ON BOUNDARY VALUES OF AN ANALYTIC TRANSFORMATION OF A CIRCLE INTO A RIEMANN SURFACE

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Introduction

Let $f(z)$ be a nonconstant analytic transformation of the unit circle $U: |z| < 1$ into a Riemann surface $\mathfrak{R}$. As an extension of a classical theorem of F. and M. Riesz, the author proved in Theorem 3.4 of [2] that if the image of U is relatively compact in $\mathfrak{R}$ and has universal covering surface of hyperbolic type, and if, at every point of a set on $|z| = 1$ of positive inner linear measure, there terminates a curve along which $f(z)$ has limit, then the set of such limits has positive inner logarithmic capacity. This theorem was followed by the first proposition in Kuramochi [1], which asserts that, if $\mathfrak{R}$ has a null boundary and the image of U excludes a set of positive logarithmic capacity on $\mathfrak{R}$ and if, at every point of a set E on $|z| = 1$, there terminates a curve along which $f(z)$ has limit in the union of a set of inner logarithmic capacity zero on $\mathfrak{R}$ and the boundary components of $\mathfrak{R}$, then the inner linear measure of E is zero.

In this note, we shall show that we can replace the condition in the Kuramochi's theorem that the image of U excludes a set of positive logarithmic capacity by the weaker condition that the image has universal covering surface of hyperbolic type. Thus the condition, in the author's theorem stated above, that the image of U is relatively compact in $\mathfrak{R}$ is dispensable. Actually our theorem below will tell more than this.

1. Let $\mathfrak{R}$ be a Riemann surface. Consider a filter $\mathfrak{B}$ on it with a base consisting of a decreasing sequence $\{D_n\}$ of open sets, and associate a new element $\mathfrak{L}$ with the filter. We introduce a topology into $\mathfrak{R} + \{\mathfrak{L}\}$ by taking $\{D_n + \{\mathfrak{L}\}\}$ as a base of neighborhoods of $\mathfrak{L}$ and preserving the original bases

Received January 4, 1956.

of neighborhoods of the points of $\mathfrak{R}$.

Suppose that there exists a function $v(P)$ on $\mathfrak{R}$ satisfying

i) it is superharmonic everywhere on $\mathfrak{R}$ or except at a point P_0 such that $v(P)$ is bounded from above in a certain neighborhood of P_0,

ii) it is bounded from below everywhere on $\mathfrak{R}$ or, if there exists an exceptional point P_0, then it is bounded from below outside every neighborhood of P_0,

iii) it tends to $+\infty$ when $P \to \mathfrak{L}$.

Then we shall say that $\mathfrak{L}$ is *of harmonic measure zero* and call $v(P)$ an *associated function of* $\mathfrak{L}$. If $v(P)$ satisfies i), ii) and, instead of iii),

iii′) it tends to $+\infty$ when and only when $P \to \mathfrak{L}$,

then we shall call $\mathfrak{L}$ *complete and of harmonic measure zero* and $v(P)$ its associated function. See Chap. II, n^0 2 of [3] for examples.

Conversely, given a function $v(P)$ which satisfies i) and ii), we obtain a complete element $\mathfrak{L}$ of harmonic measure zero associated with a filter which has $\{D_n\}$ as base, where $D_n = \{P\,;\, v(P) > n\}$, provided that no D_n is empty. We shall say that this $\mathfrak{L}$ is determined by $v(P)$. Then $v(P)$ is clearly an associated function of $\mathfrak{L}$.

Let $\mathfrak{L}$ be an element of harmonic measure zero and $v(P)$ its associated function. The element $\mathfrak{L}'$ determined by $v(P)$ has the property that if a sequence of points converges to $\mathfrak{L}$ then to $\mathfrak{L}'$.

We shall state a lemma which can be easily proved by Theorems 4 and 6 of [3].

LEMMA. *Let $F(\zeta)$ be an exceptionally ramified analytic transformation*[1] *of an angular domain $A_\zeta : 0 < \rho < \rho_0 \leqq +\infty,\ 0 < \varphi < \pi/2$ $(\zeta = \rho e^{i\varphi})$ into a Riemann surface $\mathfrak{R}$ such that $F(\zeta)$ is continuous on $A_\zeta^* : 0 < \rho < \rho_0,\ 0 \leqq \varphi < \pi/2$, and let $\mathfrak{L}$ be a complete element of harmonic measure zero added to $\mathfrak{R}$. If $F(\zeta) \to \mathfrak{L}$ as $\zeta \to 0$ along the real axis, then $F(\zeta) \to \mathfrak{L}$ as $\zeta \to 0$ in any narrower angular region $0 < \rho < \rho_0,\ 0 \leqq \varphi < \pi/2 - \varepsilon$.*

2. Now we prove

THEOREM. *Let $f(z)$ be a nonconstant exceptionally ramified analytic trans-*

[1] For the definition of exceptionally ramified transformation, we refer to Chap. II, n^0 1 of [3].

formation[1] *of the unit circle U into a Riemann surface $\mathfrak{R}$ and $\mathfrak{L}$ an element of harmonic measure zero. If, at every point of a set E on $|z|=1$, there terminates a curve along which $f(z)$ tends to $\mathfrak{L}$, then the linear measure of E is zero.*

Proof. Let $v(P)$ be a function associated with $\mathfrak{L}$, and set $g(P) = 1/\max(v(P), 1)$. Let $A(\theta, r)$ be the part of the angular domain $|\arg(1-e^{i\theta}z)| < \pi/4$ outside $|z| \leqq r$, $2^{-1/2} \leqq r < 1$. Then $\sup g(f(z))$ for $z \in A(\theta, r)$ is a lower semicontinuous function of θ for every fixed r and will be denoted by $h_r(\theta)$.

