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3-DIMENSIONAL HOMOLOGY HANDLES AND MINIMAL SECOND BETTI NUMBERS OF 4-MANIFOLDS

YOSHIHISA SATO

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1. Introduction

We consider the following problem:
For a given closed 3-manifold M, what is the minimal second Betti number of all compact 4-manifolds bounded by M?

If we add the condition that 4-manifolds are simply connected, then the answer about the above problem in the topological category can be seen from the Boyer classification theorem [1],[2]. The Boyer classification theorem states that for an oriented, closed, connected 3-manifold M, a symmetric integral bilinear form $(E, \mathcal{L})$ and a presentation $\mathcal{P}$ of $H_*(M;\mathbb{Z})$ by $(E, \mathcal{L})$, there exists an oriented, compact, simply connected, topological 4-manifold with boundary M whose intersection form is isomorphic over $\mathbb{Z}$ to $(E, \mathcal{L})$ and which represents $\mathcal{P}$ geometrically. Furthermore, Boyer gave the result about the uniqueness of such 4-manifolds up to orientation-preserving homemorphism. Here a presentation $\mathcal{P}$ of $H_*(M;\mathbb{Z})$ by $(E, \mathcal{L})$ is the following short exact sequence with some algebraic data corresponding to the relationship between the linking form of M and $(E, \mathcal{L})$, spin structures and the Kirby-Siebenmann obstruction;

$$0 \longrightarrow H_2(M;\mathbb{Z}) \longrightarrow E \overset{ad(\mathcal{L})}{\longrightarrow} E^* \longrightarrow H_1(M;\mathbb{Z}) \longrightarrow 0.$$

Hence, in the topological category, we can calculate algebraically the minimal second Betti number of all simply connected 4-manifolds bounded by M. The key to this classification theorem is the Freedman theorem [4], and in particular the fact that every homology 3-sphere can bound a contractible compact topological 4-manifold. In the topological category, it follows from this that the minimal second Betti number of all simply connected 4-manifolds bounded by a given homology 3-sphere is zero. However, the Roholin theorem and the gauge theory say that in the smooth category, a homology 3-sphere can not always bound a homology 4-ball, and so the minimal second Betti number of all simply connected 4-manifolds bounded by a homology 3-sphere is not always zero in the smooth category.

If we consider the Boyer theorem with the condition that the fundamental groups of 4-manifolds are isomorphic to the infinite cyclic group instead of simply connectedness, then the key seems to be orientable closed 3-manifolds M with the same integral homology groups as $S^1 \times S^2$, which are called *homology handles* [8]. Of course, the situation changes according as the homomorphisms of π_1 induced from inclusions are trivial or not. In this paper, we consider the case where such homomorphisms $i_\sharp : \pi_1 M \to \mathbb{Z}$ are surjective, and under this condition we consider the above problem.

By $\beta^{TOP}(M)$ and $\beta^{DIFF}(M)$, we denote the minimal second Betti number of such 4-manifolds in the topological category and in the smooth category, respectively. For example, it is clear that $\beta^{TOP}(S^1 \times S^2) = \beta^{DIFF}(S^1 \times S^2) = 0$. But it does not always hold that $\beta^{TOP}(M) = 0$, since there is a homology handle which can not bound a compact topological 4-manifold homotopy equivalent to S^1 in contrast with the case of homology 3-spheres. In this paper we show that for any positive integer n, there exist infinitely many distinct homology handles $\{M_m^{(n)}\}_{m\in\mathbb{N}}$ with $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$, and furthermore that there exists a difference between β^{TOP} and β^{DIFF}.

In §2, we introduce two operations on framed links to construct compact smooth 4-manifolds which are bounded by given 3-manifolds and whose fundamental groups are isomorphic to $\mathbb{Z}$. In §§3 and 4, we investigate β^{TOP} and β^{DIFF} of certain homology handles, and in particular homology handles obtained by 0-surgery on knots. In §4, we show that β^{TOP} and β^{DIFF} are functions onto $\mathbb{N} \cup \{0\}$ and there is a difference between β^{TOP} and β^{DIFF}.

Through this paper, we suppose that manifolds are connected and oriented, and we denote the closed interval [0,1] by I. Furthermore, the symbol b_i stands for the i-th Betti number.

2. Two kinds of 2-handle attachings

For a positive integer p, let $\rho : S^3 \to S^3$ be the $(2\pi/p)$-rotation around the z-axis and $B_j^3 (j = 0, 1, \ldots, p-1)$ small 3-balls in S^3 with $\rho(B_j^3) = B_{j+1}^3$ $(j = 0, 1, \ldots, p-2)$ and $\rho(B_{p-1}^3) = B_0^3$. Moreover, let $D_p = (S^3 - \bigcup_{j=0}^{p-1} \mathrm{int} B_j^3) \times_\rho S^1$ be the mapping torus with monodromy ρ. The compact smooth 4-manifold D_p is bounded by $S^1 \times S^2$ and has the fundamental group $\pi_1 D_p$ isomorphic to $\mathbb{Z}$. The homomorphism $i_\sharp : \pi_1(S^1 \times S^2) \to \pi_1 D_p$ has index p, where $i : S^1 \times S^2 \to D_p$ is the inclusion.

Let M be an oriented closed 3-manifold. If M bounds an oriented compact 4-manifold V such that the fundamental group $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomorphism of π_1 induced from the inclusion $i : M \to V$ is not trivial, then the first Betti number of M is positive. In this section we shall show that for any given 3-manifold M with $b_1(M) \geq 1$, M bounds an oriented compact smooth 4-manifold V such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and $i_\sharp : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$ is not trivial. To show this, we need the following two operations. Every closed 3-manifold is obtained from S^3 by an integral surgery on a link in S^3. Let M be obtained by a framed link $\mathbb{L}$.

Operation 1. Let K be a component of $\mathbb{L}$ with framing n and c a crossing on

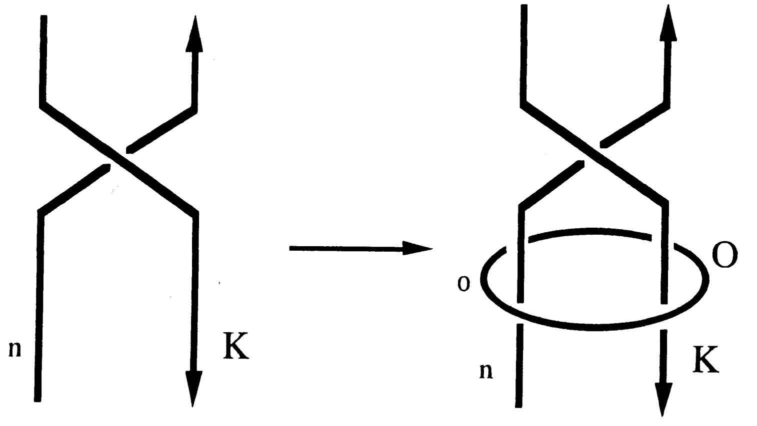


Fig. 1.

