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FINITE GROUPS WHOSE ABELIAN SUBGROUPS HAVE CONSECUTIVE ORDERS

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1. Introduction

Let G be a finite group and n be a positive integer. A group G is called an OC_n group if every element of G has order less than or equal to n and for each positive integer $m \leq n$ there exists an element of G of order m. B. H. Neumann [8] determined all OC_3 groups and R. Brandl and W. Shi [1] classified all OC_n groups. In recent years a number of papers have dealt with the question of characterizing groups G by the set of all orders of elements in G. See [1], [2] or [10].

Now we will consider the order of abelian subgroups of G instead of the order of elements of G. A group G is called an OA_n group if the order of any abelian subgroup of G is less than or equal to n and for any positive integer $m \leq n$ there exists an abelian subgroup of G of order m. For example, any abelian subgroup of the alternating group A_5 on 5 letters is isomorphic to one of the groups $\{1,\ Z_2,\ Z_3,\ Z_2 \times Z_2,\ Z_5\}$ where Z_m is a cyclic group of order m. Thus the alternating group A_5 is an OA_5 group. In this paper we will classify all OA_n groups applying the results of [6] and [14] which are proved by using the classification of finite simple groups.

Theorem. *Let G be an OA_n group. Then $n \leq 6$ and G is isomorphic to one of the symmetric groups $1,\ S_2,\ S_3,\ S_4,\ S_5$ or the alternating groups $A_4,\ A_5$.*

There are only seven isomorphism classes of OA_n groups although there are infinitely many isomorphism classes of OC_n groups.

2. Preliminaries

The prime graph $\Gamma(G)$ of G is a graph whose vertex set is the set of primes dividing $|G|$ and distinct two primes p and q are joined by an edge if there exists an element of G of order pq. Let $\nu(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and in the case where $|G|$ is even, let π_1 be the connected component containing 2. For any integer m, put $\pi(m)$ the set of all primes dividing m.

A finite group G is called a 2-Frobenius group if it has a chain $1 \subset H \subset K \subset G$

of normal subgroups, where K is a Frobenius group with Frobenius kernel H and G/H is a Frobenius group with Frobenius kernel K/H. A 2-Frobenius group is always solvable.

Theorem (Gruenberg-Kegel [7], [14]). *If $\nu(\Gamma(G)) \geq 2$, then one of the following holds.*
(1) *G is a Frobenius group or a 2-Frobenius group.*
(2) *G has normal subgroups N and G_0 with $N \subset G_0$ such that N is a nilpotent π_1-group, G_0/N is a simple group and G/G_0 is a solvable π_1-group.*
Especially if G is solvable, then $\nu(\Gamma(G)) \leq 2$.

The following theorem is well known.

Theorem (Bertrand's postulate [5, p.82]). *For any real number $t \geq 1$, there exists a prime p such that $t < p \leq 2t$.*

Let G be an OA_n group. Note that if $n \geq 2$, then $|G|$ is even and thus π_1 is not empty. The following lemma is fundamental.

Lemma 1. *Let G be an OA_n group and p be a prime.*
(1) *p divides $|G|$ if and only if $p \leq n$.*
(2) *p^2 divides $|G|$ if and only if $p^2 \leq n$.*
(3) *If $\sqrt{n} < p \leq n$, then a Sylow p-subgroup of G is cyclic of order p.*
(4) *Suppose that p is an odd prime. Then $p \leq n/2$ if and only if $p \in \pi_1$.*
(5) *If $n/2 < p \leq n$, then $\{p\}$ forms a connected component of $\Gamma(G)$ and a Sylow p-subgroup is cyclic of order p.*
(6) *Suppose that p is the largest prime dividing $|G|$. Then $n/2 < p \leq n$.*
(7) *$\nu(\Gamma(G)) \geq 2$ if $n \geq 3$.*

