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CLASSIFICATION OF SURFACES IN THREE-SPHERE IN LIE SPHERE GEOMETRY

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0. Introduction

We studied plane curves in Lie sphere geometry in [YY]. Especially we constructed Lie frames of curves in S^2 and classified them by the Lie equivalence. In this paper we are concerned with surfaces in S^3. We construct Lie frames and classify them. We moreover obtain the necessary and sufficient condition that two surfaces are Lie equivalent.

We give some basic concepts about frames before explaining our main result. Let N be a smooth n-dimensional manifold, and let $\lambda : M \to N$ and $\tilde{\lambda} : \tilde{M} \to N$ be embedded submanifolds of dimension m. We say that λ and $\tilde{\lambda}$ have *contact of at least order* k at $p \in M$ and $\tilde{p} \in \tilde{M}$ if λ and $\tilde{\lambda}$ agree up to the differentials of order k at p and $\tilde{p}$. Let G be a group of diffeomorphisms on N. We say that λ and $\tilde{\lambda}$ have *G-contact of at least order* k at p and $\tilde{p}$ if there exists a $P \in G$ such that λ and $P \circ \tilde{\lambda}$ have contact of at least order k at p and $\tilde{p}$.

Let $\lambda : M \to G/H$ be a connected, smoothly embedded n-dimensional submanifold of a homogeneous space G/H. We state the definition of Frenet frames of λ and its construction following G. R. Jensen [J]. Firstly we construct the set of zeroth order frames. A *zeroth order frame* at $p \in M$ is an element $P \in G$ such that $\pi(P) = \lambda(p)$ (where π is the natural projection $G \to G/H$). Let L_0 denote the set of all zeroth order frames on M. A *zeroth order frame field* u along λ is a smooth cross section of $L_0 \to M$.

Secondly we construct first order frames. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. We take a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{h}$, and choose a basis $e_1, \ldots, e_{m_0}$ of $\mathfrak{m}$. With respect to this basis, we consider the isotropy representation $\rho_0 : H \to GL(m_0, \mathbf{R})$ given by the adjoint action of H on $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$. There is a naturally defined smooth map λ_0 from L_0 to the Grassmann manifold $G_{m_0,n}$ given by $\lambda_0(u) = u_*^{-1}\lambda_* M_p$ where $\lambda(p) = \pi(u)$. We choose a local

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cross section W_1 of the action (H, ρ_0) on $G_{m_0,n}$. If there exists a zeroth order frame field u along λ such that $\lambda_0(u) \subset W_1$, we let $L_1 = \lambda_0^{-1}(W_1)$ and call L_1 the set of *first order frames* on λ (with respect to W_1). We define a *first order frame field* along λ to be a smooth cross section $L_1 \to M$.

Furthermore we iterate this construction of frames; $L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k \supset \cdots$. Thus we construct a set of k-th order frames L_k which gives k-th order contact under the action of G. Suppose frames of all orders can be constructed on λ. The sequence $\dim L_0 \geq \dim L_1 \geq \dim L_2 \geq \cdots$ eventually stabilizes. Thus there is the smallest integer $q \geq 1$ such that $\dim L_k = \dim L_q$ for all $k \geq q$. Then the frames of order q are called the *Frenet frames* of λ.

Let us now return to the gist of this paper. Let S^n be the unit sphere in the Euclidean space $\mathbf{E}^{n+1}$, and T_1S^n the unit tangent bundle of S^n, i.e. $T_1S^n = \{(u, v) \in S^n \times S^n ; u \cdot v = 0\}$, where $\cdot$ denotes the inner product of $\mathbf{E}^{n+1}$. An immersed hypersurface $f : M^{n-1} \to S^n$ with a unit normal field $\xi : M^{n-1} \to S^n$ naturally induces a map $\lambda = (f, \xi) : M^{n-1} \to T_1S^n$. This map λ is called the *Legendre map* induced by f with ξ.

Let $PO(n + 1, 2)$ be the projective orthogonal group of signature $(n + 1, 2)$. The group $PO(n + 1, 2)$ acts on T_1S^n transitively. Then T_1S^n is equal to a homogeneous space $PO(n + 1, 2)/H$ for the isotropy subgroup H of $PO(n + 1, 2)$ at a point. Let $\lambda : M^{n-1} \to T_1S^n$ and $\tilde{\lambda} : \tilde{M}^{n-1} \to T_1S^n$ be embedded Legendre maps. We say that λ and $\tilde{\lambda}$ are *Lie equivalent* if λ and $\tilde{\lambda}$ are $PO(n + 1, 2)$-congruent, that is, there exists a $P \in PO(n + 1, 2)$ such that $P\lambda(M)$ agree with $\tilde{\lambda}(\tilde{M})$. Frenet frames of a Legendre map λ in T_1S^n under the action of $PO(n + 1, 2)$ are called *Lie frames* of λ. We are here concerned with the case when $n = 3$.

We summarize our main results as follows: Let $f : M^2 \to S^3$ be an embedded surface with a field of unit normals $\xi : M^2 \to S^3$, and $\lambda = (f, \xi) : M^2 \to T_1S^3$ be the Legendre map induced by f with ξ. Let Ω be the Maurer-Cartan form on $PO(4, 2)$. Then for λ we can construct a Lie frame $u : M^2 \to PO(4,2)$ of one of the distinct five types (Type (a), . . . , Type (e) in Theorem 3.1) with respect to the pullback of Ω by u. Furthermore, a surface of Type (a) is an oriented sphere,
Type (b) is a cyclide of Dupin,
Type (c) or (d) is a "degenerate" surface (including canal surface) and
Type (e) is a general surface.

In Section 1, we outline basic facts in Lie sphere geometry. This section is largely based on U. Pinkall [P] an T. E. Cecil [C]. In Section 2, we give some concepts about frames according to G. R. Jensen [J]. In Section 3, we give our main theorem and its proof. In Section 4, we get some characteristics of surfaces in Lie equivalent classes. U. Pinkall showed that the class of Dupin hypersurfaces in S^n

is invariant under Lie transformations ([P]). We see here that the degeneration of Lie frames characterizes oriented spheres, cyclides of Dupin and canal surfaces in T_1S^3.

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1. Lie sphere geometry

1.1. Lie spheres and Lie transformations

Let S^n be the unit sphere in the Euclidean space $\mathbf{E}^{n+1}$, and T_1S^n the unit tangent bundle of S^n ; i.e.

$$T_1S^n = \{(u, v) \in S^n \times S^n ; u \cdot v = 0\},$$

where $\cdot$ denotes the inner product of $\mathbf{E}^{n+1}$. A hypersphere $\iota : S^{n-1} \to S^n$ with a unit normal vector field ξ along ι which gives an orientation induces a mapping $(\iota, \xi) : S^{n-1} \to T_1S^n$. We call also (ι, ξ) an *oriented hypersphere.* When ι shrinks to a point, we regard ξ as the inclusion of the fiber of T_1S^n over the point ι into T_1S^n, and call (ι, ξ) a *point sphere.* We use the term *Lie sphere* to denote an oriented hypersphere or a point sphere.

Let $\mathbf{R}_2^{n+3} = \{x = (x_1, \cdots, x_{n+3}) ; x_i \in \mathbf{R}\}$ be an $(n+3)$-dimensional real vector space with the scalar product $\langle , \rangle$ defined by

$$\langle x, y\rangle = {}^t xSy, \tag{1.1}$$

where

$$S = (S_{ij}) = \begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_{n-1} & 0 \\ -I_2 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

We denote by $\mathbf{P}^{n+2}$ the associated projective space, and by $\mathbf{Q}^{n+1}$ the quadric in $\mathbf{P}^{n+2}$ defined by $\langle x, x\rangle = 0$. Then we can identify a Lie sphere in T_1S^n with a point of $\mathbf{Q}^{n+1}$.

Let Λ^{2n-1} be the set of all projective lines on $\mathbf{Q}^{n+1}$. By $\mathrm{Line}\{Y, Z\} \in \Lambda^{2n-1}$ we denote the line generated by $[Y]$, $[Z] \in \mathbf{Q}^{n+1}$. Then

$$T_1S^n \cong \Lambda^{2n-1}.$$

A diffeomorphism $\phi : T_1S^n \to T_1S^n$ is called a *Lie transformation* if it carries Lie spheres to Lie spheres. For example a *Möbius transformation* and a *parallel transformation* are Lie transformations; the former takes point spheres to point spheres and the latter takes (ι, ξ) to $(\cos t\iota + \sin t\xi, -\sin t\iota + \cos t\xi)$, where $t \in [0, \pi)$. Lie transformations are generated by Möbius transformations and parallel transformations.

Denote the group of all Lie transformations by G. A Lie transformation ϕ can be regarded as a diffeomorphism $\phi : \mathbf{Q}^{n+1} \to \mathbf{Q}^{n+1}$ preserving lines on $\mathbf{Q}^{n+1}$, that is the restriction of a projective transformation $\Phi : \mathbf{P}^{n+2} \to \mathbf{P}^{n+2}$ preserving $\mathbf{Q}^{n+1}$. Thus,

$$G \cong PO(n+1,2) = O(n+1,2)/\{\pm 1\},$$

where $O(n+1,2) = \{P \in GL(n+3; \mathbf{R})\,;\, {}^tPSP = S\}$.

Let $o = (\mathbf{e}_1, \mathbf{e}_{n+1}) \in T_1S^n$ be the origin, where $(\mathbf{e}_1, \dots, \mathbf{e}_{n+1})$ is the natural basis of $\mathbf{E}^{n+1}$. The isotropy subgroup H of G at o given by

$$H = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} ; \right.$$
$$\left. A \in GL(2;\mathbf{R}),\ E \in O(n-1),\ B \in M_{n-1,2}(\mathbf{R}),\ C \in \mathfrak{o}(2) \right\}. \tag{1.3}$$

The group G acts on Λ^{2n-1} transitively, then

$$\Lambda^{2n-1} \cong G/H.$$

1.2. Lie frames

The Lie algebra $\mathfrak{g}$ of G is given by

$$\begin{aligned} \mathfrak{g} &= \{X \in \mathfrak{gl}(n+3;\mathbf{R})\,;\, {}^tXS + SX = 0\} \\ &= \left\{ \begin{pmatrix} \alpha & \delta & \zeta \\ \beta & \varepsilon & {}^t\delta \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \begin{matrix} \alpha \in M_{2,2}(\mathbf{R}),\ \beta,\ {}^t\delta \in M_{n-1,2}(\mathbf{R}), \\ \gamma,\ \zeta \in \mathfrak{o}(2),\ \varepsilon \in \mathfrak{o}(n-1) \end{matrix} \right\}, \end{aligned} \tag{1.4}$$

and the Lie algebra $\mathfrak{h}$ of H is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \varepsilon & 0 \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \begin{matrix} \alpha \in M_{2,2}(\mathbf{R}),\ \beta \in M_{n-1,2}(\mathbf{R}), \\ \gamma \in \mathfrak{o}(2),\ \varepsilon \in \mathfrak{o}(n-1) \end{matrix} \right\}. \tag{1.5}$$

A *Lie frame* $(Y_1, \dots, Y_{n+3})$ is an ordered set of vectors in $\mathbf{R}_2^{n+3}$ satisfying the relations

(1.6) $$\langle Y_i, Y_j \rangle = S_{ij}$$

for $1 \leq i, j \leq n + 3$. The space of all Lie frames can be identified with $O(n + 1, 2)$.