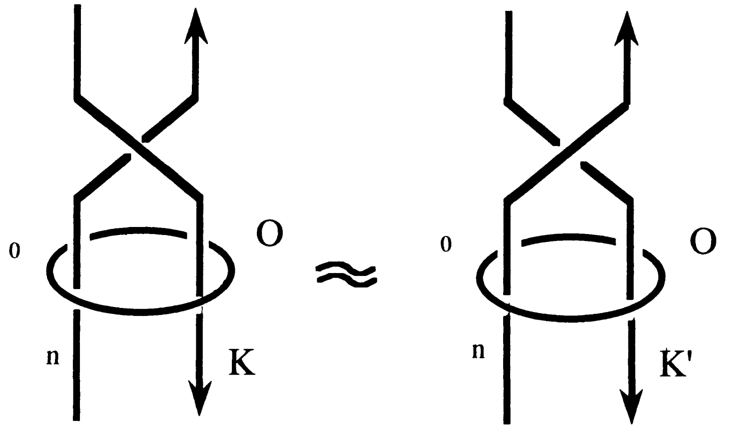


Fig. 2.

a diagram of $K \subset \mathbb{L}$. Add a trivial knot O with framing 0 to $\mathbb{L}$ at c so that the linking number $lk(O, K)$ between O and K is zero. See Fig. 1. Let K' be a knot obtained from K by crossing-change at c. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $\mathbb{L} \cup O$ is orientation-preserving homeomorphic to the 3-manifold obtained by a framed link $\mathbb{L}'$ containing a new component O with framing 0 and the component K' with framing n instead of K with framing n. See Fig. 2.

Operation 2. Let K and L be two components of $\mathbb{L}$ with framing m and n, respectively. Let c be a crossing of K and L on a diagram of $\mathbb{L}$. Give the framing 0 to a meridional curve O of L. See Fig. 3. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $\mathbb{L} \cup O$ is orientation-

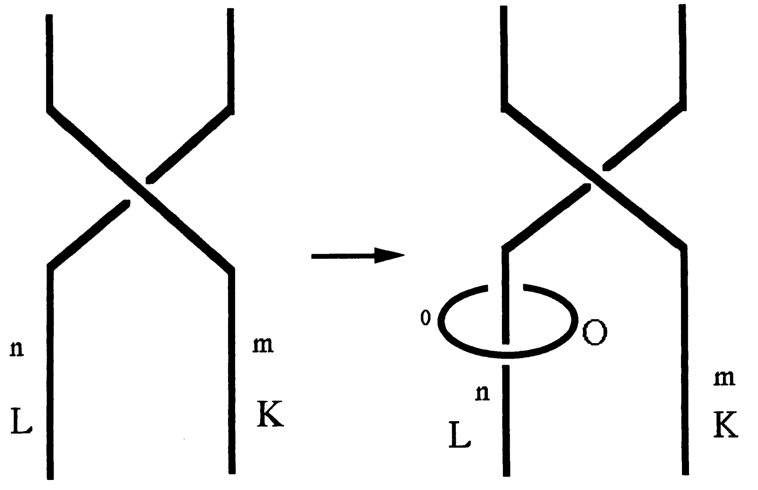


Fig. 3.

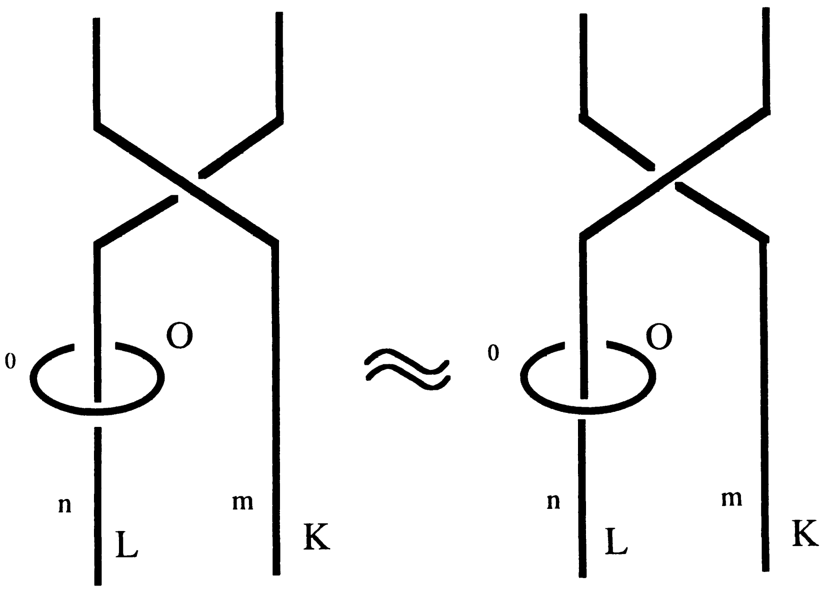


Fig. 4.

preserving homeomorphic to the 3-manifold obtained by a framed link $\mathbb{L}'$ which contains a new component O with framing 0 and which has an opposite crossing at c. See Fig. 4. Note that this operation leaves the knot type of K invariant, since O is trivial.

We use Operations 1 and 2 to make a knot trivial and to split geometrically a component of a link from other components, respectively.

Proposition 1. *For any positive integer p and for any given* 3*-manifold* M *with* $b_1(M) \geq 1$, *there exists an oriented compact smooth* 4*-manifold* V *bounded by* M *such that*
(1) $\pi_1 V$ *is isomorphic to* $\mathbb{Z}$, *and*
(2) *the index,* $(\pi_1 V : \mathrm{Im} i_\sharp)$, *of* $\mathrm{Im}\{i_\sharp : \pi_1 M \to \pi_1 V\}$ *in* $\pi_1 V$ *is p.*

Every oriented 3-manifold is obtained from S^3 by an integral surgery on a link in S^3, but this link is not always an algebraically split link. Here, we say that a link $\mathbb{L} = K_1 \cup K_2 \cup \cdots \cup K_\mu$ is *an algebraically split link* if for each pair of distinct components K_i, $K_j (i \neq j)$ of $\mathbb{L}$, the linking number $lk(K_i, K_j)$ is zero.

We use the following lemma.