Proof. (1) If $|G|$ is divisible by p, then there exists a cyclic subgroup of order p. Then we have $p \leq n$. Conversely, if $p \leq n$ then there exists an abelian subgroup of order p in G by the definition of OA_n groups. This yields that p divides $|G|$.
(2) Since a group of order p^2 is abelian, we have the result by using similar arguments in the proof of (1).
(3) If $\sqrt{n} < p \leq n$, G does not have an abelian subgroup of order p^2 since $n < p^2$. This yields that a Sylow p-subgroup of G is cyclic of order p.
(4) If $p \leq n/2$, there exists an abelian subgroup of order $2p$ by the definition of OA_n groups. Hence $p \in \pi_1$. Conversely, if $p \in \pi_1$, there exists a prime $q \in \pi_1$ such that G has an element of order pq, that is, G has an abelian subgroup of order pq. Since G is an OA_n group, $pq \leq n$. Since $2p \leq pq$, we have $p \leq n/2$.
(5) If there exists a prime q such that G has an abelian subgroup of order pq, then

$pq \leq n$ because G is an OA_n group. We have $2p \leq pq \leq n$, a contradiction. Hence $\{p\}$ is a connected component of $\Gamma(G)$ and a Sylow p-subgroup is cyclic of order p.
(6) By Bertrand's postulate, there exists a prime r such that $n/2 < r \leq n$. We see that r divides $|G|$ by (1). Since p is the largest prime dividing $|G|$, we have $r \leq p$. This yields that $n/2 < p \leq n$.
(7) Because there is a prime r such that $n/2 < r \leq n$ by Bertrand's postulate, $\nu(\Gamma(G)) \geq 2$ if $n \geq 3$. □

Proposition 1. *If $n \geq 47$, then $\sharp\{p : prime | n/2 < p \leq n\} \geq 6$.*

Proof. See [1, p.395] □

Theorem (Williams [14], Iiyori-Yamaki [6]). *For any finite group G, $\nu(\Gamma(G)) \leq 6$.*

As a corollary, we have the following:

Corollary 1. *If G is an OA_n group, then $n \leq 46$.*

Proof. Suppose that $n \geq 47$. Then Lemma 1 (5) and Proposition 1 imply that $\nu(\Gamma(G)) \geq 7$. This contradicts the theorem of Williams and Iiyori-Yamaki. □

3. The Proof of the Main Theorem

Proposition 2. *Let G be a solvable OA_n group. Then $G \simeq 1$, Z_2, S_3, A_4 or S_4.*

Proof. By Gruenberg-Kegel's theorem, if G is solvable then $\nu(\Gamma(G)) \leq 2$. If $n \neq 1, 2, 3, 4, 6, 10$, then there exist primes p and q such that $n/2 < p < q \leq n$ (See [1, p.396, TABLE I]). Then $\nu(\Gamma(G)) \geq 3$ by Lemma 1 (4). This is a contradiction. If $n = 10$, there exists a Hall $\{3, 5, 7\}$-subgroup H of G because G is solvable. Then $\nu(\Gamma(H)) = 3$. This is a contradiction. If $n = 6$, then $|G| = 2^a \cdot 3 \cdot 5$ for some integer a. A Hall $\{3, 5\}$-subgroup H is cyclic of order 15, a contradiction. Hence $n \leq 4$. If $\nu(\Gamma(G)) = 1$, then $G \simeq Z_2$. If $\nu(\Gamma(G)) = 2$, again by Gruenberg-Kegel's theorem, G is a Frobenius group or a 2-Frobenius group. If G is Frobenius, then its Frobenius kernel N must be isomorphic to $Z_2 \times Z_2$ or Z_3. Then we have $G \simeq A_4$ or S_3. If G is 2-Frobenius, there exist normal subgroups K and H such that K is a Frobenius group with Frobenius kernel H and G/H is a Frobenius group with Frobenius kernel K/H. Then $H \simeq Z_2 \times Z_2$ or Z_3. Since K/H is a Frobenius kernel of G/H and it is also isomorphic to a Frobenius complement of K, K/H must be a cyclic subgroup of odd order. This yields that $H \simeq Z_2 \times Z_2$ and $K/H \simeq Z_3$. This implies that $G \simeq S_4$. □