Let $\Omega = (\omega_j^i)$ be the Maurer-Cartan form introduced by the equation

(1.7) $$dY_i = \sum_{j=1}^{n+3} \omega_i^j Y_j.$$

Taking the exterior derivative of (1.8), we get the Maurer-Cartan equations

(1.8) $$d\omega_i^j = \sum_{j=1}^{n+3} \omega_i^k \wedge \omega_k^j.$$

By using (1.8), we find that $\omega_{n+3}^1 \wedge (d\omega_{n+3}^1)^{n-1} \neq 0$. Hence T_1S^n is a contact manifold with a contact form ω_{n+3}^1.

1.3. Legendre submanifolds and Dupin submanifolds

An immersed $(n - 1)$-dimensional integral submanifold of the contact distribution D is called a *Legendre submanifold.*

An immersed hypersurface $f : M^{n-1} \to S^n$ with a unit normal field $\xi : M^{n-1} \to S^n$ naturally induces a Legendre submanifold $\lambda = (f, \xi) : M^{n-1} \to T_1S^n$. This map λ is called the *Legendre map* induced by f with ξ. Conversely a Legendre submanifold $\lambda = (f, \xi) : M^{n-1} \to T_1S^n$ naturally induces a smooth map $f : M^{n-1} \to S^n$, which may have singularities; however, a Legendre submanifolds is locally transformed by a parallel transformation to be a Legendre map. (See [P Theorem 1].)

Let Y and Z are smooth maps from M^{n-1} into $\mathbf{Q}^{n+1}$. By $\mathrm{Line}\{Y(p), Z(p)\}$ we denote the line generated by the points $[Y(p)]$ and $[Z(p)]$ in $\mathbf{Q}^{n+1}$ for $p \in M^{n-1}$. Let $\lambda = \mathrm{Line}\{Y, Z\} : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Then,

(1.9) $$\langle dY, Z \rangle = 0.$$

(For the necessary and sufficient condition that a smooth map $\lambda = \mathrm{Line}\{Y, Z\} : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold, see [C Theorem 2.3].)

We call the sphere

(1.10) $$[K(p)] = [rY(p) + sZ(p)]$$

a *curvature sphere* of λ at $p \in M^{n-1}$, if there exist a non-zero vector X in T_pM^{n-1} and $r, s \in \mathbf{R}$ with $(r, s) \neq (0,0)$ such that

(1.11) $$r\, dY(X) + s\, dZ(X) \in \mathrm{Span}\{Y(p), Z(p)\}.$$

A curvature sphere is invariant under Lie transformations. The vector X is called a *principal vector* corresponding to $[K]$. At each point $p \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $[K_1], \ldots, [K_g]$. The principal vectors corresponding to the curvature sphere $[K_i]$ form a subspace T_i of the tangent space T_pM^{n-1}, and $T_pM^{n-1} = T_1 \oplus \cdots \oplus T_g$. If the dimension of T_i (which we call the *multiplicity* of $[K_i]$) is constant on an open subset U of M^{n-1}, then the distribution T_i is integrable on U. A connected submanifold $\mathcal{S}$ of M^{n-1} is called a *curvature surface* if at each $p \in \mathcal{S}$, the tangent space $T_p\mathcal{S}$ is equal to a T_i.

A Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is called a *Dupin submanifold* if along each curvature surface, the corresponding curvature sphere is constant. We say that a Dupin submanifold is *proper* if multiplicities of curvature spheres are constant on M^{n-1}. If $\mathcal{S}$ is a curvature surface of dimension $m > 1$ in a Legendre submanifold, then the corresponding curvature sphere is constant along $\mathcal{S}$. This fact shows that we have only to check the Dupin condition along curvature surfaces with dimension one.

1.4. The second fundamental form

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Let $(Y_1, \ldots, Y_{n+3})$ be a smooth Lie frame on an open set U of M^{n-1} such that for each $x \in U$, $\lambda(x) = \mathrm{Line}\{Y_{n+2}, Y_{n+3}\}$. We can choose the Lie frame so that Y_{n+2} is not a curvature sphere at U. By (1.4) and (1.9) we find that the forms $\omega_3^1, \omega_4^1, \ldots, \omega_{n+1}^1$ are linearly independent; i.e.

$$\omega_3^1 \wedge \omega_4^1 \wedge \cdots \wedge \omega_{n+1}^1 \neq 0. \tag{1.12}$$

The condition (1.9) is equal to

$$\omega_{n+3}^1 = 0. \tag{1.13}$$

Taking the exterior derivative of (1.13) and using the Maurer-Cartan equations (1.8), we obtain that

$$d\omega_{n+3}^1 = \sum_{\alpha=3}^{n+1} \omega_\alpha^2 \wedge \omega_\alpha^1 = 0 \tag{1.14}$$

By Cartan's lemma and (1.12) it follows that

$$\omega_\alpha^2 = \sum_{\beta=3}^{n+1} h_{\alpha\beta}\omega_\beta^1 \quad \text{with } h_{\alpha\beta} = h_{\beta\alpha}. \tag{1.15}$$

We define the *second fundamental form* of λ determined by Y_{n+2} to be the quadratic differential form

$$II(Y_{n+2}) = \sum_{\alpha,\beta=3}^{n+1} h_{\alpha\beta}\omega^1_\alpha\omega^1_\beta. \tag{1.16}$$

2. Frenet frames

2.1. Contact

Let N be a smooth n-dimensional manifold, and let $\lambda: M \to N$ and $\tilde{\lambda}: \tilde{M} \to N$ be embedded submanifolds of dimension m. We say that λ and $\tilde{\lambda}$ have *contact of at least order* 0 at $p \in M$ and $\tilde{p} \in \tilde{M}$ if $\lambda(p) = \tilde{\lambda}(\tilde{p})$, and that λ and $\tilde{\lambda}$ have *contact of at least order* 1 a p and $\tilde{p}$ if $\lambda(p) = \tilde{\lambda}(\tilde{p}) = x$ and $\lambda_* M_p = \tilde{\lambda}_* \tilde{M}_{\tilde{p}}$ as subspaces of N_x.

We reformulate the definition above. Let $G_{n,m}(N)$ denote the Grassmann bundle of tangent m-plane on N. The immersion λ induces the smooth mapping $T_\lambda: M \to G_{n,m}(N)$ given by $T_\lambda(p) = \lambda_* M_p$, which is a m-plane in $N_{\lambda(p)}$. Then λ and $\tilde{\lambda}$ have contact of at least order 1 at p and $\tilde{p}$ if and only if $T^{(1)}_\lambda(p) = T^{(1)}_{\tilde{\lambda}}(\tilde{p})$.

We iterate this construction and define higher order contact as follows: Let $N^{(0)} = N$, $n_0 = n$ and for $k \geq 0$ let $N^{(k+1)} = G_{n_k,m}(N^{(k)})$ where $n_{k+1} = \dim N^{(k+1)}$. Let $T^{(0)}_\lambda = \lambda$ and $T^{(k+1)}_\lambda = T_{T^{(k)}_\lambda}: M \to G_{n_k m}(N^{(k)})$. Then for $k \geq 0$ we say that λ and $\tilde{\lambda}$ have *contact of at least order* k at p and $\tilde{p}$ if $T^{(k)}_\lambda(p) = T^{(k)}_{\tilde{\lambda}}(\tilde{p})$. If λ and $\tilde{\lambda}$ have contact of at least order k at $p \in M$ and $\tilde{p} \in \tilde{M}$ but not of order $k+1$, then we say that λ and $\tilde{\lambda}$ have *contact of order* k at p and $\tilde{p}$.

We say that λ and $\tilde{\lambda}$ have *G-contact of least order* k at $\tilde{p}$ if there exists a $P \in G$ such that λ and $P \circ \tilde{\lambda}$ have contact of at least order k at p and $\tilde{p}$.

2.2. Construction of Frenet frames

Let G be a transitive Lie transformation group on a manifold N. We fix an origin o of N, and denote the isotropy subgroup of G at o by H. Then the map $\pi: G \to N$ given by $\pi(P) = P(o)$ induces diffeomorphism $G/H \cong N$.

Let $\lambda: M \to G/H \cong N$ be a connected, smoothly embedded n-dimensional submanifold of a homogeneous space G/H. Firstly we construct the set of zeroth order frames. A *zeroth order frame* at $p \in M$ is an element $P \in G$ such that $\pi(P) = \lambda(p)$.

Choosing a basis $e_1, \ldots, e_{m_0}$ of the tangent space N_o, we have a natural bundle map $h_0: G \to L(N)$ defined by $h_0(P) = P_*(e_1, \ldots, e_{m_0})$, where $L(N) \to N$ is the principal $GL(m, \mathbf{R})$-bundle of linear frames on N. We identify $P \in G$ with $h_0(P) \in L(N)$, so we call P a frame.

Let L_0 denote the set of all zeroth order frames on M. A *zeroth order frame field* u along λ is a smooth cross section of $L_0 \to M$; i.e. is a smooth map $u: M \to$

G such that $\pi \circ u = \lambda$.

Secondly we construct first order frames. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. We take a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{h}$, and choose basis $e_1, \ldots, e_{m_0}$ of $\mathfrak{m}$. With respect to this basis, we consider the linear isotropy representation $\rho_0 : H \to GL(m_0, \mathbf{R})$ given by the adjoint action of H on $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$. There is a naturally defined smooth map λ_0 from L_0 to the Grassmann manifold $G_{m_0,n}$ given by $\lambda_0(P) = P_*^{-1}\lambda_* M_p$ where $\lambda(p) = \pi(P)$. We choose a local cross section W_1 of the action (H, ρ_0) on $G_{m_0,n}$. We say that λ has the type of W_1 if there exists a zeroth order frame field u along λ such that $\lambda_0(u) \subset W_1$. If λ has the type of W_1, we let $L_1 = \lambda_0^{-1}(W_1)$ and call L_1 the set of *first order frames* on λ (with respect to W_1). We define a *first order frame field* along λ to be a smooth cross section $L_1 \to M$.

The smooth map $\lambda_0 \circ u : M \to W_1$ does not depend on the choice of first order frame field u along λ. Choose a coordinate system $x^1, \ldots, x^{\mu_1}$ on W_1, where $\mu_1 = \dim W_1$. We call the functions $k^i = x^i \circ \lambda_0 \circ u (i = 1, \ldots, \mu_1)$ the *first order invariants* of λ.

The set of first order frames L_1 gives first order contact under the action of G. To put it more precisely, let $\lambda : M \to G/H$ and $\tilde{\lambda} : \tilde{M} \to G/H$ be smoothly embedded n-dimensional submanifold on which first order frames can be constructed. Then λ and $\tilde{\lambda}$ have G-contact of at least order 1 at $p \in M$ and $\tilde{p} \in \tilde{M}$ if and only if they are both the type of a local cross section W_1 of ρ_0, and they have the same first order invariants at p and $\tilde{p}$. (See [J 1.6 Theorem 1].)

Furthermore we iterate this construction of frames: $L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k \supset \cdots$. Thus we construct a set of k-th order frames L_k which gives k-th order contact under the action of G. The sequence $\dim L_0 \geq \dim L_1 \geq \dim L_2 \geq \cdots$ eventually stabilized. Thus there is the smallest integer $q \geq 1$ such that $\dim L_k = \dim L_q$ for all $k \geq q$. Then the frames of order q are called the *Frenet frames* of λ.