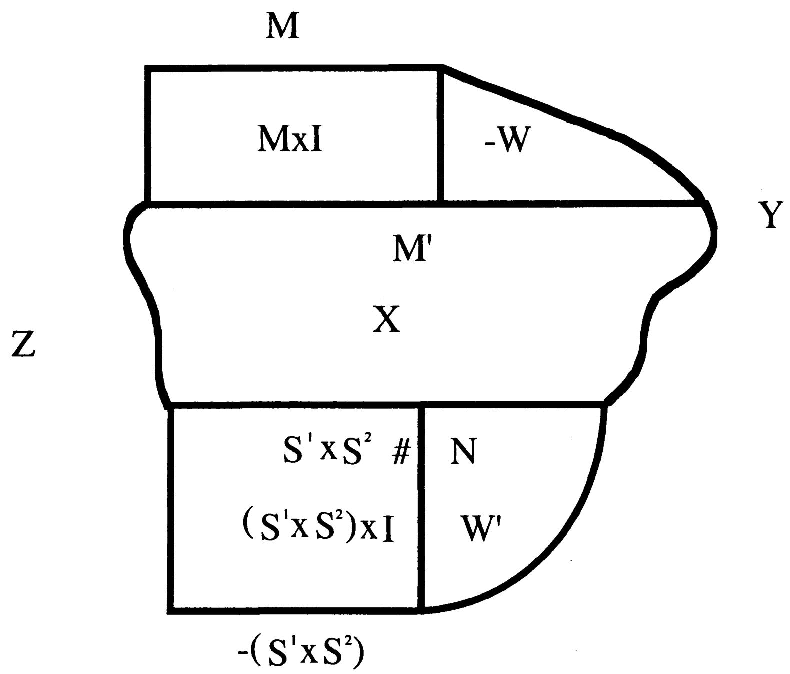
Lemma 1 ([13]). *Any integral symmetric matrix is made diagonalizable over* $\mathbb{Z}$ *by taking block sums of some* 1×1*-matrices* (p_j).

We can translate Lemma 1 into geometric terms : Let M be an oriented closed 3-manifold. Then, there are some lens spaces $L(p_j, 1)$ $(j = 1, 2, \cdots, k)$ such that after taking connected sums of $L(p_j, 1)$ $(j = 1, 2, \cdots, k)$, the 3-manifold $M \sharp L(p_1, 1) \sharp L(p_2, 1) \sharp \cdots \sharp L(p_k, 1)$ has a surgery description by a framed algebraically split link.

Proof of Proposition 1. By Lemma 1, there are some lens spaces $L(p_j, 1)$ $(j = 1, 2, \cdots, k)$ such that the 3-manifold $M' = M \sharp L(p_1, 1) \sharp L(p_2, 1) \sharp \cdots \sharp L(p_k, 1)$ is obtained by an integral surgery on an algebraically split link $\mathbb{L}$. Let $r (\geq 1)$ be the first Betti number of M. Then, the linking matrix of $\mathbb{L}$ is an $(r+n) \times (r+n)$-matrix

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & m_1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ 0 & \dots & 0 & & & m_n \end{pmatrix}$$

where $|m_1 m_2 \cdots m_n|$ is not zero and the order of the torsion part of $H_1(M'; \mathbb{Z})$. Generators of $H_1(M'; \mathbb{Z})$ are given by meridional curves of the components of $\mathbb{L}$. Let K_i $(i = 1, 2, \cdots, r)$ be the components of $\mathbb{L}$ with framing 0 and L_j $(j = 1, 2, \cdots, n)$ the other components of $\mathbb{L}$. The 3-manifold $L(p_1, 1) \sharp L(p_2, 1) \sharp \cdots \sharp L(p_k, 1)$ bounds an oriented simply connected compact smooth 4-manifold W, for example the $\natural$-sum of k D^2-bundles over S^2. Then the smooth 4-manifold $(M \times I) \natural (-W)$ is bounded by $M \coprod (-M')$. We shall make K_1 a trivial knot which is split geometrically from the other components of $\mathbb{L}$.



Schema 1.

Step 1. If K_1 is not trivial, then we can make K_1 a trivial knot K_1' by a finite sequence of Operation 1 at some crossings of K_1. Then the framed link $\mathbb{L}$ changes into another framed link $\mathbb{L}'$, which is algebraically split. The trivial knot K_1' has framing 0.

In general, K_1' is not split geometrically from the other components of $\mathbb{L}'$.

Step 2. By a finite sequence of Operation 2, we can split geometrically K_1' from the other components of $\mathbb{L}'$ keeping K_1' trivial and without changing the framing of K_1'. By $\mathbb{L}''$ we denote the framed link obtained by the operations as above. Let $\mathbb{L}_2''$ be the link consisting of the other components of $\mathbb{L}''$ except K_1', that is, $\mathbb{L}'' = K_1' \cup \mathbb{L}_2''$. Then the 3-manifold given by the framed link $\mathbb{L}''$ is $S^1 \times S^2 \sharp N$, where N is the 3-manifold given by $\mathbb{L}_2''$.

Hence it follows that by attaching 2-handles to $M' \times \{1\} \subset M' \times I$ in ways corresponding to Steps 1 and 2, we get an oriented compact smooth 4-manifold X whose boundary is $M' \coprod (-(S^1 \times S^2 \sharp N))$. Set $Y = ((M \times I) \natural (-W)) \bigcup_{M'} X$. Let W' be an oriented simply connected compact smooth 4-manifold bounded by N, for example, the 4-manifold consisting of one 0-handle and some 2-handles given by the

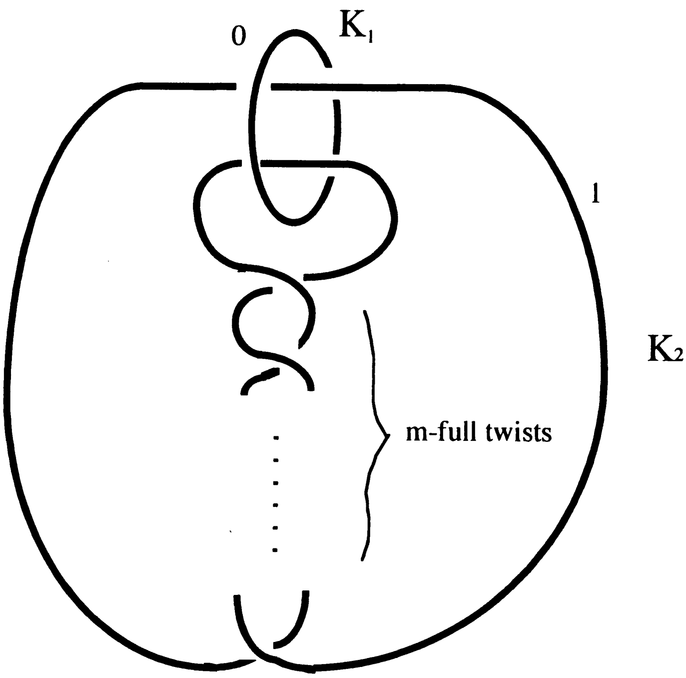


Fig. 5.

framed link $\mathbb{L}_2''$. Then $Z = Y \bigcup ((S^1 \times S^2) \times I \natural W')$ is an oriented compact smooth 4-manifold with boundary $\partial Z = M \coprod (-S^1 \times S^2)$. See Schema 1. Now let V be the 4-manifold $Z \cup_\partial D_p$, which is an oriented compact smooth 4-manifold with boundary $\partial V = M$. By van Kampen's theorem, $\pi_1 V$ is isomorphic to $\mathbb{Z}$. If we let t be a generator of $\pi_1 D_p$, then a loop coming from a meridional curve of K_1 represents $t^{\pm p}$ in $\pi_1 D_p$, and so $(\pi_1 V : \mathrm{Im} i_\sharp) = p$. $\square$