Lemma 2. *Let G be a nonsolvable OA_n group. Then G is not a Frobenius group.*

Proof. By Lemma 1 (7), we see $\nu(\Gamma(G)) \geq 2$. Suppose that $G = NH$ is a nonsolvable Frobenius group with Frobenius kernel N and Frobenius complement H. Then H has a subgroup $H_0 \simeq SL(2,5) \times M$ with $(H : H_0) \leq 2$, where M is a group in which every Sylow subgroup is cyclic and $|M|$ is not divisible by 2, 3 and 5 (See [9, p.204]). Let p be the largest prime dividing $|G|$. Since $p \notin \pi_1$ by Lemma 1, p does not divide $|H|$. Therefore p divides $|N|$. If $|N|$ is divisible by a prime $q \neq p$, N has an abelian subgroup of order $pq \geq 2p > n$ because N is nilpotent. This is a contradiction. Hence N is a p-group and $N \simeq Z_p$ by Lemma 1. Since $|N| - 1 \geq |H|$, we have $p \geq 121$. This contradicts Corollary 1 and completes the proof. □

Lemma 3. *Let G be a nonsolvable OA_n group. Then $F(G) = 1$, where $F(G)$ is the Fitting subgroup of G.*

Proof. By Lemma 1 (7), we see $\nu(\Gamma(G)) \geq 2$. By Gruenberg-Kegel's theorem, G has normal subgroups N and G_0 with $N \subset G_0$ such that N is a nilpotent π_1-group, G_0/N is a simple group and G/G_0 is a solvable π_1-group since G is not a Frobenius group by Lemma 2. We see that $N = F(G)$. Suppose that $N \neq 1$. Let N_0 be a minimal normal subgroup of G_0. Then N_0 is an elementary abelian p-group for some $p \in \pi_1$. Let q be the largest prime dividing $|G|$. Then we see that $q \geq 5$, $n/2 < q \leq n$ and q divides $|G_0|$ by Gruenberg-Kegel's theorem. By Lemma 1 (5), $\{q\}$ is a connected component of $\Gamma(G)$ and a Sylow q-subgroup is cyclic of order q. Then N_0Q is a Frobenius group for some $Q \in Syl_q(G)$ since $C_{N_0}(x) = 1$ for any $x \in Q - \{1\}$. Hence q divides $|N_0| - 1$. If p is odd, then $|N_0| - 1$ is even. We have $q \leq (|N_0| - 1)/2 \leq (n-1)/2 < n/2$, a contradiction. Hence we have $p = 2$. Then $|N_0| = 2, 4, 8, 16$ or 32 by Corollary 1. If $|N_0| = 32$, then $q = 31$. In this case, G has an abelian subgroup H of order 29 since $32 \leq n$. Since H can not act on N fixed point freely, N_0H has an element of order $58 > 46$, a contradiction. If $|N_0| = 16$, then $q = 5$ because $q \geq 5$. In this case, G has an abelian subgroup H of order 13 since $16 \leq n$. This contradicts the choice of q. If $|N_0| = 2$ or 4, then $q \leq |N_0| - 1 \leq 3$, a contradiction. If $|N_0| = 8$ then $q = 7$. Since $q = 7$ is the largest prime dividing G and G has an abelian subgroup N_0 of order 8, we have $8 \leq n < 11$. Furthermore we have $5 \in \pi_1$, since a Sylow 5-subgroup of G does not act on N_0 fixed point freely. This implies that $n = 10$. In this case, $C_{G_0}(N_0)$ is a 2 group. In fact, if $C_{G_0}(N_0)$ has an element x of odd prime order, then $N_0\langle x \rangle$ is an abelian subgroup whose order is more than 24. This is a contradiction. Since G_0 has a nonsolvable simple factor and $G_0/C_{G_0}(N_0)$ is isomorphic to a subgroup of $GL(3,2)$, $G_0/C_{G_0}(N_0) \simeq GL(3,2)$ and $N \simeq C_{G_0}(N_0)$. We see that 5 does not divide $|G|$ since orders of $Aut(GL(3,2))$ and $C_{G_0}(N_0)$ are not divisible by 5. This is a contradiction. This completes the proof. □

The above lemma implies that if G is a nonsolvable OA_n group, then there exists a simple group G_0 such that $G_0 \subseteq G \subseteq Aut(G_0)$. We will use this notation in the following propositions.