We have the following congruence and existence theorem: Let $\lambda : M \to G/H$ and $\tilde{\lambda} : \tilde{M} \to G/H$ be smoothly embedded n-dimensional submanifold. Then λ and $\tilde{\lambda}$ are G-congruent if and only if their Frenet frames are of the same order q, they are both the type of a local cross section W_q and there exist a one-to-one correspondence $\varphi : M \to \tilde{M}$ such that $\tilde{k}^a \circ \varphi = k^a$ where k^a, $\tilde{k}^a$ are invariants of order $< q$. (See [J I.11 Theorem 3].)

3. Lie frames of Legendre maps in T_1S^3 under $PO(4,2)$

3.1. Main theorem

Our main theorem is the following:

THEOREM 3.1. *Let $\lambda : M^2 \to T_1S^3$ be a Legendre map which is induced by an embedded oriented surface $f : M^2 \to S^3$ with a field of unit normals $\xi : M^2 \to S^3$. Let Ω be the Maurer-Cartan form on $G = PO(4,2)$ and ϕ_1, ϕ_2 coframe fields on M^2. Then the Legendre map λ belongs to one of the following fine types.*

Type (a): *We can take a Lie frame $u : M^2 \to G$ of λ such that*

$$u^*\Omega = \begin{pmatrix} 0 & 0 & \phi_1 & \phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_1 & 0 \\ 0 & 0 & 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Type (b): *We can take a Lie frame $u : M^2 \to G$ of λ such that*

$$u^*\Omega = \begin{pmatrix} 0 & 0 & \phi_1 & \phi_2 & 0 & 0 \\ 0 & 0 & \phi_1 & -\phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_1 & \phi_1 \\ 0 & 0 & 0 & 0 & \phi_2 & -\phi_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

Type (c): *We can take a Lie frame $u : M^2 \to G$ of λ such that*

$$\begin{aligned} u^*\Omega &= \begin{pmatrix} \alpha & \delta & 0 \\ \beta & 0 & {}^t\delta \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix}, \\ \alpha &= \begin{pmatrix} -3k^1\phi_1 - 3k^2\phi_2 & k^1\phi_1 + k^2\phi_2 \\ (k^1+1)\phi_1 + k^2\phi_2 & (-3k^1+1)\phi_1 - 3k^2\phi_2 \end{pmatrix}, \\ \beta &= \begin{pmatrix} k^1\phi_1 + (k^2-k^3)\phi_2 & k^1\phi_1 + (k^2-k^3)\phi_2 \\ 2k^3\phi_1 + k^4\phi_2 & -2k^3\phi_1 - k^4\phi_2 \end{pmatrix}, \\ \gamma &= \begin{pmatrix} 0 & k^4\phi_1 \\ -k^4\phi_1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1 & -\phi_2 \end{pmatrix}. \end{aligned} \tag{3.3}$$

In (3.3), k^i $(i = 1, 2, 3, 4)$ *are smooth functions on* M^2 *that satisfy the following integrability conditions*:

$$
(3.4)\qquad
\begin{aligned}
&k^2 + 6k^1k^3 + k_1^3 = 0,\\
&12k^2k^3 - 8k^1k^4 + (2k_2^3 - k_1^4) = 0.\\
&8k^2k^4 + k_2^4 = 0,\\
&2k^1k^2 + k^3 - k_2^1 + k_1^2 = 0,
\end{aligned}
$$

where k_j^i $(i = 1, 2, 3, 4,\ j = 1,2)$ *are smooth functions on* M^2 *such that* $dk^i = k_1^i\phi_1 + k_2^i\phi_2$.

Type (d): *We can take a Lie frame* $\boldsymbol{u} : M^2 \to G$ *of* λ *such that*

$$
(3.5)\qquad
\begin{aligned}
&u^*\Omega = \begin{pmatrix} \alpha & \delta & 0 \\ \beta & 0 & {}^t\delta \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix},\\
&\alpha = \begin{pmatrix} 3k^1\phi_1 + 3k^2\phi_2 & k^1\phi_1 + k^2\phi_2 \\ k^1\phi_1 + (k^2 + 1)\phi_2 & 3k^1\phi_1 + (3k^2 - 1)\phi_2 \end{pmatrix},\\
&\beta = \begin{pmatrix} k^4\phi_1 - 2k^3\phi_2 & k^4\phi_1 - 2k^3\phi_2 \\ (k^1 + k^3)\phi_1 + k^2\phi_2 & (-k^1 - k^3)\phi_1 - k^2\phi_2 \end{pmatrix},\\
&\gamma = \begin{pmatrix} 0 & k^4\phi_2 \\ -k^4\phi_2 & 0 \end{pmatrix},\quad \delta = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1 & -\phi_2 \end{pmatrix}.
\end{aligned}
$$

In (3.5), k^i $(i = 1, 2, 3, 4)$ *are smooth functions on* M^2 *that satisfy the following integrability conditions*:

$$
(3.6)\qquad
\begin{aligned}
&k^1 - 6k^2k^3 + k_2^3 = 0,\\
&12k^1k^3 + 8k^2k^4 + (-2k_1^3 - k_2^4) = 0.\\
&8k^1k^4 - k_1^4 = 0,\\
&2k^1k^2 - k^3 - k_2^1 + k_1^2 = 0,
\end{aligned}
$$

where k_j^i $(i = 1,2,3,4,\ j = 1,2)$ *are smooth functions on* M^2 *such that* $dk^i = k_1^i\phi_1 + k_2^i\phi_2$.

Type (e): *We can take a Lie frame* $\boldsymbol{u} : M^2 \to G$ *of* λ *such that*

$$
(3.7)\qquad u^*\Omega = \begin{pmatrix} \alpha & \delta & 0 \\ \beta & 0 & {}^t\delta \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix},
$$

$$\alpha = \begin{pmatrix} (3k^1+1)\phi_1 - (3k^2+1)\phi_2 & k^1\phi_1 + k^2\phi_2 \\ (k^1+1)\phi_1 + (k^2+1)\phi_2 & (3k^1+2)\phi_1 - (3k^2+2)\phi_2 \end{pmatrix},$$

$$\beta = \begin{pmatrix} k^5\phi_1 + k^3\phi_2 & k^5\phi_1 + k^3\phi_2 \\ k^4\phi_1 + k^6\phi_2 & -k^4\phi_1 - k^6\phi_2 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 0 & k^6\phi_1 + k^5\phi_2 \\ -k^6\phi_1 - k^5\phi_2 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1 & -\phi_2 \end{pmatrix}.$$

In (3.7), k^i $(i = 1, \ldots, 6)$ *are smooth functions on* M^2 *that satisfy the following integrability conditions*:

$$(3.8)\qquad \begin{aligned} k^3 &= (k^1 + k^2)^2 + 2k^1_2 + k^2_1, \\ k^4 &= (k^1 + k^2)^2 - k^1_2 + 2k^2_1, \\ 6k^1k^3 &+ 4k^2k^5 + 3k^3 + 2k^5 - k^3_1 + k^5_2 = 0, \\ 4k^1k^6 &+ 6k^2k^4 + 3k^4 + 2k^6 + k^4_2 - k^6_1 = 0 \\ 4k^1k^5 &+ 4k^2k^6 + 2k^5 + 2k^6 + k^5_1 - k^6_2 = 0, \end{aligned}$$

where k^i_j $(i = 1, \ldots, 6,\ j = 1, 2)$ *are smooth functions on* M^2 *such that* $dk^i = k^i_1\phi_1 + k^i_2\phi_2$.

3.2. Proof of Theorem 3.1

Before turning to the proof of Theorem 3.1, we indicate its process in Figure 1.

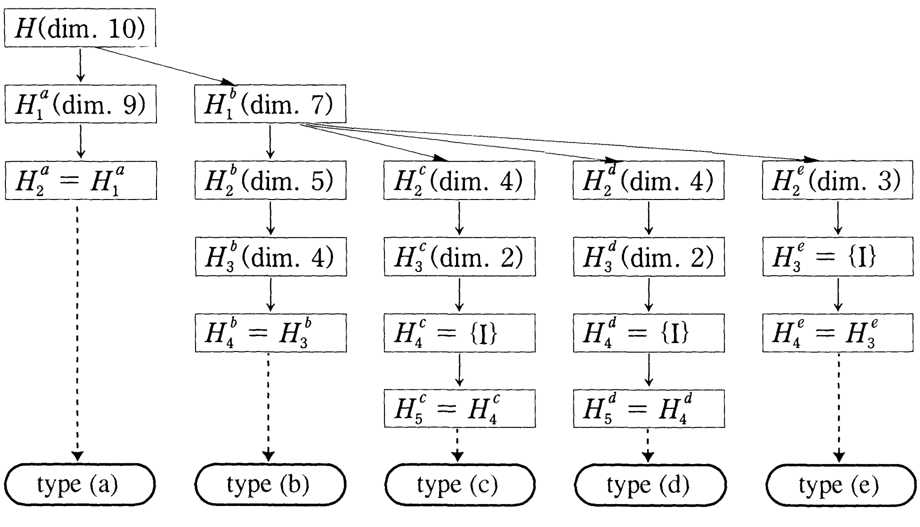

Figure 1

The detail of the proof of Theorem 3.1 is as follows:

Construction of first order frames

We denote by H the isotropy subgroup of G at the origin o, and $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and H respectively. (See (1.3), (1.4) and (1.5).) As a vector subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$, we take the $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$. For a basis of $\mathfrak{m}$ we choose the following e_1, e_2, e_3, e_4, e_5 :

$$e_1 = \begin{pmatrix} 0 & 0 & E_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & E_2 & 0 \\ 0 & 0 & {}^tE_2 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & E_3 & 0 \\ 0 & 0 & {}^tE_3 \\ 0 & 0 & 0 \end{pmatrix},$$

(3.9)

$$e_4 = \begin{pmatrix} 0 & E_4 & 0 \\ 0 & 0 & {}^tE_4 \\ 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & E_5 & 0 \\ 0 & 0 & {}^tE_5 \\ 0 & 0 & 0 \end{pmatrix},$$

where

(3.10)

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The isotropy representation $\rho_0 : H \to GL(5, \mathbf{R})$ is

$$\rho_0(P) = \begin{pmatrix} \det A & 0 & 0 \\ * & a_1^1 E & a_2^1 E \\ * & a_1^1 E & a_2^2 E \end{pmatrix} = \begin{pmatrix} \det A & 0 & 0 \\ * & a_1^1 I_2 & a_2^1 I_2 \\ * & a_1^2 I_2 & a_2^2 I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & E & 0 \\ * & 0 & E \end{pmatrix} \tag{3.11}$$

for $P \in H, A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \in GL(2, \mathbf{R})$, $E \in O(2)$. Let $\bar{A}$ denote the former matrix determined by A in (3.11), and $\bar{E}$ the latter matrix determined by E in (3.11). With respect to $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ we decompose the Maurer-Cartan form Ω into $\Omega_0 + \Theta_0$, and we set $\Theta_0 = \sum \theta_i e_i$ $(i = 1,2,3,4,5)$ then

$$\theta_1 = \omega_6^1,\ \theta_2 = \omega_3^1,\ \theta_2 = \omega_4^1,\ \theta_4 = \omega_3^2,\ \theta_5 = \omega_4^2. \tag{3.12}$$

We need to choose a local cross section W_1 of the action (H, ρ_0) on $G_{5,2}$ to obtain a first order frames on λ. To get orbits of the action of (H, ρ_0), we consider the Maurer-Cartan form Ω on G. Let u be a zeroth order frame field along λ, and ϕ_1, ϕ_2 coframe fields on M. Set $u^*\theta_i = x_i\phi_1 + y_i\phi_2$ for some smooth functions x_i, y_i on M, then

$$\lambda_0(u) = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \end{bmatrix} : M \to G_{5,2}.$$

(For further details, see [J].) Hence u is a first order frame field along λ with respect to W_1 if and only if

$${}^t\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} \in W_1.$$

PROPOSITION 3.2. *We can take two types of* W_1:

$$W_1^a = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \right\}, \quad W_1^b = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \right\}. \tag{3.13}$$

Proof. We see that $x_1 = y_1 = 0$, because λ is a Legendre map and ω_6^1 is a contact form. We choose coframe fields ϕ_1, ϕ_2 on M such that $u^*\theta_2 = \phi_1$, $u^*\theta_3 = \phi_2$. From (1.15),

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ I_2 \\ H \end{bmatrix}.$$

We first consider the orbit of the action of $\bar{E}$, where $E \in O(2)$.

$$\bar{E}\begin{bmatrix} 0 \\ I_2 \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ E \\ EH \end{bmatrix} = \begin{bmatrix} 0 \\ I_2 \\ EHE^{-1} \end{bmatrix}.$$

Thus those orbits meet ${}^t\begin{bmatrix} 0 & 1 & 0 & \lambda_1 & 0 \\ 0 & 0 & 1 & 0 & \lambda_2 \end{bmatrix}$.