EXAMPLE 1. Let m be an integer. Let $M(m)$ be the homology handle given by the following framed link $K_1 \cup K_2$ in Fig. 5. The link $K_1 \cup K_2$ is an algebraically split link. Let $\tilde{M}(m)$ be the universal abelian covering of $M(m)$, that is, the infinite cyclic covering of $M(m)$ associated to the kernel of the Hurewitz homomorphism $\alpha : \pi_1 M(m) \to H_1(M(m); \mathbb{Z}) \cong \mathbb{Z}$. Then $\tilde{M}(m)$ is obtained from the universal covering $q : \mathbb{R} \times S^2 \to S^1 \times S^2$ by 1-surgeries on the preimage of K_2 via q as in Fig. 6. See [14]. By $\Lambda = \mathbb{Z}\langle t \rangle$ we denote the ring of Laurent polynomials with integer coffecients. Thus $H_1(\tilde{M}(m); \mathbb{Z})$ has a Λ-module structure by the group of deck transformations and is isomorphic to $\Lambda/(mt^{-1} - (2m-1) + mt)$ as Λ-modules. Here $(f(t))$ stands for the principal ideal generated by $f(t) \in \Lambda$. Now attach one 2-handle $h^{(2)}$ to $M(m) \times I$ so that the attaching circle of $h^{(2)}$ is a meridional curve of K_2 and the framing of $h^{(2)}$ is zero. Let W be the resultant 4-manifold. By Op-

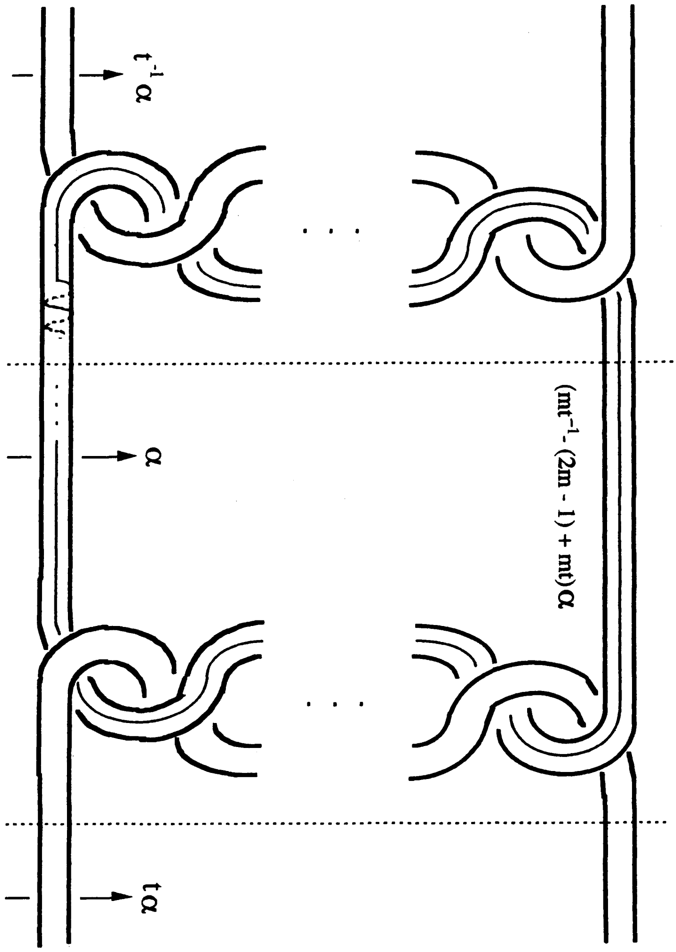


Fig. 6.

eration 1, it is seen that W is bounded by $M(m)\coprod(-S^1\times S^2)$. See Fig. 7. Thus $V=W\cup_{S^1\times S^2}D_p$ is an oriented compact smooth 4-manifold bounded by $M(m)$ with $\pi_1V\cong\mathbb{Z}$, $(\pi_1V:\mathrm{Im}i_\sharp)=p$, and $H_2(V;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}_p$. In §§3 and 4 we show that in the case of $p=1$ this 4-manifold V gives the minimal second Betti number of all oriented compact topological 4-manifolds X bounded by $M(m)$ with $\pi_1X\cong\mathbb{Z}$ and $(\pi_1X:\mathrm{Im}i_\sharp)=1$.

We have the following proposition for a 3-manifold M such that $H_1(M;\mathbb{Z})$ has a torsion subgroup.

Proposition 2. *Let p be any positive integer and $\mathbb{L}=K_1\cup K_2$ a 2-component*

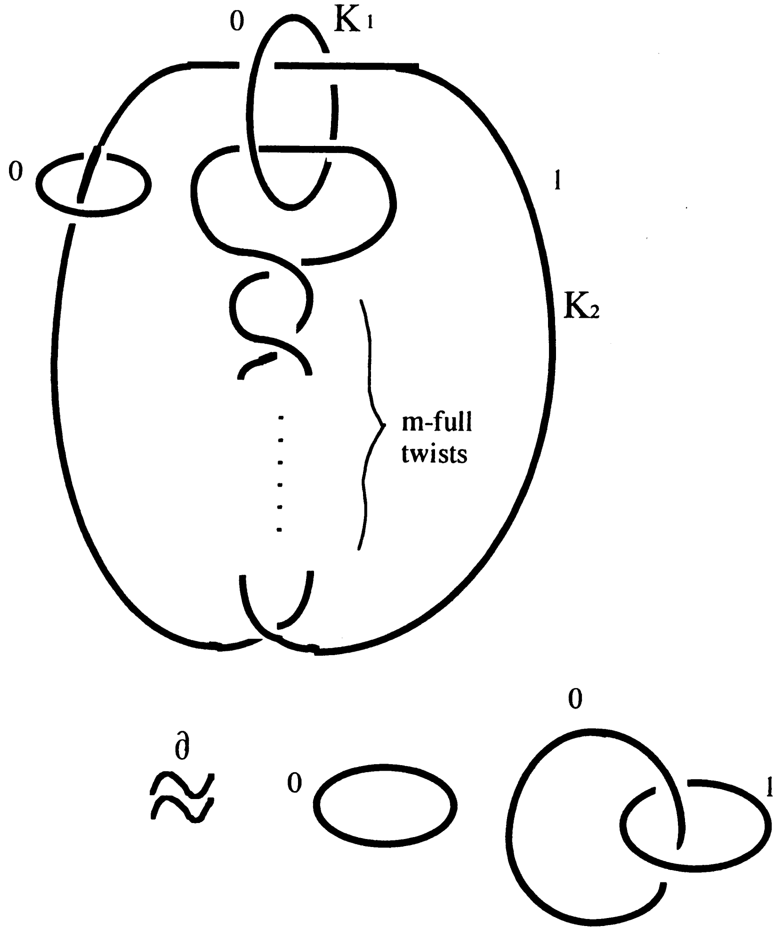


Fig. 7.

framed link such that

(1) *K_1 is a trivial knot,*

(2) *the linking number $lk(K_1, K_2)$ is zero, and*

(3) *the framings of K_1 and K_2 is 0 and n, respectively.*

Let M be the resultant 3-manifold obtained by surgery on the framed link $\mathbb{L}$. If $|n| > 1$, then the smooth 4-manifold V constructed in the manner of Example 1 gives the minimal second Betti number of all oriented compact topological 4-manifolds X bounded by M with $\pi_1 X \cong \mathbb{Z}$ and $(\pi_1 X : \mathrm{Im} i_\sharp) = p$. Note that $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$.