Proposition 3. *Let G be a nonsolvable OA_n group.*

(1) *If G_0 is an alternating group A_m on m letters, then $m = 5$. Conversely, A_5 is an OA_5 group and S_5 is an OA_6 group.*

(2) *G_0 is not a sporadic simple group.*

Proof. (1) If $G_0 \simeq A_m$, $\nu(\Gamma(G_0)) \leq 3$ by [14]. Hence $2 \leq \nu(\Gamma(G)) \leq 3$. This yields that $5 \leq n \leq 16$, $n \neq 13$ by counting the number of primes p with $n/2 < p \leq n$. (See Lemma 1 (5) and [1, p.396, TABLE I].) If $G_0 \simeq A_5$, then $n < 7$ since 7 does not divide $|Aut(G_0)|$. Clearly A_5 is an OA_5 group and S_5 is an OA_6 group. If $G_0 \simeq A_6$, then $n < 7$. On the other hand, A_6 has an abelian subgroup of order 9. This is a contradiction. If $G_0 \simeq A_7$ or A_8, then $n < 11$. But $G_0 \supseteq A_7 \supset \langle(1,2)(3,4),(1,3)(2,4)\rangle \times \langle(5,6,7)\rangle$ which is abelian of order 12. If $G_0 \supseteq A_9$, then $A_9 \supset \langle(1,2)(3,4),(1,3)(2,4)\rangle \times \langle(5,6,7,8,9)\rangle$ which is abelian of order 20. This is a contradiction since $n \leq 16$.

(2) See [4]. □

Proposition 4. *Let G be a nonsolvable OA_n group and G_0 a simple group of Lie type over the field of q elements. Then $G_0 \simeq A_1(4)$.*

Proof. Suppose that $\nu(\Gamma(G_0)) \geq 4$. By the classification of the prime graph components of finite simple groups, $G_0 \simeq E_8(q)$, $A_2(4)$, ${}^2B_2(q)$ or ${}^2E_6(2)$ (See [6, p.337, TABLE III] and [14, p.492, TABLE Ie]). The groups $A_2(4)$, ${}^2E_6(2)$ and their automorphism groups are not OA_n groups (See [4]). If $G_0 \simeq E_8(q)$, G_0 has a maximal torus of order $q^8 - q^4 + 1 \geq 2^8 - 2^4 + 1 > 46$, a contradiction (See [3]). Clearly $G_0 \not\simeq {}^2B_2(8)$ and $G_0 \not\simeq {}^2B_2(32)$ (See [4]). If $G_0 \simeq {}^2B_2(q)$ where $q = 2^{2m+1}$ and $m \geq 3$, then G_0 has a maximal torus of order $q + \sqrt{2q} + 1 \geq 2^7 + 2^4 + 1 > 46$, a contradiction (See [12]). Suppose that $\nu(\Gamma(G_0)) = 3$. This implies that $\nu(\Gamma(G)) \leq 3$ and therefore $5 \leq n \leq 16$, $n \neq 13$ (See [1, p.396, TABLE I]). If the characteristic is more than or equal to 5, then q is a prime because q divides $|G_0|$ and $n \leq 16$. Since q^2 does not divide $|G_0|$, $G_0 \simeq A_1(q)$. Clearly $G_0 \not\simeq A_1(7)$, $A_1(11)$, and $A_1(13)$ (See [4]). We have $G_0 \simeq A_1(5) \simeq A_5$. Suppose now that the characteristic is 3. If $n \leq 8$, in a similar way, we have $G_0 \simeq A_1(3)$, which is not simple. If $n \geq 9$, then G_0 is isomorphic to one of groups in [14, p.492, TABLE Id], that is, $G_0 \simeq A_1(q)$ $(q \equiv 1\ (4))$, $A_1(q)$ $(q \equiv -1\ (4))$, $E_7(3)$, $G_2(q)$ $(q \equiv 0\ (3))$, ${}^2G_2(q)$ $(q = 3^{2m+1},\ m \geq 1)$, or ${}^2D_p(3)$ $(p = 2^n+1,\ n \geq 2)$. Clearly $G_0 \not\simeq E_7(3)$ (See [14]). If