Next we consider the orbit of the action of $\bar{A}$, where $A \in GL(2, \mathbf{R})$.

$$\bar{A}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a_1^1 + \lambda_1 a_2^1 & 0 \\ 0 & a_1^1 + \lambda_2 a_2^1 \\ a_1^2 + \lambda_1 a_2^2 & 0 \\ 0 & a_1^2 + \lambda_2 a_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \dfrac{a_1^2 + \lambda_1 a_2^2}{a_1^1 + \lambda_1 a_2^1} & 0 \\ 0 & \dfrac{a_1^2 + \lambda_2 a_2^2}{a_1^1 + \lambda_2 a_2^1} \end{bmatrix}.$$

In case that $\lambda_1 = \lambda_2$, those orbits meet ${}^t\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, and in case that $\lambda_1 \neq \lambda_2$, those orbits meet ${}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$. □

We say that $\lambda : M \to T_1 S^3$ is of type (a) if it has the type of W_1^a, and that λ is of type (B) if it has the type of W_1^b.

Type (a)

Let $\lambda : M \to T_1 S^3$ be of type (a); i.e. there exists a zeroth order frame field u along λ such that

$$u^*\omega_6^1 = u^*\theta_1 = 0,\ u^*\omega_3^1 = u^*\theta_1 = \phi_1,\ u^*\omega_4^1 = u^*\theta_3 = \phi_2, \tag{3.14}$$

$$u^*\omega_3^2 = u^*\theta_4 = 0,\ u^*\omega_4^2 = u^*\theta_5 = 0. \tag{3.15a}$$

Construction of second order frames of type (a)

The isotropy subgroup H_1^a of H at a point of W_1^a is

$$H_1^a = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}; \right. \tag{3.16a}$$

$$\left. A = \begin{pmatrix} a_1^1 & a_2^1 \\ 0 & a_2^2 \end{pmatrix} \in GL(2, \mathbf{R}),\ E \in O(2),\ B \in M_{2,2}(\mathbf{R}),\ C \in \mathfrak{o}(2) \right\}.$$

The Lie algebra $\mathfrak{h}_1^a$ of H_1^a is

$$\mathfrak{h}_1^a = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \varepsilon & 0 \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix};\ \alpha = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ 0 & \alpha_2^2 \end{pmatrix},\ \beta \in M_{2,2}(\mathbf{R}),\ \gamma,\ \varepsilon \in \mathfrak{o}(2) \right\}. \tag{3.17a}$$

We decompose $\mathfrak{h} = \mathfrak{h}_1^a + \mathfrak{m}_1^a$, and for a basis of $\mathfrak{m}_1^a$ we take

(3.18a) $$e_6 = \begin{pmatrix} E_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^tE_4 \end{pmatrix}.$$

The representation $\rho_1^a : H_1^a \to GL(6, \mathbf{R})$ is

(3.19a) $$\rho_1^a(P) = \begin{pmatrix} * & 0 & 0 & * & * & 0 \\ * & \multicolumn{2}{c}{\multirow{2}{*}{$a_1^1 E$}} & * & * & 0 \\ * & & & * & * & 0 \\ * & 0 & 0 & * & * & 0 \\ * & 0 & 0 & * & * & 0 \\ * & 0 & 0 & * & * & \dfrac{a_2^2}{a_1^1} \end{pmatrix},$$

where $P \in H_1^a$. With respect to $\mathfrak{h} = \mathfrak{h}_1^a + \mathfrak{m}_1^a$ we have $\Omega_0 = \Omega_1^a + \Theta_1^a$, and we set $\Theta_1^a = \theta_6^a e_6$ then

(3.20a) $$\theta_6^a = \omega_1^2.$$

Taking the exterior derivative of (3.15a), and using (3.14) and the Maurer-Cartan equation (1.8), we obtain the following equations:

(3.21a) $$du^*\theta_4 = \phi_1 \wedge u^*\theta_6^a = 0, \quad du^*\theta_5 = \phi_2 \wedge u^*\theta_6^a = 0,$$

where u is a first order frame field along λ. It follows that

(3.22a) $$u^*\theta_6^a = 0.$$

As a result, we take a local cross section $W_2^a = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \right\}$ of the action (H_1^a, ρ_1^a) on $G_{6,2}$. For this reason, a first order frame field u along λ is a second order frame field along λ with respect to W_2^a ; i.e.

$$u^*(\Theta_0 + \Theta_1^a) = \phi_1 e_2 + \phi_2 e_3.$$

The isotropy subgroup H_2^a of H_1^a at a point of W_2^a is equal to H_1^a. In this way we get a second order frame field $u : M \to G/H_1^a$ along $\lambda : M \to G/H$. Adding an extra step we lift u from G/H_1^a to G, then we get a Lie frame of λ. By choosing a lifting $\tilde{u}$ of u such that $\tilde{u}^*\Omega_1^a = 0$, we obtain the Lie frame of type (a) in Theorem 3.1.

Type (B)

Let $\lambda : M \to T_1S^3$ be of type (B); i.e. there exists a zeroth order frame field u along λ which satisfies (3.14) and the following equations:

$$u^*\omega_3^2 = u^*\theta_4 = \phi_1, \quad u^*\omega_4^2 = u^*\theta_5 = -\phi_2. \tag{3.15b}$$

Construction of second order frames of type (B)

The isotropy subgroup H_1^b of H at a point of W_1^b is

$$H_1^b = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \pm I_2 & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} ; \right. \tag{3.16b}$$
$$\left. A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \in GL(2, \mathbf{R}),\ B \in M_{2,2}(\mathbf{R}),\ C \in \mathfrak{o}(2) \right\}.$$

The Lie algebra $\mathfrak{h}_1^b$ of H_1^b is

$$\mathfrak{h}_1^b = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},\ \beta \in M_{2,2}(\mathbf{R}),\ \gamma \in \mathfrak{o}(2) \right\}. \tag{3.17b}$$

We decompose $\mathfrak{h} = \mathfrak{h}_1^b + \mathfrak{m}_1^b$, and for a basis of $\mathfrak{m}_1^b$ we take

$$e_6 = \left\{ \begin{pmatrix} E_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^tE_4 \end{pmatrix}, e_7 = \begin{pmatrix} E_5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^tE_5 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. . \tag{3.18b}$$

The representation $\rho_1^b : H_1^b \to GL(8, \mathbf{R})$ is

(3.19b)

$$\rho_1^b(P) = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & a_1 & 0 & a_2 & 0 & 0 & 0 & 0 \\ * & 0 & a_1 & 0 & a_2 & 0 & 0 & 0 \\ * & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 \\ * & 0 & a_2 & 0 & a_1 & 0 & 0 & 0 \\ * & pb_2^1 - qb_1^1 & pb_2^2 - qb_1^2 & -pb_1^1 + qb_2^1 & -pb_1^2 + qb_2^2 & p & -q & 0 \\ * & -qb_2^1 + pb_1^1 & -qb_2^2 + pb_1^2 & qb_1^1 - pb_2^1 & qb_1^2 - qb_2^2 & -q & p & 0 \\ * & -b_1^2 & b_1^1 & -b_2^2 & b_2^1 & 0 & 0 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & a_1 & 0 & a_2 & 0 & 0 & 0 & 0 \\ * & 0 & a_1 & 0 & a_2 & 0 & 0 & 0 \\ * & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 \\ * & 0 & a_2 & 0 & a_1 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & p & -q & 0 \\ * & 0 & 0 & 0 & 0 & -q & p & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & b_2^1 & b_2^2 & -b_1^1 & -b_1^2 & 1 & 0 & 0 \\ * & b_1^1 & b_1^2 & -b_2^1 & -b_2^2 & 0 & 1 & 0 \\ * & -b_1^2 & b_1^1 & -b_2^2 & b_2^1 & 0 & 0 & 1 \end{pmatrix},
$$

where $P \in H_1^b$, $p = \dfrac{(a_1)^2 + (a_2)^2}{\det A}$, $q = \dfrac{2a_1 a_2}{\det A}$. Let $\tilde{A}$ denote the former matrix determined by A, and $\tilde{B}$ the latter matrix determined by B. With respect to $\mathfrak{h} = \mathfrak{h}_1^b + \mathfrak{m}_1^b$ we have $\Omega_0 = \Omega_1^b + \Theta_1^b$, and we set $\Theta_1^b = \sum \theta_i^b e_i^b$ $(i = 6, 7, 8)$ then

$$
\text{(3.20b)} \qquad \theta_6^b = \omega_1^2 - \omega_2^1, \quad \theta_7^b = \omega_2^2 - \omega_1^1, \quad \theta_8^b = \omega_4^3.
$$

By taking the exterior derivative of (3.15b), and by using (3.14) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$
\text{(3.21b)} \qquad \begin{aligned} d(\phi_1 - u^*\theta_4) &= -\phi_1 \wedge u^*\theta_6^b - \phi_1 \wedge u^*\theta_7^b - 2u^*\theta_8^b \wedge \phi_2 = 0, \\ d(\phi_2 + u^*\theta_5) &= -u^*\theta_6^b \wedge \phi_2 + u^*\theta_7^b \wedge \phi_2 - 2\phi_1 \wedge u^*\theta_8^b = 0. \end{aligned}
$$

Set $u^*\theta_i^b = x_i\phi_1 + y_i\phi_2$ $(i = 6, 7, 8)$ for some smooth functions x_i, y_i on M. From (3.21b),

$$
y_6 + y_7 + 2x_8 = 0, \quad x_6 - y_7 + 2y_8 = 0.
$$

We put

$$
x_6 + y_8 = x_7 - x_8 = X, \quad y_6 + x_8 = -y_7 - x_8 = Y.
$$

We consider orbits and local sections W_2 of the action of (H_1^6, ρ_1^b) on $G_{8,2}$.