Proof. Suppose that $b_2(V) = 1$ is not minimal. Namely, there is an oriented compact topological 4-manifold X as above with $b_2(X) = 0$. By considering the homology exact sequence of the pair (X, M), we have the following short exact sequence;

$$0 \to \mathbb{Z} \to H_2(M;\mathbb{Z}) \to \mathbb{Z}_p \to 0 \xrightarrow{\partial} H_1(M;\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_p \to 0.$$

Because of $|n| > 1$, $H_1(M;\mathbb{Z})$ has a torsion subgroup. This contradicts that $H_1(M;\mathbb{Z}) \to \mathbb{Z}$ is injective. □

3. Minimal second Betti numbers for homology handles

Through §§3 and 4, we consider the case of $p = 1$, namely, the case where the homomorphisms on π_1 induced from inclusions are surjective. If M is an oriented closed 3-manifold with $H_*(M;\mathbb{Z}) \cong H_*(S^1 \times S^2;\mathbb{Z})$, then we call M a *homology handle*. See [8]. Since a homology handle M has $H^1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, M admits two spin structures τ_0 and τ_1. By $\mu(M,\tau)$ we denote the Roholin invariant of M with respect to a spin structure τ.

Proposition 3. *Let M be a homology handle with spin structures τ_0 and τ_1. Suppose that $\mu(M,\tau_0) = 0$ and $\mu(M,\tau_1) = 1$. Then, there is no orientable compact topological spin 4-manifold V bounded by M such that $\pi_1 V \cong \mathbb{Z}$ and the homomorphism $i_\sharp : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$ is surjective.*

Proof. Suppose that there would be such a 4-manifold V. Because of $\pi_1 V \cong \mathbb{Z}$, V admits two spin structures σ_0 and σ_1. Since $i_\sharp : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$ is surjective, $\pi_1(V, M) = 0$ and so $H^1(E(\tau_V), E(\tau_M);\mathbb{Z}_2) = 0$. Here $E(\tau_M)$ and $E(\tau_V)$ are the total spaces of the principal STop(3)-bundle and the principal STop(4)-bundle associated with stable topological tangent bundles over M and V, respectively. From the following cohomology exact sequence of the pair $(E(\tau_V), E(\tau_M))$,

$$0 = H^1(E(\tau_V), E(\tau_M);\mathbb{Z}_2) \to H^1(E(\tau_V);\mathbb{Z}_2) \to H^1(E(\tau_M);\mathbb{Z}_2) \xrightarrow{\delta},$$

if follows that the restrictions of σ_0 and σ_1 to M are τ_0 and τ_1, say $\sigma_0|_M = \tau_0$ and $\sigma_1|_M = \tau_1$. By [5, Chapter 10], we can calculate the Kirby-Siebenmann obstruction $ks(V) \in H^4(V, M;\mathbb{Z}_2)$ of V from (V, σ_0) and we have that

$$\begin{aligned} 8ks(V) &\equiv signature(V) + \mu(M,\tau_0) \pmod{16} \\ &\equiv signature(V) \pmod{16}. \end{aligned}$$

From (V, σ_1) it follows that

$$8ks(V) \equiv signature(V) + 1 \pmod{16},$$

and this equation contradicts that one. □

For any given homology handle M, we would like to investigate the minimal second Betti number of 4-manifolds bounded by M.

Let M be a homology handle. By $\beta^{TOP}(M)$ we denote the minimal second Betti number of all oriented compact topological 4-manifolds V bounded by M such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomorphism $i_\sharp : \pi_1 M \to \pi_1 V$ is surjective. Furthermore, we denote by $\beta^{DIFF}(M)$ the minimal second Betti number of all oriented compact smooth 4-manifolds as above. Then it is clear that $\beta^{DIFF}(M) \geq \beta^{TOP}(M) \geq 0$.

REMARK. If we define $\beta^{TOP}(M)$ and $\beta^{DIFF}(M)$ for a general 3-manifold M in the same manner, then it follows from the homology exact sequence of the pair (V, M) that $\beta^{DIFF}(M) \geq \beta^{TOP}(M) \geq \mathrm{rank}_{\mathbb{Z}} H_1(M;\mathbb{Z}) - 1$.

Corollary 1. *Let M be a homology handle as in Proposition* 3. *Then, $\beta^{TOP}(M) \geq 1$.*

Corollary 2. *Let $\mathbb{L} = K_1 \cup K_2$ be a 2-component framed link such that*
(1) *K_1 is a trivial knot,*
(2) *the linking number $lk(K_1, K_2)$ is* 0, *and*
(3) *the framings of K_1 and K_2 is* 0 *and* ± 1, *respectively.*
Let M be the homology handle obtained by surgery on $\mathbb{L}$. If M admits two spin structures τ_0 and τ_1 with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$, then $\beta^{DIFF}(M) = \beta^{TOP}(M) = 1$.

Proof. We can construct a smooth 4-manifold V bounded by M with $H_2(V;\mathbb{Z}) \cong \mathbb{Z}$ in the same manner as Example 1. Hence, it follows from Corollary 1 that $\beta^{DIFF}(M) = \beta^{TOP}(M) = 1$. □

EXAMPLE 2. Let $M(m)$ be the homology handle in Example 1. If m is odd, then $M(m)$ admits two spin structures τ_0 and τ_1 with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$. If m is even, then $M(m)$ admits two spin structures τ_0 and τ_1 with $\mu(M, \tau_0) = \mu(M, \tau_1) = 0$, Hence, if m is odd, then $\beta^{DIFF}(M(m)) = \beta^{TOP}(M(m)) = 1$.

For what homology handle M does it hold that $\beta^{TOP}(M) = 0$ or $\beta^{DIFF}(M) = 0$? Note that $\beta^{TOP}(M) = 0$ if and only if M bounds an oriented compact topological 4-manifold homotopy equivalent to S^1. Freedman and Quinn give a necessary and sufficient condition to hold that $\beta^{TOP}(M) = 0$ in [5, Proposition 11.6A and 11.6C].