Let $e^{i\theta}$ be a point where terminates a curve l along which $f(z)$ tends to $\mathfrak{L}$. We shall prove that $g(f(z))$ tends to 0 as $z \to e^{i\theta}$ from $A(\theta, r)$. We map conformally two domains lying between l and two arcs of $|z|=1$ onto angular domains $A_\zeta : 0 < \rho < \rho_0$, $0 < \varphi < \pi/2$ and $A'_\zeta : 0 < \rho < \rho'_0$, $0 < \varphi < \pi/2$ respectively in the ζ-plane so that $\zeta = 0$ corresponds to $e^{i\theta}$ and l corresponds to the side $\varphi = 0$ of A_ζ and to the side $\varphi = \pi/2$ of A'_ζ. We note that if narrower angular domains $0 < \rho < \rho'_0$, $0 < \varphi < \pi/2 - \varepsilon$ and $0 < \rho < \rho'_0$, $\varepsilon < \varphi < \pi/2$ are taken sufficiently close to A_ζ and A'_ζ respectively, then the union of their inverse images together with l covers $A(\theta, r)$ (see the discussion in p. 130 of [3]). Let $\mathfrak{L}'$ be the complete element determined by $v(P)$. In virtue of Lemma, it follows that $f(z(\zeta))$ tends to $\mathfrak{L}'$ as $\zeta \to 0$ in any narrower angular regions: $0 < \rho < \rho_0$, $0 \leqq \varphi < \pi/2 - \varepsilon$ and $0 < \rho < \rho'_0$, $\varepsilon < \varphi \leqq \pi/2$. Thus $f(z)$ tends to $\mathfrak{L}'$ as $z \to e^{i\theta}$ in $A(\theta, r)$. Therefore, $v(f(z)) \to +\infty$ and hence $g(f(z)) \to 0$ as $z \to e^{i\theta}$ in $A(\theta, r)$.

Now we see that the set E_0 of all points $e^{i\theta}$, where $h_r(\theta)$ decreases to zero as $r \to 1$, includes E. We shall show that $m(E_0) = 0$. Suppose, to the contrary, that $m(E_0) > 0$. By Egoroff's theorem, we can find a closed set $F \subset E_0$ with $m(F) > 0$ such that $g(f(z)) \to 0$ uniformly as $z \to e^{i\theta} \in F$ inside of $A(\theta, 2^{-1/2})$. Denote by D the union of $|z| \leqq 2^{-1/2}$ and $\bigcup_{e^{i\theta} \in F} A(\theta, 2^{-1/2})$. Its boundary is rectifiable and contains F as the intersection with $|z|=1$. Hence F is of positive harmonic measure with respect to D by Riesz's theorem in [4]. Consider the case that there is a point P_0 at which $v(P)$ is not superharmonic. Since the convergence is uniform as $D \ni z \to F$, the inverse image of a certain neighborhood of P_0 has a positive distance from F in view of condition i) on $v(P)$. If we exclude this inverse image from D, then F has a positive harmonic measure with respect to the remaining open set D_1. In case $v(P)$ is superharmonic everywhere on $\mathfrak{R}$, we set $D_1 = D$. The function $v(f(z))$ is bounded from below, superharmonic in D_1 and tends to $+\infty$ as $z \to F$. Therefore, the harmonic measure of F with respect to D_1 must be zero. Thus a contradiction is led and the theorem is proved.

3. If we take diverse examples of $\mathfrak{L}$, we shall have corresponding corollaries. We shall give an example.

Let $\overline{\mathfrak{R}}$ be a space with topology $\mathfrak{T}$ in which $\mathfrak{R}$ is imbedded in such a manner that $\mathfrak{R}$ is everywhere dense in $\overline{\mathfrak{R}}$. We shall call $\mathfrak{R}^b = \overline{\mathfrak{R}} - \mathfrak{R}$ *boundary* of $\mathfrak{R}$. We take a compact set K corresponding to a closed parameter circle. The *outer harmonic measure* of a subset $\mathfrak{E}$ of the boundary $\mathfrak{R}^b$ of $\mathfrak{R}$ with respect to the domain $\mathfrak{R} - K$ is defined by the lower envelope of the class of positive superharmonic functions whose lower limits at $\mathfrak{E}$, taken in accordance with $\mathfrak{T}$, are not less than 1. If $\mathfrak{E}$ is of outer harmonic measure zero, then there exists $v(P)$ satisfying conditions i) and ii) and tending to $+\infty$ as $P \to \mathfrak{E}$. This is shown in the same way as in Chap. II, n^0 2 of [3].

Thus we have

COROLLARY 1. *Let $\mathfrak{R}$ be a Riemann surface with boundary $\mathfrak{R}^b$, $\mathfrak{E}$ a subset of $\mathfrak{R}^b$ of outer harmonic measure zero in the above sense, and $f(z)$ an exceptionally ramified analytic transformation of U into $\mathfrak{R}$. If, at every point of a set E on $|z| = 1$, there terminates a curve which is transformed to a curve on $\mathfrak{R}$ converging to $\mathfrak{E}$, then the linear measure of E is zero.*

From this Corollary follows immediately the following result of Tsuji [5]:

COROLLARY 2. *The image of the ideal boundary of an open Riemann surface with null boundary has linear measure zero on $|z| = 1$ under the conformal mapping of the universal covering surface of $\mathfrak{R}$ onto U, provided that this covering surface is of hyperbolic type.*

Here, we may understand by the image of the ideal boundary the set of points on $|z| = 1$ such that at each point terminates at least one curve whose image tends to the ideal boundary.

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