PROPOSITION 3.3. *We can take four types of* W_2:

$$
\text{(3.22b)} \qquad \begin{aligned}
W_2^b &= \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \right\}, \\
W_2^c &= \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \right\}, \\
W_2^d &= \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix} \right\}, \\
W_2^e &= \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix} \right\},
\end{aligned}
$$

Proof. We first consider the orbit of the action of $\tilde{B}$, where $B \in M_{2,2}(\mathbf{R})$.

$$\tilde{B}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ x_6 & y_6 \\ x_7 & y_7 \\ x_8 & y_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -(b_1^1 - b_2^1) + x_6 & (b_1^2 + b_2^2) + y_6 \\ (b_1^1 - b_2^1) + x_7 & (b_1^2 + b_2^2) + y_7 \\ -(b_1^2 + b_2^2) + x_8 & (b_1^1 - b_2^1) + y_8 \end{bmatrix}.$$

Thus those orbits meet ${}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & 0 & -1 & Y & -Y & 0 \end{bmatrix}$.

Next we consider the orbit of the action of $\tilde{A}$, where $A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \in GL(2, \mathbf{R})$.

$$\tilde{A}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ X & Y \\ X & -Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a_1 + a_2 & 0 \\ 0 & a_1 - a_2 \\ a_1 + a_2 & 0 \\ 0 & -a_1 + a_2 \\ (p-q)X & (p+q)Y \\ (p-q)X & -(p+q)Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ \dfrac{p-q}{a_1+a_2}X & \dfrac{p+q}{a_1-a_2}Y \\ \dfrac{p-q}{a_1+a_2}X & -\dfrac{p+q}{a_1-a_2}-Y \\ 0 & 0 \end{bmatrix}.$$

In case that $X = 0,\ Y = 0$, those orbits meet the point of W_2^b, in case that $X \neq 0$, $Y = 0$, those orbits meet the point of W_2^c, in case that $X = 0,\ Y \neq 0$, those orbits meet the point of W_2^d and in case that $X \neq 0,\ Y \neq 0$, those orbits meet the point of W_2^e. □

We say that $\lambda : M \to T_1S^3$ is of type (B-b) if it has the type of W_2^b. (We abbreviate "type (B-b)" to "type (b)".) And we say that λ is type (c), type (d) or type (e) if it has the type or W_2^c, W_2^d or W_2^e respectively.

Type (b)

Let $\lambda : M \to T_1S^3$ be of type (b); i.e. there exists a first order frame field u along λ which satisfies (3.14), (3.15b) and the following equations:

(3.23b) $u^*(\omega_1^2 - \omega_2^1) = u^*\theta_6^b = 0,\ u^*(\omega_2^2 - \omega_1^1) = u^*\theta_7^b = 0,\ u^*\omega_4^3 = u^*\theta_8^b = 0.$

Construction of third order frames of type (b)

The isotropy subgroup H_2^b of H_1^b at a point of W_2^b is

(3.24b)
$$H_2^b = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} ; \right.$$
$$\left. A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \in GL(2, \mathbf{R}),\ B = \begin{pmatrix} b_1 & b_1 \\ b_2 & -b_2 \end{pmatrix},\ C \in \mathfrak{o}(2) \right\}.$$

The Lie algebra $\mathfrak{h}_2^b$ of H_2^b is

$$\text{(3.25b)}\quad \mathfrak{h}_2^b = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{pmatrix},\ \gamma \in \mathfrak{o}(2) \right\}.$$

We decompose $\mathfrak{h}_1^b = \mathfrak{h}_2^b + \mathfrak{m}_2^b$, and for a basis of $\mathfrak{m}_2^b$ we take

$$\text{(3.26b)}\qquad e_9^b = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ E_3 & 0 & 0 \\ 0 & {}^tE_3 & 0 \end{pmatrix} \quad e_{10}^b = \begin{pmatrix} 0 & 0 & 0 \\ E_5 & 0 & 0 \\ 0 & {}^tE_5 & 0 \end{pmatrix} \right\}.$$

The representation $\rho_2^b : H_2^b \to GL(10, \mathbf{R})$ is

$$\text{(3.27b)}\quad \rho_2^b(P) = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
* & a_1 & 0 & a_2 & 0 & * & * & * & 0 & 0 \\
* & 0 & a_1 & 0 & a_2 & * & * & * & 0 & 0 \\
* & a_2 & 0 & a_1 & 0 & * & * & * & 0 & 0 \\
* & 0 & a_2 & 0 & a_1 & * & * & * & 0 & 0 \\
* & \dfrac{sb_1}{t} & -\dfrac{tb_2}{s} & -\dfrac{sb_1}{t} & -\dfrac{tb_2}{s} & * & * & * & 0 & 0 \\
* & \dfrac{sb_1}{t} & \dfrac{tb_2}{s} & -\dfrac{sb_1}{t} & \dfrac{tb_2}{s} & * & * & * & 0 & 0 \\
* & -b_2 & b_1 & b_2 & b_1 & * & * & * & 0 & 0 \\
* & \dfrac{-(b_2)^2 - c}{s} & \dfrac{2b_1b_2}{s} & \dfrac{(b_2)^2 - c}{s} & \dfrac{2b_1b_2}{s} & * & * & * & \dfrac{1}{s} & 0 \\
* & -\dfrac{2b_1b_2}{t} & \dfrac{(b_1)^2 - c}{t} & \dfrac{2b_1b_2}{t} & \dfrac{(b_1)^2 + c}{t} & * & * & * & 0 & \dfrac{1}{t}
\end{pmatrix},$$

where $P \in H_2^b$, $s = a_1 - a_2$, $t = a_1 + a_2$. With respect to $\mathfrak{h}_1^b = \mathfrak{h}_2^b + \mathfrak{m}_2^b$ we have $\Omega_1^b = \Omega_2^b + \Theta_2^b$, and we set $\Theta_2^b = \sum \theta_i^b e_i^b$ $(i = 9, 10)$ then

(3.28b) $$\theta_9^b = \omega_2^3 - \omega_1^3, \quad \theta_{10}^b = \omega_2^4 + \omega_1^4.$$

By taking the exterior derivative of (3.23b), and by using (3.14), (3.15b) and the Maurer-Cartan equation (1.8), we obtain the following equations:

(3.29b) $$\begin{aligned} du^*\theta_6^b &= \phi_1 \wedge u^*\theta_9^b - u^{*b}_{10} \wedge \phi_2 = 0, \\ du^*\theta_7^b &= -\phi_1 \wedge u^*\theta_9^b - u^{*b}_{10} \wedge \phi_2 = 0, \\ du^*\theta_8^b &= -\phi_1 \wedge u^*\theta_{10}^b + u^{*b}_9 \wedge \phi_2 = 0. \end{aligned}$$

Set $u^*\theta_i^b = x_i\phi_1 + y_i\phi_2$ $(i = 9, 10)$ for some smooth functions x_i, y_i on M. From (3.29b),

$$x_9 = y_{10}, \quad x_{10} = y_9 = 0.$$

We consider orbits and local sections W_3^b of the action of (H_2^b, ρ_2^b) on $G_{10,2}$.

$$\rho_2^b(P)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ x_9 & 0 \\ 0 & x_9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t & 0 \\ 0 & s \\ t & 0 \\ 0 & -s \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{-2c+x_9}{s} & 0 \\ 0 & \dfrac{-2c+x_9}{t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{-2c+x_9}{\det A} & 0 \\ 0 & \dfrac{-2c+x_9}{\det A} \end{bmatrix}.$$

Thus we take

(3.30b) $$W_3^b = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\};$$

i.e.

(3.31b) $$u^*\theta_9^b = 0, \quad u^*\theta_{10}^b = 0.$$

Construction of fourth order frames of type (b)

The isotropy subgroup H_3^b of H_2^b at a point of W_3^b is

(3.32b) $$H_3^b = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \right.;$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \in GL(2, \mathbf{R}),\ B = \begin{pmatrix} b_1 & b_1 \\ b_2 & -b_2 \end{pmatrix} \Bigg\}.$$

The Lie algebra $\mathfrak{h}_3^b$ of H_3^b is

$$\text{(3.33b)} \qquad \mathfrak{h}_3^b = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ 0 & {}^t\beta & -{}^t\alpha \end{pmatrix};\ \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{pmatrix} \right\}.$$

We decompose $\mathfrak{h}_2^b = \mathfrak{h}_3^b + \mathfrak{m}_3^b$, and for a basis of $\mathfrak{m}_3^b$ we take

$$\text{(3.34b)} \qquad e_{11}^b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_1 & 0 & 0 \end{pmatrix}.$$

The representation $\rho_3^b : H_3^b \to GL(11, \mathbf{R})$ is
(3.35b)

$$\rho_3^b(P) = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 \\
* & a_1 & 0 & a_2 & 0 & * & * & * & * & * & 0 \\
* & 0 & a_1 & 0 & a_2 & * & * & * & * & * & 0 \\
* & a_2 & 0 & a_1 & 0 & * & * & * & * & * & 0 \\
* & 0 & a_2 & 0 & a_1 & * & * & * & * & * & 0 \\
* & \frac{sb_1}{t} & -\frac{tb_2}{s} & -\frac{sb_1}{t} & -\frac{tb_2}{s} & * & * & * & * & * & 0 \\
* & \frac{sb_1}{t} & \frac{tb_2}{s} & -\frac{sb_1}{t} & \frac{tb_2}{s} & * & * & * & * & * & 0 \\
* & -b_2 & b_1 & b_2 & b_1 & * & * & * & * & * & 0 \\
* & -\frac{(b_2)^2}{s} & \frac{2b_1b_2}{s} & \frac{(b_2)^2}{s} & \frac{2b_1b_2}{s} & * & * & * & * & * & 0 \\
* & -\frac{2b_1b_2}{t} & \frac{(b_1)^2}{t} & \frac{2b_1b_2}{t} & \frac{(b_1)^2}{t} & * & * & * & * & * & 0 \\
* & -\frac{b_1(b_2)^2}{st} & \frac{(b_1)^2b_2}{st} & \frac{b_1(b_2)^2}{st} & \frac{(b_1)^2b_2}{st} & * & * & * & * & * & \frac{1}{st}
\end{pmatrix},$$

where $P \in H_3^b$, $s = a_1 - a_2$, $t = a_1 + a_2$. With respect to $\mathfrak{h}_2^b = \mathfrak{h}_3^b + \mathfrak{m}_3^b$ we have $\Omega_2^b = \Omega_3^b + \Theta_3^b$, and we set $\Theta_3^b = \theta_{11}^b e_{11}^b$ then

$$\text{(3.36b)} \qquad \theta_{11}^b = \omega_2^5.$$

By taking the exterior derivative of (3.31b), and by using (3.14), (3.15b), (3.23b) and the Maurer-Cartan equation (1.8), we obtain the following equations:

(3.37b) $$du^*\theta_9^b = -2\phi_1 \wedge u^*\theta_{11}^b = 0, \quad du^*\theta_{10}^b = 2u^*\theta_{11}^b \wedge \phi_2 = 0.$$

It follows that

(3.38b) $$u^*\theta_{11}^b = 0.$$

Thus we take a local cross section

(3.39b) $$W_4^b = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

of the action (H_3^a, ρ_3^a) on $G_{11,2}$. For this reason, third order frame fields u along λ are fourth order frame fields along λ with respect to W_4^b ; i.e.

$$u^*(\Theta_0 + \Theta_1^b + \Theta_2^b + \Theta_3^b) = \phi_1 e_2 + \phi_2 e_3 + \phi_1 e_4 - \phi_2 e_5.$$

The isotropy subgroup H_4^b of H_3^b at a point of W_4^b is equal to H_3^b. In this way we get a third order frame field $u : M \to G/H_3^b$ along $\lambda : M \to G/H$. Adding an extra step we lift u from G/H_3^b to G, then we get a Lie frame of λ. By choosing a lifting $\tilde{u}$ of u such that $\tilde{u}^*\Omega_3^b = 0$, we obtain the Lie frame of type (b) in Theorem 3.1.