Theorem 2 ([5]). *Let M be a homology handle. Let $C = [\pi_1 M, \pi_1 M]$ be the commutator subgroup of $\pi_1 M$. Then, $\beta^{TOP}(M) = 0$ if and only if C is perfect.*

Since the universal abelian convering $\widetilde{M}$ of a homology handle M is the infinite cyclic covering associated to the kernel of the Hurewicz homomorphism $\pi_1 M \to H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(\widetilde{M};\mathbb{Z})$ is isomorphic to $C/[C,C]$. Theorem 2 implies that $\beta^{TOP}(M) = 0$ if and only if $H_1(\widetilde{M};\mathbb{Z}) = 0$. Furthermore, the group of deck transformation of $\widetilde{M}$ gives a Λ-modules structure to $H_1(\widetilde{M};\mathbb{Z})$, which is isomorphic to $H_1(M;\Lambda)$ as Λ-modules. So, one can define the Alexander polynomials $\Delta_M(t) \in \Lambda$ for homology handles M as well as for knots. Kawauchi gave in [8, 9] a characterization of the Alexander polynomials of homology handles and how to calculate the Alexander polynomials. Thus $H_1(\widetilde{M};\mathbb{Z}) = 0$, that is, $\beta^{TOP}(M) = 0$ if and only if the Alexander polynomial $\Delta_M(t)$ of M is trivial, that is, a unit of Λ.

4. Minimal second Betti numbers for homology handles obtained by 0-surgery on knots

Consider a homology handle M obtained by 0-surgery on a knot K in S^3. Note that the class $\ell \in \pi_1(S^3 - K)$ represented by the preferred longitude for K belongs to the commutator subgroup $[\pi_1(S^3 - K), \pi_1(S^3 - K)]$ of $\pi_1(S^3 - K)$ and that $\pi_1 M$ is isomorphic to $\pi_1(S^3 - K)/\langle \ell \rangle$, where $\langle \ell \rangle$ is the smallest normal subgroup generated by ℓ. Thus we have the following.

Lemma 2. *Let K be a knot with exterior $E(K)$, and $\widetilde{E(K)}$ the universal abelian covering of $E(K)$. Let M be the homology handle obtained by* 0*-surgery on K. Then, $H_1(\widetilde{M};\mathbb{Z})$ is isomorphic to $H_1(\widetilde{E(K)};\mathbb{Z})$ as Λ-modules. In particular, the Alexander polynomial $\Delta_M(t)$ of M is equal to the Alexander polynomial $\Delta_K(t)$ of K (See Lemma* 2.6-(III) *in* [8].).

Hence, we have the following.

Corollary 3. *Let M be the homology handle obtained by* 0*-surgery on a knot K. The minimal second Betti number $\beta^{TOP}(M) = 0$ if and only if the Alexander polynomial $\Delta_K(t)$ of K is trivial.*

EXAMPLE 3. Let $M(m)$ be the homology handle in Example 1. In Example 1 we see that $H_1(\widetilde{M(m)};\mathbb{Z})$ is isomorphic to $\Lambda/(mt^{-1} - (2m-1) + mt)$ as Λ-modules. In fact, it follows from the Kirby calculus that $M(m)$ is also obtained by 0-surgery on the following knot in Fig. 8. Thus the Alexander polynomial for $M(m)$ is $mt^{-1} - (2m - 1) + mt$ and $\beta^{TOP}(M(m)) \neq 0$. Therefore, in the case when m is even, it also holds that $\beta^{TOP}(M(m)) = \beta^{DIFF}(M(m)) = 1$, since we can construct a required 4-manifold in the same manner as Example 1. See Example 2.

We can estimate $\beta^{DIFF}(M)$ by the unknotting number $u(K)$ of a knot K.

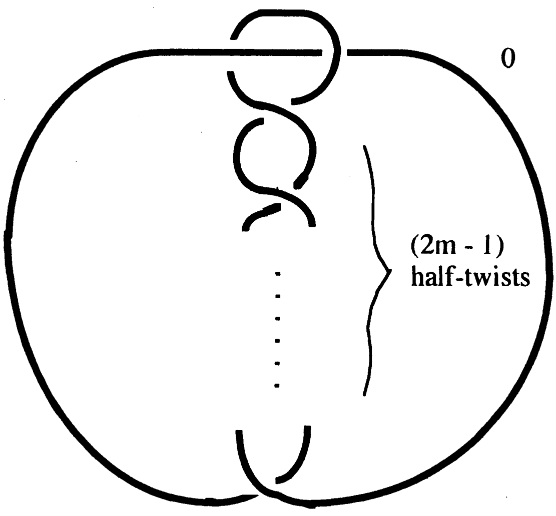


Fig. 8.

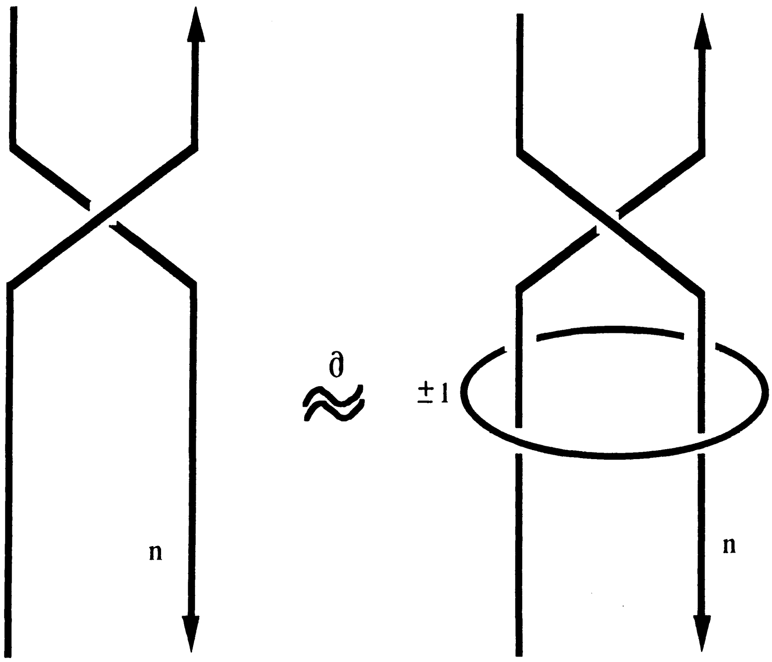


Fig. 9.

Proposition 4. *Let M be the homology handle obtained by* 0*-surgery on a knot K with unknotting number $u(K)$. Then, $u(K) \geq \beta^{DIFF}(M)$.*

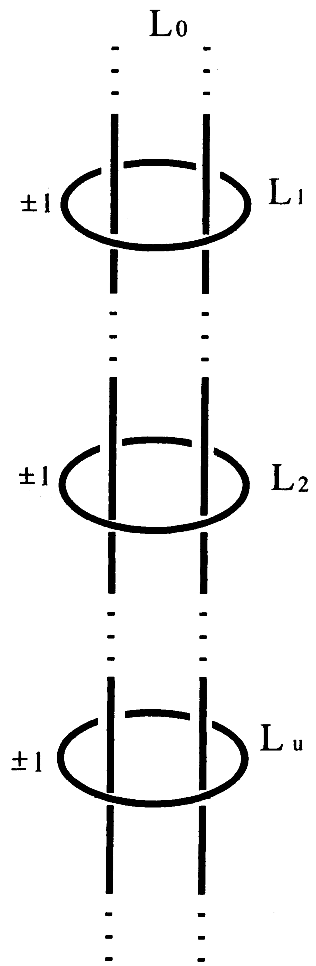


Fig. 10.

