-z|^\alpha}\,\mathrm{d}\theta < \infty \quad \text{for} \quad 0 < \alpha < 1. \qquad \square$$

**Corollary 3.6.** *Let $f \in \mathcal{C}$ be close-to-convex function, satisfy* (1.2) *with $\alpha = 0$ in $\mathbb{D}$ and map $\mathbb{D}$ onto a domain of finite area $A$. Then by* Theorem 3.2, $\beta = 1$, *we have*

$$L(r) = \mathcal{O}\left\{\left(\log \frac{1}{1-r}\right)^{3/2}\right\} \quad as \quad r \to 1.$$

Notice that D.K. Thomas in Theorem 2 [13, p. 431]. has shown that

$$L(r) = \mathcal{O}\left\{\left(\log \frac{1}{1-r}\right)\right\} \quad \text{as} \quad r \to 1.$$

when $f \in \mathcal{C}$, satisfies (1.2) with $\alpha = 0$ and $f$ is bounded in $\mathbb{D}$.

**Corollary 3.7.** *Let $f \in \mathcal{C}$ be close-to-convex function, satisfy* (1.2) *with $\alpha = 0$ in $\mathbb{D}$. Then by Theorem* 3.2, $\beta = 1$, *we have*

$$L(r) = \mathcal{O}\left\{A(r)\left(\log \frac{1}{1-r}\right)^{3/2}\right\} \quad as \quad r \to 1.$$

In [13] it was shown that

$$L(r) = \mathcal{O}\left\{M(r)\left(\log \frac{1}{1-r}\right)\right\} \quad \text{as} \quad r \to 1,$$

when $f \in \mathcal{C}$, satisfies (1.2) with $\alpha = 0$. Compare also Theorems 1.1–1.8 in the introduction.

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