Type (c) and Type (d)

Let $\lambda : M \to T_1S^3$ be of type (c); i.e. there exists a first order frame field u along λ which satisfies (3.14), (3.15b) and the following equations:

(3.23c) $$u^*(\omega_1^2 - \omega_2^1) = u^*\theta_6^b = \phi_1, \ u^*(\omega_2^2 - \omega_1^1) = u^*\theta_7^b = \phi_1, \ u^*\omega_4^3 = u^*\theta_8^b = 0.$$

Construction of third order frames of type (b)

The isotropy subgroup H_2^c of H_1^b at a point of W_2^c is

(3.24c)

$$H_2^c = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} ; \right.$$
$$\left. A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \ a_1 - a_2 = (a_1 + a_2)^2 \neq 0, \ B = \begin{pmatrix} b_1 & b_1 \\ b_2 & -b_2 \end{pmatrix}, \ C \in \mathfrak{o}(2) \right\}.$$

The Lie algebra $\mathfrak{h}_2^c$ of H_2^c is

(3.25c)

$$\mathfrak{h}_2^c = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \alpha = \begin{pmatrix} -3\alpha_2 & \alpha_2 \\ \alpha_2 & -3\alpha_2 \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{pmatrix}, \ \gamma \in \mathfrak{o}(2) \right\}.$$

We decompose $\mathfrak{h}_1^b = \mathfrak{h}_2^c + \mathfrak{m}_2^c$, and for a basis of $\mathfrak{m}_2^c$ we take

$$(3.26c) \qquad e_9^c = \begin{pmatrix} 0 & 0 & 0 \\ E_3 & 0 & 0 \\ 0 & {}^tE_3 & 0 \end{pmatrix}, \quad e_{10}^c = \begin{pmatrix} 0 & 0 & 0 \\ E_5 & 0 & 0 \\ 0 & {}^tE_5 & 0 \end{pmatrix}. \quad e_{11}^c = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^tI_2 \end{pmatrix}.$$

The representation $\rho_2^c : H_2^c \to GL(11, \mathbf{R})$ is

(3.27c)

$$\rho_2^c(P) = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
* & a_1 & 0 & a_2 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
* & 0 & a_1 & 0 & a_2 & 0 & 0 & * & 0 & 0 & 0 \\
* & a_2 & 0 & a_1 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
* & 0 & a_2 & 0 & a_1 & 0 & 0 & * & 0 & 0 & 0 \\
* & tb_1 & -\frac{b_2}{t} & -tb_1 & -\frac{b_2}{t} & p & -q & * & 0 & 0 & 0 \\
* & tb_1 & \frac{b_2}{t} & -tb_1 & \frac{b_2}{t} & -q & p & * & 0 & 0 & 0 \\
* & -b_2 & b_1 & b_2 & b_1 & 0 & 0 & * & 0 & 0 & 0 \\
* & \frac{-b_2^2-c}{t^2} & \frac{2b_1b_2}{t^2} & \frac{b_2^2-c}{t^2} & \frac{2b_1b_2}{t^2} & -\frac{b_1}{t^2} & \frac{b_1}{t^2} & * & \frac{1}{t^2} & 0 & 0 \\
* & -\frac{2b_1b_2}{t} & \frac{b_1^2-c}{t} & \frac{2b_1b_2}{t} & \frac{b_1^2+c}{t} & -\frac{b_2}{t} & -\frac{b_2}{t} & * & 0 & \frac{1}{t} & 0 \\
* & -\frac{4a_1b_1}{t} & \frac{2a_1b_2}{t^2} & -\frac{4a_2b_1}{t} & \frac{2a_2b_2}{t^2} & \frac{a_1a_2-3a_2^2}{t^3} & \frac{-a_2^2+3a_1a_2}{t^3} & * & 0 & 0 & 1
\end{pmatrix},$$

where $P \in H_2^c$, $t = a_1 + a_2$, $p = \dfrac{a_1^2 + a_2^2}{t^3}$, $q = \dfrac{2a_1a_2}{t^3}$. With respect to $\mathfrak{h}_1^b = \mathfrak{h}_2^c + \mathfrak{m}_2^c$ we have $\Omega_1^b = \Omega_2^c + \Theta_2^c$, and we set $\Theta_2^c = \sum \theta_i^c e_i^c$ $(i = 9, 10, 11)$ then

$$(3.28c) \qquad \theta_9^c = \omega_2^3 - \omega_1^3,\ \theta_{10}^c = \omega_2^4 + \omega_1^4,\ \theta_{11}^b = \omega_1^1 + 3\omega_2^1.$$

By taking the exterior derivative of (3.23c), and by using (3.14), (3.15b) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$(3.29c) \qquad \begin{aligned} d(u^*\theta_6^b - \phi_1) &= \phi_1 \wedge u^*\theta_9^c - u^*\theta_{10}^c \wedge \phi_2 - \phi_1 \wedge u^*\theta_{11}^c = 0, \\ d(u^*\theta_7^b - \phi_2) &= -\phi_1 \wedge u^*\theta_9^c - u^*\theta_{10}^c \wedge \phi_2 - \phi_1 \wedge u^*\theta_{11}^c = 0, \\ du^*\theta_8^b &= u^*\theta_9^c \wedge \phi_2 - \phi_1 \wedge u^*\theta_{10}^c = 0. \end{aligned}$$

Set $u^*\theta_i^c = x_i\phi_1 + y_i\phi_2$ $(i = 9, 10, 11)$ for some smooth functions x_i, y_i on M. From (3.29c),

$$x_9 = y_{10}, \quad x_{10} = -y_{11}, \quad y_9 = 0.$$

We consider orbits and local sections W_3^c of the action of (H_2^c, ρ_2^c) on $G_{11,2}$.

$$\rho_2^c(P)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ x_9 & 0 \\ x_{10} & x_9 \\ x_{11} & -x_{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t & 0 \\ 0 & t^2 \\ t & 0 \\ 0 & -t^2 \\ t & 0 \\ t & 0 \\ 0 & 0 \\ \dfrac{-2c+x_9}{t^2} & 0 \\ \dfrac{-2b_2+x_{10}}{t} & \dfrac{-2c+x_9}{t} \\ \dfrac{4(a_2-tb_1)+tx_{11}}{t} & 2b_2-x_{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ \dfrac{-2c+x_9}{t^3} & 0 \\ \dfrac{-2b_2+x_{10}}{t^2} & \dfrac{-2c+x_9}{t^3} \\ \dfrac{4(a_2-tb_1)+tx_{11}}{t^2} & \dfrac{2b_2-x_{10}}{t^2} \end{bmatrix}.$$

Thus we take

(3.30c) $$W_3^c = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\};$$

i.e.

(3.31c) $$u^*\theta_9^c = 0, \quad u^*\theta_{10}^c = 0, \quad u^*\theta_{11}^c = 0.$$

Construction of fourth order frames of type (c)

The isotropy subgroup H_3^c of H_2^c at a point of W_3^c is

(3.32c) $$H_3^c = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} ; A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \right.$$
$$\left. a_1 - a_2 = (a_1 + a_2)^2 \neq 0, B = \begin{pmatrix} b_1 & b_1 \\ 0 & 0 \end{pmatrix}, b_1 = \frac{a_2}{a_1 + a_2} \right\}.$$

The Lie algebra $\mathfrak{h}_3^c$ of H_3^c is

(3.33c) $$\mathfrak{h}_3^c = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ 0 & {}^t\beta & -{}^t\alpha \end{pmatrix} ; \alpha = \begin{pmatrix} -3\alpha_2 & \alpha_2 \\ \alpha_2 & -3\alpha_2 \end{pmatrix}, \beta = \begin{pmatrix} \alpha_2 & \alpha_2 \\ 0 & 0 \end{pmatrix} \right\}.$$

We decompose $\mathfrak{h}_2^c = \mathfrak{h}_3^c + \mathfrak{m}_3^c$, and for a basis of $\mathfrak{m}_3^c$ we take

(3.34c)
$$e_{12}^c = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \quad e_{13}^c = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&-1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&-1&0&0 \end{pmatrix},$$
$$e_{14}^c = \begin{pmatrix} 0&0&0\\ 0&0&0\\ E_1&0&0 \end{pmatrix}.$$

The representation $\rho_3^c : H_3^c \to GL(13, \mathbf{R})$ is

(3.35c)
$$\rho_3^c(P) = \begin{pmatrix} \bar\rho_1^1 & \bar\rho_2^1 \\ \bar\rho_1^2 & \bar\rho_2^2 \end{pmatrix},$$
$$\bar\rho_1^1 = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ * & a_1 & 0 & a_2 & 0 & 0 & 0 & * & * & * \\ * & 0 & a_1 & 0 & a_1 & 0 & 0 & * & * & * \\ * & a_2 & 0 & a_1 & 0 & 0 & 0 & * & * & * \\ * & 0 & a_2 & 0 & a_1 & 0 & 0 & * & * & * \\ * & a_2 & 0 & -a_2 & 0 & p & -q & * & * & * \\ * & a_2 & 0 & -a_2 & 0 & -q & p & * & * & * \\ * & 0 & \frac{a_2}{t} & 0 & \frac{a_2}{t} & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & -\frac{a_2}{t^3} & \frac{a_2}{t^3} & * & * & * \\ * & 0 & \frac{a_2^2}{t^3} & 0 & \frac{a_2^2}{t^3} & 0 & 0 & * & * & * \\ * & -\frac{4a_1a_2}{t^2} & 0 & -\frac{4a_2^2}{t^2} & 0 & \frac{a_1a_2 - 3a_2^2}{t^3} & \frac{-a_2^2 + 3a_1a_2}{t^3} & * & * & * \end{pmatrix},$$
$$\bar\rho_2^1 = 0 \in M_{10,3},$$
$$\bar\rho_1^2 = \begin{pmatrix} * & \frac{-a_2^2 + 2ta_1a_2}{2t^3} & 0 & \frac{-a_2^2 + 2ta_2^2}{2t^3} & 0 & \frac{a_1a_2 + ta_2^2}{t^4} & \frac{-ta_1a_2 - a_2^2}{t^4} & * & * & * \\ * & 0 & \frac{a_2^2}{2t^3} & 0 & \frac{a_2^2}{2t^3} & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & -\frac{a_2^2}{2t^5} & \frac{a_2^2}{2t^5} & * & * & * \end{pmatrix},$$

$$\bar{\rho}_2^2 = \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 \\ 0 & 0 & \frac{1}{t^3} \end{pmatrix},$$

where $P \in H_3^c$, $t = a_1 + a_2$, $p = \dfrac{a_1^2 + a_2^2}{t^3}$, $q = \dfrac{2a_1a_2}{t^3}$. With respect to $\mathfrak{h}_2^c = \mathfrak{h}_3^c \cdot \mathfrak{m}_3^c$ we have $\Omega_2^c = \Omega_3^c + \Theta_3^c$, and we set $\Theta_3^c = \sum \theta_i^c e_i^c$ $(i = 12, 13, 14)$ then

$$\theta_{12}^c = \omega_1^3 - \omega_2^1, \quad \theta_{13}^c = \omega_1^4, \quad \theta_{14}^c = \omega_2^5. \tag{3.36c}$$