Proof. Note that by the Kirby calculus the 3-manifolds in Fig. 9. are homeomorphic. Let u be the unknotting number of K. Then after taking cross-changing at certain u crossings of a diagram of K, K becomes a trivial knot L_0. Hence, M has a surgery description by a framed link $\mathbb{L} = L_0 \cup L_1 \cup \cdots \cup L_u$ such that all $L_j (j = 0, 1, \cdots, u)$ are trivial knots, the framing of L_0 is zero and the framings of $L_j (j = 1, 2, \cdots, u)$ are ± 1. See Fig. 10. If we apply Operation 2 to each $L_j (j = 1, 2, \cdots, u)$, then we get a new framed link $\mathbb{L}'$. See Fig. 11. The 3-manifold given by $\mathbb{L}'$ is $S^1 \times S^2$. By attaching u 2-handles $h_j^{(2)}$ $(j = 1, 2, \cdots, u)$ as above to $M \times I$ and identifying one component of the boundary of the resultant smooth 4-manifold with the boundary of $S^1 \times B^3$, we get a 4-manifold V with second Betti number u and with boundary M such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomor-

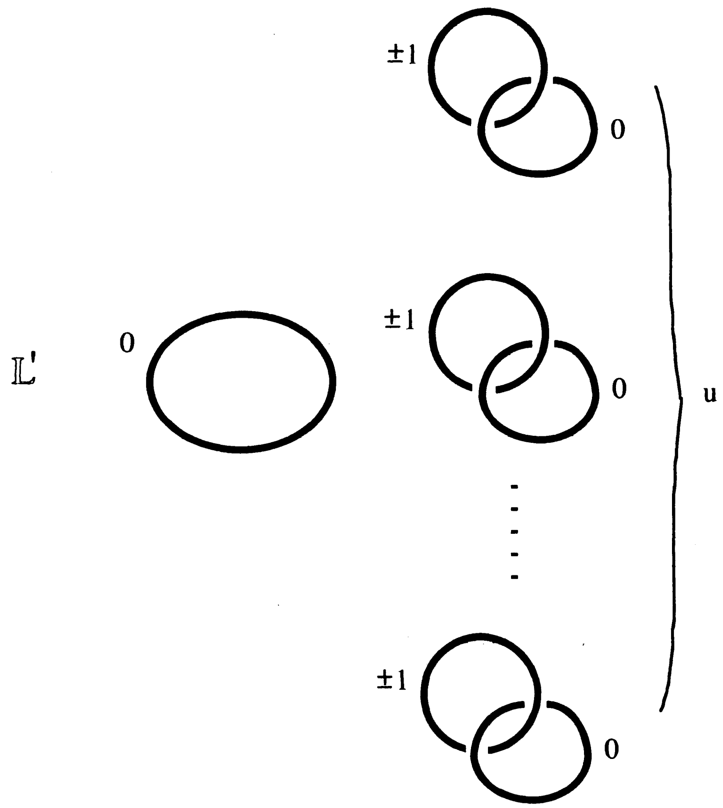


Fig. 11.

phism $i_\sharp : \pi_1 M \to \pi_1 V$ is surjective. Hence, $\beta^{DIFF}(M) \le u$. □

For example, the knots K_m in Fig. 8 are unknotting number 1 knots. Hence, $1 = u(K_m) \ge \beta^{DIFF}(M(m)) \ge \beta^{TOP}(M(m)) \ge 1$, and so $\beta^{TOP}(M(m)) = \beta^{DIFF}(M(m)) = 1$.

We generalize Examples 2 and 3 as follows.

Theorem 3. *For any positive integer n, there exist infinitely many distinct homology handles $\{M_m^{(n)}\}_{m\ge 1}$ with $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$.*

To show Theorem 3, we use the local signatures of homology handles, which are introduced by Kawauchi [8] and defined by generalizing local signatures of knots. See also [12]. In [9], Kawauchi considered the embedding problem of 3-manifolds into 4-manifolds. In particular, he gave an estimation of second Betti numbers and signatures of 4-manifolds by local signatures of their boundaries : Let M be a homology handle and X a compact topological 4-manifold bounded by M. Then, he showed that for any $a \in [-1,1]$,

$$\left|\Sigma_{x\in(a,1]}\sigma_x(M)\right| \leq b_2(X) + |signature(X)|. \tag{4.1}$$

Here $\sigma_x(M)$ is a local signature of M. Since $b_2(X) + |signature(X)| \leq 2b_2(X)$, we have

$$\left|\Sigma_{x\in(a,1]}\sigma_x(M)\right| \leq 2b_2(X) \quad \text{for any } a \in [-1,1], \tag{4.2}$$

and so

$$\left|\Sigma_{x\in(a,1]}\sigma_x(M)\right| \leq 2\beta^{TOP}(M) \quad \text{for any } a \in [-1,1]. \tag{4.3}$$

Proof of Theorem 3. For each positive integer m, let K_m be a knot in Fig. 8. Then, the Alexander polynomial $\Delta_{K_m}(t)$ of K_m is $mt^2 - (2m-1)t + m$ up to units in Λ and the unknotting number $u(K_m)$ of K_m is 1. Because of $\Delta_{K_m}(t)/m = t^2 - 2\{(2m-1)/(2m)\}t+1$, it follows from Assertion 11 in [12] that the signature $\sigma(K_m)$ of K_m is ± 2. Hence, it folds that for the local signature $\sigma_x(K_m)(x \in [-1,1])$,

$$\sigma_x(K_m) = \begin{cases} \pm 2, & \text{if} \quad x = (2m-1)/(2m), \\ 0 & \text{if} \quad x \neq (2m-1)/(2m). \end{cases}$$

Let $K_m^{(n)}$ be the connected sum of n copies of K_m, that is, $K_m^{(n)} = K_m \sharp K_m \sharp \cdots \sharp K_m$. Let $M_m^{(n)}$ be the homology handle obtained by 0-surgery on $K_m^{(n)}$. Since $\Delta_{K_m^{(n)}}(t) = (\Delta_{K_m}(t))^n \neq (\Delta_{K_{m'}}(t))^n = \Delta_{K_{m'}^{(n)}}(t)$ $(m \neq m')$, $M_m^{(n)}$ and $M_{m'}^{(n)}$ $(m \neq m')$ are not homeomorphic. Noting that the quadratic form of the universal abelian covering $\widetilde{M_m^{(n)}}$ is the orthogonal sum of n copies of the quadratic form of K_m, it follows that for the local signature $\sigma_x(M_m^{(n)})(x \in [-1,1])$,

$$\sigma_x(M_m^{(n)}) = \begin{cases} \pm 2n, & \text{if} \quad x = (2m-1)/(2m), \\ 0 & \text{if} \quad x \neq (2m-1)/(2m). \end{cases}$$