$G_0 \simeq A_1(q)$, then we have $q = 3$ or 9 since a Sylow q-subgroup of G_0 is abelian and $n \leq 16$. Since G_0 is simple, $G \not\simeq A_1(3)$ and we see $G_0 \not\simeq A_1(9) \simeq A_6$ by Proposition 3. Clearly $G_0 \not\simeq G_2(3)$ (See [4]). If $G_0 \simeq G_2(q)$ $(q \equiv 0(3))$ and $q \geq 3^2$ then G_0 has a maximal torus of order $q^2+q+1 \geq 3^4+3^2+1 > 16$, a contradiction (See [3]). If $G_0 \simeq {}^2G_2(q)$ $(q = 3^{2m+1},\ m \geq 1)$, G_0 has a maximal torus of order $q+\sqrt{3q}+1 > 16$, a contradiction (See [13]). If $G_0 \simeq {}^2D_p(3)$ $(p = 2^n+1$ is a prime, $n \geq 2)$, then G_0 has a maximal torus of order $(3^p+1)/4 > 16$, a contradiction (See [12] or [14]). Suppose now that the characteristic is 2. Then $G_0 \simeq A_1(q)$, $A_2(2)$, ${}^2A_5(2)$, $E_7(2)$, ${}^2F_4(q)$ or $F_4(q)$ by [6, p.336, TABLE II]. Clearly $G_0 \not\simeq A_2(2)$, ${}^2A_5(2)$, $E_7(2)$, $A_1(8)$ and $A_1(16)$ (See [4]). If $G_0 \simeq A_1(q)$, we have $q \leq 16$ since a Sylow 2-subgroup of G_0 is abelian. We have $G_0 \simeq A_1(4) \simeq A_5$ (See [4]). If $G_0 \simeq {}^2F_4(q)$ $(q = 2^{2m+1},\ m \geq 1)$, then G_0 has a maximal torus of order $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1 > 16$, a contradiction (See [11]). Clearly $G_0 \not\simeq F_4(2)$ (See [4]). If $G_0 \simeq F_4(q)$, then G_0 has a maximal torus of order $q^4+1 > 16$, a contradiction (See [3]). This completes the case where $\nu(\Gamma(G_0)) = 3$. Suppose that $\nu(\Gamma(G_0)) = 2$. Then $n = 6$ or 10 (See [1, p.396, TABLE I]). If the characteristic is more than or equal to 5, we have $G_0 \simeq A_1(q)$, a contradiction since $\nu(\Gamma(A_1(q))) = 3$. We have that the characteristic is 2 or 3. Suppose now that the characteristic is 3. By an argument similar to that in the case where $\nu(\Gamma(G_0)) = 3$, we see $n = 10$. Notice that G_0 has prime graph components $\pi_1 = \{2,3,5\}$ and $\{7\}$. And G_0 is isomorphic to one of groups in [14, p.490, TABLE Ib, p.491, TABLE Ic] whose characteristic is 3. We see that there exist no groups satisfying our condition in this case. Suppose now that the characteristic is 2. Then $n = 6$ and the connected components are $\pi_1 = \{2,3\}$ and $\{5\}$ or $n = 10$ and the connected components are $\pi_1 = \{2,3,5\}$ and $\{7\}$. And G_0 is isomorphic to one of groups in [6, p.336, TABLE Ia, Ib]. We see that only ${}^2A_3(2)$ has the connected components $\pi_1 = \{2,3\}$ and $\{5\}$. However we see $G_0 \not\simeq {}^2A_3(2)$ by [4]. Also we see that only $A_3(2)$, $C_3(2)$ and $D_4(2)$ have the connected components $\pi_1 = \{2,3,5\}$ and $\{7\}$. However we see that $G_0 \not\simeq A_3(2)$, $C_3(2)$ and $D_4(2)$ by [4]. This yields that there exist no groups satisfying our conditions in this case. This completes the proof. □

Proof of Theorem. Straightforward from Propositions 2, 3 and 4. □

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