By taking the exterior derivative of (3.31c), and by using (3.14), (3.15b), (3.23c) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$\begin{aligned} du^*\theta_9^c &= -2\phi_1 \wedge u^*\theta_{14}^c = 0, \\ du^*\theta_{10}^c &= -2\phi_1 \wedge u^*\theta_{13}^c + 2u^*\theta_{14}^c \wedge \phi_2 = 0, \\ du^*\theta_{11}^c &= -4\phi_1 \wedge u^*\theta_{12}^c - 2u^*\theta_{13}^c \wedge \phi_2 = 0. \end{aligned} \tag{3.37c}$$

Set $u^*\theta_i^c = x_i\phi_1 + y_i\phi_2$ $(i = 12, 13, 14)$ for some smooth functions x_i, y_i on M. From (3.37c),

$$x_{13} = -2y_{12} = 2k^3, \quad x_{14} = y_{13} = k^4, \quad y_{14} = 0. \tag{3.38c}$$

We consider orbits and local sections W_4^c of the action of (H_3^c, ρ_3^c) on $G_{13,2}$.

$$\rho_3^c(P)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ x_{12} & -k^3 \\ 2k^3 & k^4 \\ k^4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t & 0 \\ 0 & t^2 \\ t & 0 \\ 0 & -t^2 \\ t & 0 \\ t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{a_1a_2}{t^3} + \frac{1}{t}x_{12} & -\frac{1}{t}k^3 \\ \frac{2}{t^2}k^3 & \frac{1}{t^2}k^4 \\ \frac{1}{t^3}k^4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{a_1a_2}{t^4} + \frac{1}{t^2}x_{12} & -\frac{1}{t^3}k^3 \\ \frac{2}{t^3}k^3 & \frac{1}{t^4}k^4 \\ \frac{1}{t^4}k^4 & 0 \end{bmatrix}$$

Thus we take

$$(3.39c)\quad W_4^c = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2k^3 & k^4 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -k^3 & k^4 & 0 \end{bmatrix} \right\};$$

i.e

$$(3.40c)\quad u^*\theta_{12}^c = -k^3\phi_2, \quad u^*\theta_{13}^c = 2k^3\phi_1 + k^4\phi_2, \quad u^*\theta_{14}^c = k^4\phi_1.$$

Construction of fifth order frames of type (c)

The isotropy subgroup H_4^c of H_3^c is equal to $\{I_6\}$. For a basis of $\mathfrak{h}_3^c$ we take

$$(3.41c)\quad e_{15}^c = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 3 \end{pmatrix}.$$

We set $\Omega_3^c = \theta_{15}^c e_{15}^c$ then

$$(3.42c)\quad \theta_{15}^c = \omega_2^1.$$

Set

$$(3.43c)\quad u^*\theta_{15}^c = k^1\phi_1 + k^2\phi_2,$$

$$(3.44c)\quad dk^i = k_1^i\phi_1 + k_2^i\phi_2 \ (i = 1, 2, 3, 4),$$

for some smooth functions k^i, k_j^i on M. By taking the exterior derivative of (3.40c), and by using (3.14), (3.15b), (3.23c), (3.31c), (3.44c) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$(3.45c)\quad \begin{aligned} &d(u^*\theta_{12}^c + k^3\phi_2) = \phi_1 \wedge u^*\theta_{15}^c + 6k^3u^*\theta_{15}^c \wedge \phi_2 + k_1^3\phi_1 \wedge \phi_2 = 0, \\ &d(u^*\theta_{13}^c - (2k^3\phi_1 + k^4\phi_2)) \\ &\qquad = 12k^3\phi_1 \wedge u^*\theta_{15}^c - 8k^4u^*\theta_{15}^c \wedge \phi_2 + (2k_2^3 - k_1^4)\phi_1 \wedge \phi_2 = 0, \\ &d(u^*\theta_{14}^c - k^4\phi_1) = 8k^4\phi_1 \wedge u^*\theta_{15}^c + k_2^4\phi_1 \wedge \phi_2 = 0. \end{aligned}$$

From (3.43c), (3.44c), and (3.45c),

$$(3.46c)\quad \begin{aligned} &k^2 + 6k^1k^3 + k_1^3 = 0, \\ &12k^2k^3 - 8k^1k^4 + (2k_2^3 - k_1^4) = 0, \\ &8k^2k^4 + k_2^4 = 0. \end{aligned}$$

Thus we take a local cross section

$$W_5^c = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2k^3 & k^4 & k^1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -k^3 & k^4 & 0 & k^2 \end{bmatrix} \right\} \tag{3.47c}$$

of the trivial adjoint action (H_4^c, ρ_4^c) on $G_{15,2}$, where k^i are smooth functions on M that satisfy the conditions (3.46c). By taking the exterior derivative (3.43c), and by using (3.14), (3.15b), (3.23c), (3.31c), (3.40c) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$d(u^*\theta_{15}^c - (k^1\phi_1 + k^2\phi_2)) = (-2k^1k^2 + k^3 + k_2^1 - k_1^2)\phi_1 \wedge \phi_2 = 0. \tag{3.48c}$$

Then

$$2k^1k^2 - k^3 - k_2^1 + k_1^2 = 0. \tag{3.49c}$$

Thus the Lie frame of a Legendre map $\lambda : M \to T_1S^3$ of type (c) is the fifth order frame field u along λ with respect to W_5^c. In this way we obtain the Lie frame of type (c) in Theorem 3.1.

In precisely the same fashion as the case of type (c), we obtain the Lie frame of type (d) in Theorem 3.1.

Type (e)

Let $\lambda : M \to T_1S^3$ is of type (e); i.e. there exists order frame field u along λ which satiafies (3.14), (3.15b) and the following equations:

$$\begin{aligned} &u^*(\omega_1^2 - \omega_2^1) = u^*\theta_6^b = \phi_1 + \phi_2, \; u^*(\omega_2^2 - \omega_1^1) = u^*\theta_7^b = \phi_1 - \phi_2, \\ &u^*\omega_4^3 = u^*\theta_8^b = 0. \end{aligned} \tag{3.23e}$$

Construction of third order frames of type (e)

The isotropy subgroup H_2^e of H_1^b at a point of W_2^e is

(3.24e)

$$H_2^e = \left\{ \exp\begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & {}^tB & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} b_1 & b_1 \\ b_2 & -b_2 \end{pmatrix}, \, C \in \mathfrak{o}(2) \right\}.$$

The Lie algebra $\mathfrak{h}_2^e$ of H_2^e is

$$\mathfrak{h}_2^e = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & {}^t\beta & 0 \end{pmatrix} ; \beta = \begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{pmatrix}, \, \gamma \in \mathfrak{o}(2) \right\}. \tag{3.25e}$$

We decompose $\mathfrak{h}_1^b = \mathfrak{h}_2^e + \mathfrak{m}_2^e$, and for a basis of $\mathfrak{m}_2^e$ we take

$$(3.26e)\qquad e_9^e=\begin{pmatrix}0&0&0\\E_3&0&0\\0&{}^tE_3&0\end{pmatrix},\quad e_{10}^e=\begin{pmatrix}0&0&0\\E_5&0&0\\0&{}^tE_5&0\end{pmatrix}.$$

$$e_{11}^e=\begin{pmatrix}I_2&0&0\\0&0&0\\0&0&-{}^tI_2\end{pmatrix}.\quad e_{12}^e=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&-1&0\end{pmatrix}.$$

The representation $\rho_2^e: H_2^e \to GL(15, \mathbf{R})$ is

(3.27e)

$$\rho_2^e(P)=\begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & 1 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & 0 & 0 & 1 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & 0 & 0 & 0 & 1 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & b_1 & -b_2 & -b_1 & -b_2 & 1 & 0 & * & 0 & 0 & 0 & 0\\
* & b_1 & b_2 & -b_1 & b_2 & 0 & 1 & * & 0 & 0 & 0 & 0\\
* & -b_2 & b_1 & b_2 & b_1 & 0 & 0 & * & 0 & 0 & 0 & 0\\
* & -b_2^2-c & 2b_1b_2 & b_2^2-c & 2b_1b_2 & -b_1 & +b_1 & * & 1 & 0 & 0 & 0\\
* & -2b_1b_2 & b_1^2-c & 2b_1b_2 & b_1^2+c & -b_2 & -b_2 & * & 0 & 1 & 0 & 0\\
* & -b_1 & -b_2 & 0 & 0 & 0 & 0 & * & 0 & 0 & 1 & 0\\
* & -b_1 & b_2 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 1
\end{pmatrix},$$

where $P \in H_2^e$. With respect to $\mathfrak{h}_1^b = \mathfrak{h}_2^e + \mathfrak{m}_2^e$ we have $\Omega_1^b = \Omega_2^e + \Theta_2^e$, and we set $\Theta_2^e = \sum \theta_i^e e_i^e$ $(i = 9, 10, 11, 12)$ then

$$(3.28e)\qquad \theta_9^e=\omega_2^3-\omega_1^3,\ \theta_{10}^e=\omega_2^4+\omega_1^4,\ \theta_{11}^e=\omega_1^1,\ \theta_{12}^e=\omega_2^1.$$

By taking the exterior derivative of (3.23e), and by using (3.14), (3.15b) and the Maurer-Cartan equation (1.8), we obtain the following equations:

$$\begin{aligned}
&d(u^*\theta_6^b-(\phi_1+\phi_2))\\
&\quad=\phi_1\wedge(u^*\theta_9^e-u^*\theta_{11}^e-3u^*\theta_{12}^e)+(-u^*\theta_{10}^e+u^*\theta_{11}^e-3u^*\theta_{12}^e)\wedge\phi_2\\
&\qquad-2\phi_1\wedge\phi_2=0,
\end{aligned}$$

$$\begin{aligned}
(3.29e)\quad &d(u^*\theta_7^b-(\phi_1+\phi_2))\\
&\quad=\phi_1\wedge(-u^*\theta_9^e-u^*\theta_{11}^e-3u^*\theta_{12}^e)+(-u^*\theta_{10}^e-u^*\theta_{11}^e+3u^*\theta_{12}^e)
\end{aligned}$$

$$\wedge \phi_2 = 0,$$
$$du^*\theta_8^b = -\phi_1 \wedge u^*\theta_{10}^e + u^*\theta_9^e \wedge \phi_2 = 0.$$

Set $u^*\theta_i^e = x_i\phi_1 + y_i\phi_2$ $(i = 9, 10, 11, 12)$ for some smooth functions x_i, y_i on M. From (3.29e),

$$x_{10} + y_{11} + 3y_{12} + 1 = 0, \quad y_9 + x_{11} - 3x_{12} - 1 = 0, \quad x_9 = y_{10}.$$

We consider orbits and local sections W_3^e of the action of (H_2^e, ρ_2^e) on $G_{12,2}$.

$$\rho_2^e(P)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \\ x_9 & y_9 \\ x_{10} & y_{10} \\ x_{11} & y_{11} \\ x_{12} & x_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \\ -2c + x_9 & -2b_1 + y_9 \\ -2b_2 + x_{10} & -2c + y_{10} \\ -b_1 + x_{11} & -b_2 + y_{11} \\ -b_1 + x_{12} & b_2 + x_{12} \end{bmatrix}.$$

Thus we take

$$\text{(3.30e)} \quad W_3^e = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 3k^1+1 & k^1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & -3k^2-1 & k^2 \end{bmatrix} \right\};$$

i.e

$$\text{(3.31e)} \quad \begin{aligned} &u^*\theta_9^e = 0, \quad u^*\theta_{10}^e = 0, \\ &u^*\theta_{11}^e = (3k^1 + 1)\phi_1 + (-3k^3 - 1)\phi_2, \quad u^*\theta_{12}^e = k^1\phi_1 + k^2\phi_2. \end{aligned}$$