Hence, we have

$$\left|\Sigma_{x\in(0,1]}\sigma_x(M_m^{(n)})\right| = \left|\sigma_{(2m-1)/(2m)}(M_m^{(n)})\right| = 2n.$$

Thus, by the inequality (4.3) we have

$$n = \frac{1}{2}\left|\Sigma_{x\in(0,1]}\sigma_x(M_m^{(n)})\right| \leq \beta^{TOP}(M_m^{(n)}).$$

By noting that $u(K_m^{(n)}) \leq n$ because of $u(K_m) = 1$, it follows from Proposition 4 that $\beta^{DIFF}(M_m^{(n)}) \leq u(K_m^{(n)}) \leq n$. Therefore, $n \leq \beta^{TOP}(M_m^{(n)}) \leq \beta^{DIFF}(M_m^{(n)}) \leq n$, and so $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$. □

REMARK. (1) The unknotting number $u(K_m^{(n)})$ is n because of $n = |\sigma(K_m^{(n)})|/2 \leq u(K_m^{(n)}) \leq n$.

(2) Consider a short exact sequence of Λ-modules

$$0 \to E \to F \to \Lambda/(f_1) \oplus \Lambda/(f_2) \oplus \cdots \oplus \Lambda/(f_n) \to 0,$$

where E and F are free Λ-modules of the same rank. If each f_{i+1} can be divided by f_i, then $\mathrm{rank}_\Lambda E \geq n$. Let V be an oriented compact 4-manifold bounded by $M_m^{(n)}$ such that $\pi_1 V \cong \mathbb{Z}$ and the homomorphism $i_\sharp : \pi_1 M_m^{(n)} \to \pi_1 V$ is surjective. Then we have the following homology exact sequence with local coefficient Λ,

$$0 \to H_2(V;\Lambda) \to H_2(V, M_m^{(n)};\Lambda) \to H_1(M_m^{(n)};\Lambda) \to 0.$$

The homology groups $H_2(V;\Lambda)$ and $H_2(V, M_m^{(n)};\Lambda)$ are free Λ-modules of the same rank. Since $H_1(M_m^{(n)};\Lambda) \cong \bigoplus_{i=1}^n (\Lambda/(mt-(2m-1)+mt^{-1}))_i = \Lambda/(mt-(2m-1)+mt^{-1})\oplus\cdots\oplus\Lambda/(mt-(2m-1)+mt^{-1})$, $\mathrm{rank}_\Lambda H_2(V;\Lambda) = \mathrm{rank}_\Lambda H_2(V, M_m^{(n)};\Lambda) \geq n$. Hence it follows that $\beta^{TOP}(M_m^{(n)}) \geq n$.

Next we give two definitions on sliceness of knots.

DEFINITION 1. If a knot K bounds a smooth disk D in the 4-ball B^4 such that $(B^4, D) \times I$ is a trivial ball pair, then K is a *super slice knot*. See [7].

For example, untwisted doubles of slice knots are super slice [7].

DEFINITION 2. A knot K is *pseudo-slice*, if there exists a pair (W, D) for K such that W is a smooth 4-manifold homemorphic to B^4 and D is a smooth disk in W bounded by K.

Proposition 5. *Let K be a super slice knot, and M the homology handle obtained by 0-surgery on K. Then, $\beta^{TOP}(M) = \beta^{DIFF}(M) = 0$.*

Proof. Let D be a slice disk for K such that $(B^4, D) \times I$ is a trivial ball pair. Let $N(D)$ be a closed tubular neighborhood of D in B^4. Then, M is the boundary of the smooth 4-manifold $V = B^4 - intN(D)$. The 4-manifold V is homotopy equivalent to $V \times I = B^4 \times I - intN(D) \times I$. Since $(B^4, D) \times I$ is trivial, V is homotopy equivalent to S^1. Thus V is a required 4-manifold. □

Is there a difference between β^{TOP} and β^{DIFF} ? Now we answer this question.

Theorem 4. *Let K be a knot which is not pseudo-slice and whose Alexander polynomial Δ_K is trivial. Let M be the homology handle obtained by 0-surgery on K. Then, $0 = \beta^{TOP}(M) < \beta^{DIFF}(M)$.*

Proof. Since Δ_K is trivial, it follows from Corollary 3 that $\beta^{TOP}(M) = 0$. Suppose that $\beta^{DIFF}(M) = 0$. Then M bounds a smooth 4-manifold V homotopy equivalent to S^1. By attaching to $M \times I$ one 2-handle $h^{(2)}$ whose attaching circle is a meridian of K and whose framing is zero, we get the 4-manifold $(M \times I) \cup h^{(2)}$ whose boundary is $M \coprod (-S^3)$. See Operation 1. Furthermore, by identifying ∂V with one component M of the boundary of $(M \times I) \cup h^{(2)}$, we get a compact smooth 4-manifold W bounded by S^3. Then, since W is simply-connected and $H_*(W; \mathbb{Z}) \cong H_*(B^4; \mathbb{Z})$, W is homeomorphic to B^4. The co-core of the above 2-handle $h^{(2)}$ gives a smooth disk D in W with $\partial(W, D) = (S^3, K)$. Since K is not pseudo-slice, this is a contradiction. □

EXAMPLE 4. In [3], Cochran and Gompf showed that there are untwisted doubles which are not pseudo-slice. For example, the untwisted double K of the trefoil knot is such a knot. Note that the Alexander polynomials of nontrivial untwisted doubles are trivial and their unknotting numbers are 1. Thus, for the homology handle M obtained by 0-surgery on K, $1 = u(K) \geq \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$, and so $1 = \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$.

EXAMPLE 5. Let $K(-3,5,7)$ be the pretzel knot of type $(-3,5,7)$. Then $K(-3,5,7)$ has a trivial Alexander polynomial. Furthermore, in [6] Fintushel and Stern showed that $K(-3,5,7)$ is not pseudo-slice. Thus, for the homology handle M obtained by 0-surgery on $K(-3,5,7)$, $\beta^{DIFF}(M) > \beta^{TOP}(M) = 0$.

It follows from [11] that $K(-3,5,7)$ is not an unknotting number 1 knot. One can make $K(-3,5,7)$ a trivial knot by crossing-change at certain 3 crossings. Hence, $2 \leq u(K(-3,5,7)) \leq 3$. Thus it follows that $1 \leq \beta^{DIFF}(M) \leq 3$. What is $\beta^{DIFF}(M)$?

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Department of Mathematics,
Faculty of Educations,
Yamaguchi University
Yoshida Yamaguchi, 753 Japan