Construction of fourth order frames of type (e)

The isotropy subgroup H_3^e of H_2^e is equal to $\{I_6\}$. For a basis of $\mathfrak{h}_2^e$ we take

$$\text{(3.32e)} \quad e_{13}^e = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}, \quad e_{14}^e = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&-1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&-1&0&0 \end{pmatrix},$$
$$e_{15}^e = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ E_1&0&0 \end{pmatrix}$$

We set $\Omega_2^c = \sum \theta_i^e e_i^e$ $(i = 13, 14, 15)$ then

(3.33e) $$\theta_{13}^e = \omega_1^3, \quad \theta_{14}^e = \omega_1^4, \quad \theta_{15}^e = \omega_2^5.$$

Set

(3.34e) $$dk^i = k_1^i\phi_1 + k_2^i\phi_2 \ (i = 1,2).$$

By taking the exterior derivative of (3.31e), and by using (3.14), (3.15b), (3.23e), (3.34e) and the Maurer-Cartan equation (1.8), we obtain the following equations:

(3.35e) $$\begin{aligned} &du^*\theta_9^e = -2\phi_1 \wedge u^*\theta_{15}^e + 2u^*\theta_{13}^e \wedge \phi_2 = 0,\\ &du^*\theta_{10}^e = -2\phi_1 \wedge u^*\theta_{14}^e + 2u^*\theta_{15}^e \wedge \phi_2 = 0,\\ &d(u^*\theta_{11}^b - ((3k^1+1)\phi_1 + (-3k^2-1)\phi_2))\\ &\quad = -\phi_1 \wedge u^*\theta_{13}^e + u^*\theta_{14}^e \wedge \phi_2 + (3k_2^1 + 3k_1^2)\phi_1 \wedge \phi_2 = 0,\\ &d(u^*\theta_{12}^e - (k^1\phi_1 + k^2\phi_2))\\ &\quad = -\phi_1 \wedge u^*\theta_{13}^e - u^*\theta_{14}^e \wedge \phi_2 + (k_2^1 - k_1^2 + 2k^1 + 2k^2 + 4k^1k^2)\phi_1 \wedge \phi_2\\ &\quad = 0. \end{aligned}$$

Set $u^*\theta_i^e = x_i\phi_1 + y_i\phi_2$ $(i = 13, 14, 15)$ for some smooth functions x_i, y_i on M. From (3.35e),

(3.36e) $$\begin{aligned} &x_{13} = y_{15} = k^5, \quad x_{15} = y_{14} = k^6,\\ &x_{13} = (k^1 + k^2)^2 + 2k_2^1 + k_1^2 = k^3, \quad x_{14} = (k^1 + k^2)^2 - k_2^1 + 2k_1^2 = k^4. \end{aligned}$$

Thus we take a local cross section

(3.37e)
$$W_4^e = \left\{ {}^t\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 3k^1+1 & k^1 & k^5 & k^4 & k^6 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -3k^2-1 & k^2 & k^3 & k^6 & k^5 \end{bmatrix} \right\}$$

of the trivial adjoint action (H_3^e, ρ_3^e) on $G_{15,2}$; i.e.

(3.38e) $$u^*\theta_{13}^e = k^5\phi_1 + k^3\phi_2, \quad u^*\theta_{14}^e = k^4\phi_1 + k^6\phi_2, \quad u^*\theta_{15}^e = k^6\phi_1 + k^5\phi_2.$$

By taking the exterior derivative of (3.38e), and by using (3.14), (3.15b), (3.23e), (3.34e) and the Maurer-Cartan equation (1.8), we obtain the following equations:

(3.39e) $$\begin{aligned} &du^*(\theta_{13}^e - (k^5\phi_1 + k^3\phi_2))\\ &\quad = (6k^1k^3 + 4k^2k^5 + 2k^5 + 3k^3 + k_2^5 - k_1^3)\phi_1 \wedge \phi_2 = 0,\\ &du^*(\theta_{14}^e - (k^4\phi_1 + k^6\phi_2))\\ &\quad = (4k^1k^6 + 6k^2k^4 + 2k^6 + 3k^4 + k_2^4 - k_1^6)\phi_1 \wedge \phi_2 = 0, \end{aligned}$$

$$
\begin{aligned}
&d(u^*\theta^e_{15} - (k^6\phi_1 + k^5\phi_2)) \\
&\qquad = (-4k^2k^6 - 4k^1k^5 - 2k^5 - 2k^6 + k^5_1 + k^6_2)\phi_1 \wedge \phi_2 = 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&6k^1k^3 + 4k^2k^5 + 2k^5 + 3k^3 + k^5_2 - k^3_1 = 0, \\
&4k^1k^6 + 6k^2k^4 + 2k^6 + 3k^4 + k^4_2 - k^6_1 = 0, \qquad (3.40e)\\
&-4k^2k^6 - 4k^1k^5 - 2k^5 - 2k^6 - k^5_1 + k^6_2 = 0.
\end{aligned}
$$

Thus the Lie frame of a Legendre map $\lambda : M \to T_1S^3$ of typy (e) is the fourth order frame field u along λ with respsct to W^e_4. In this way we obtain the Lie frame of type (e) in Theorem 3.1.

We have thus proved Theorem 3.1.

4. Classification of Legendre maps in T_1S^3 in view of curvature spheres

Let us consider the classification of surfaces obtained in Theorem 3.1 in view of curvature spheres. The curvature sphere of $\lambda : M^2 \to T_1S^3$ of type (a) is $[K] = [Y_6]$, which has multiplicity 2. The curvature spheres of λ of type (b), (c), (d) and (e) are

$$
[K_1(p)] = [Y_5(p) + Y_6(p)], \quad [K_2(p)] = [-Y_5(p) + Y_6(p)]. \qquad (4.1)
$$

COROLLARY 4.1. (1) *A Legendre map of type* (a) *is a oriented hypershpere.*
(2) *A Legendre map of type* (b) *is a cyclide of Dupin.*
(3) *A Legendre map of type* (c) (*type* (d)) *is a canal surface if the function* $y \equiv 0$ ($x \equiv 0$).

Proof. (1) $[K]$ is constant since $dY_6 = 0$, so λ is a oriented hypersphere.

(2) Let X_1, X_2 are the principal vectors corresponding to $[K_1]$, $[K_2]$ respectively; i.e. X_1, X_2 are vectors in T_pM such that $\phi_1(X_1) = 0$, $\phi_2(X_2) = 0$. If λ is of type (b), then

$$
dK_1 = 2\phi_1 Y_3, \quad dK_2 = -\phi_2 Y_4,
$$

and hence $dK_1(X_1) = dK_2(X_2) = 0$. Thus along every line of curvature in M the corresponding curvature sphere is constant; i.e. λ is a cyclide of Dupin.

(3) Let λ be of type (c). We change the functions k^1, k^2 in (2.3) so that $h^1 = 4k^1$, $h^2 = 2k^2$. Then

$$
dK_1 = 2\left\{\phi_1 Y_3 + \left(\frac{h^1 - 2}{4}\phi_1 + \frac{h^2}{2}\phi_2\right)Y_5 + \left(\frac{h^1 - 2}{4}\phi_1 + \frac{h^2}{2}\phi_2\right)Y_6\right\} \qquad (4.2)
$$

$$dK_2 = 2\Big\{-\phi_2 Y_4 + \Big(-\frac{h^1+1}{2}\phi_1 - h^2\phi_2\Big)Y_5 + \Big(\frac{h^1-1}{2}\phi_1 + h^2\phi_2\Big)Y_6\Big\}.$$

Let λ be of type (d). We change the functions k^1, k^2 in (2.5) so that $h^1 = -2k^1$, $h^2 = -4k^2$. Then

$$(4.3)\qquad \begin{aligned} dK_1 &= 2\Big\{\phi_1 Y_3 + \Big(h^1\phi_1 + \frac{h^2-1}{2}\phi_2\Big)Y_5 + \Big(h^1\phi_1 + \frac{h^2+2}{2}\phi_2\Big)Y_6\Big\} \\ dK_2 &= 2\Big\{-\phi_2 Y_4 + \Big(-\frac{h^1}{2}\phi_1 - \frac{h^2+2}{4}\phi_2\Big)Y_5 + \Big(\frac{h^1}{2}\phi_1 + \frac{h^2+2}{4}\phi_2\Big)Y_6\Big\}. \end{aligned}$$

Thus if λ is of type (c) and $k^2 = h^2 \equiv 0$, then only $[K_1]$ is constant along X_1, and if λ is of type (d) and $k^1 = h^1 \equiv 0$, then only $[K_2]$ is constant along X_2; such λ is classically called a canal surface. □

Let λ be of type (e). We change the functions k^1, k^2 in (2.7) so that $h^1 = -2k^1 - 1$, $h^2 = 2k^2 + 1$. Then

$$(4.4)\qquad \begin{aligned} dK_1 &= 2\Big\{\phi_1 Y_3 + \Big(h^1\phi_1 + \frac{h^2-1}{2}\phi_2\Big)Y_5 + \Big(h^1\phi_1 + \frac{h^2+2}{2}\phi_2\Big)Y_6\Big\} \\ dK_2 &= 2\Big\{-\phi_2 Y_4 + \Big(-\frac{h^1+1}{2}\phi_1 - h^2\phi_2\Big)Y_5 + \Big(\frac{h^1-1}{2}\phi_1 + h^2\phi_2\Big)Y_6\Big\}. \end{aligned}$$

Hence we can distinguish the type of a given Legendre map $\lambda : M_2 \to T_1S^3$ which has two curvature spheres $[K_1]$, $[K_2]$ with multiplicity 1 in the following way. Set

$$(4.5)\qquad \begin{aligned} dK_1 &= 2\phi_1 Y_3 + (A\phi_1 + B\phi_2)K_1 + C\phi_2 K_2, \\ dK_2 &= -2\phi_2 Y_4 + D\phi_1 K_1 + (E\phi_1 + F\phi_2)K_2, \end{aligned}$$

where A, B, C, D, E, F are some smooth functions of M. If $C = 0$, $D = 0$, then λ is of type (b); i.e. a cyclide of Dupin. If $C = 0$, $D \neq 0$ ($C \neq 0$, $D = 0$), then λ is of type (c) (type (d)); moreover if $B = 0$ ($E = 0$), then λ is a canal surface. If $C \neq 0$, $D \neq 0$, then λ is of type (e).

Finally we obtain the necessary and sufficient condition that two surfaces are Lie equivalent by virtue of Theorem 3.1 and the theorem in Section 2.2.

Corollary 4.2. (a) *All oriented spheres in T_1S^3 are Lie equivalent.*
(b) *All cyclides of Dupin in T_1S^3 are Lie equivalent.*
(c) *Let $\lambda : M \to T_1S^2$, $\tilde\lambda : \tilde M \to T_1S^2$ be smooth surfaces of type* (c), (d) *or* (e). *Let k^i, $\tilde k^i$ be the smooth functions which are defined in Theorem* 2.1 *with respect to λ, $\tilde\lambda$ respectively. Surfaces λ and $\tilde\lambda$ are Lie equivalent if and only if there exists a one-to-one*

correspondance $\varphi : M \to \tilde{M}$ *such that* $k^i = \varphi^* \tilde{k}^i$ *for all* i.

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