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MULTIPLIER HERMITIAN STRUCTURES ON KÄHLER MANIFOLDS

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Abstract. The main purpose of this paper is to make a systematic study of a special type of conformally Kähler manifolds, called multiplier Hermitian manifolds, which we often encounter in the study of Hamiltonian holomorphic group actions on Kähler manifolds. In particular, we obtain a multiplier Hermitian analogue of Myers' Theorem on diameter bounds with an application (see [M5]) to the uniqness up to biholomorphisms of the "Kähler-Einstein metrics" in the sense of [M1] on a given Fano manifold with nonvanishing Futaki character.

§1. Introduction

For a connected complete Kähler manifold (M, ω_0) of complex dimension n, let $\mathcal{K}$ denote the set of all Kähler forms on M expressible as

$$\omega_\varphi := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi \tag{1.1}$$

for some real-valued smooth function $\varphi \in C^\infty(M)_\mathbb{R}$ on M. In this paper, we fix once for all a holomorphic vector field $X \neq 0$ on M, and M is assumed to be compact except in Section 4 and in Theorem B below. Put

$$\mathcal{K}_X := \{\omega \in \mathcal{K} \,;\, L_{X_\mathbb{R}}\omega = 0\},$$

where $X_\mathbb{R} := X + \bar{X}$ denotes the real vector field on M associated to the holomorphic vector field X. Let $\mathcal{H}_X$ denote the set of all $X_\mathbb{R}$-invariant functions φ in $C^\infty(M)_\mathbb{R}$ such that ω_φ is in $\mathcal{K}_X$. Let $\mathcal{K}_X \neq \emptyset$, so that we may assume without loss of generality that

$$\omega_0 \in \mathcal{K}_X.$$

In terms of a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M above, we write each Kähler form ω in $\mathcal{K}_X$ as

$$\omega = \sqrt{-1}\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, dz^\alpha \wedge dz^{\bar{\beta}}.$$

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Throughout this paper, we assume that X is *Hamiltonian*, i.e., to each $\omega \in \mathcal{K}_X$, we can associate a function $u_\omega \in C^\infty(M)_\mathbb{R}$ such that X is expressible as

$$\operatorname{grad}^{\mathbb{C}}_\omega u_\omega := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial u_\omega}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^\alpha}.$$

Then u_ω is an $X_\mathbb{R}$-invariant function, and the image I_X of the function u_ω on M is an interval in $\mathbb{R}$. For an arbitrary nonconstant real-valued smooth function

$$\sigma : I_X \longrightarrow \mathbb{R}, \quad s \longmapsto \sigma(s),$$

we define functions $\dot\sigma = \dot\sigma(s)$ and $\ddot\sigma = \ddot\sigma(s)$ on I_X as the derivatives $\dot\sigma := (\partial/\partial s)\sigma$ and $\ddot\sigma := (\partial^2/\partial s^2)\sigma$, respectively. We further define a function $\psi_\omega \in C^\infty(M)_\mathbb{R}$ by

$$\psi_\omega = \sigma(u_\omega), \tag{1.2}$$

which is obviously $X_\mathbb{R}$-invariant. The function σ is said to be *strictly convex* or *weakly convex*, according as $\ddot\sigma > 0$ on I_X or $\ddot\sigma \geq 0$ on I_X. By abuse of terminology, σ is said to be *convex* if either σ is strictly convex or σ satisfies $\dot\sigma \leq 0 \leq \ddot\sigma$ on I_X.

Let $G := \mathrm{Aut}^0(M)$ be the identity component of the group of all holomorphic automorphisms of M. Let

Q : closure in G of the real one-parameter group $\{\exp(tX_\mathbb{R})\ ;\ t \in \mathbb{R}\}$.

Under the assumption of the compactness of M, we require the function u_ω to satisfy the equality $\int_M u_\omega \omega^n = 0$, and applying the theory of moment maps to the action on M of the compact torus Q, we obtain

$$I_X = [\alpha_X, \beta_X],$$

where both $\alpha_X := \min_M u_\omega$ and $\beta_X := \max_M u_\omega$ are independent of the choice of ω in $\mathcal{K}_X$. To each $\omega \in \mathcal{K}_X$, we associate the corresponding Laplacian $\square_\omega$ of the Kähler manifold (M, ω), and define an operator $\tilde\square_\omega$ on $C^\infty(M)_\mathbb{R}$ by

$$\tilde\square_\omega := \sum_{\alpha,\beta} g^{\bar\beta\alpha} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} - \sum_{\alpha,\beta} g^{\bar\beta\alpha} \frac{\partial \psi_\omega}{\partial z^\alpha} \frac{\partial}{\partial z^{\bar\beta}} = \square_\omega + \sqrt{-1}\,\dot\sigma(u_\omega)\bar{X}. \tag{1.3}$$

The natural connection, induced by ω, on the holomorphic tangent bundle TM of M is denoted by ∇. To each ω in $\mathcal{K}_X$, we associate a conformally Kähler metric $\tilde{\omega}$ by

$$\tilde{\omega} := \omega \exp(-\psi_\omega/n), \tag{1.4}$$

which is called a *multiplier Hermitian metric* (*of type* σ). Here, a Hermitian form and the corresponding Hermitian metric are used interchangeably. The Hermitian metric $\tilde{\omega}$ naturally induces a Hermitian connection $\tilde{\nabla} : \mathcal{A}^0(TM) \to \mathcal{A}^1(TM)$ such that

$$\tilde{\nabla} = \nabla - \frac{\partial \psi_\omega}{n} \, \mathrm{id}_{TM},$$

where $\mathcal{A}^q(TM)$ denotes the sheaf of germs of TM-valued C^∞ q-forms on M. By abuse of terminology, the Ricci form of $(\tilde{\omega}, \tilde{\nabla})$ is denoted by $\mathrm{Ric}^\sigma(\omega)$. Then (see [L2], [K1], [Mat])

$$\mathrm{Ric}^\sigma(\omega) = \sqrt{-1}\,\bar{\partial}\partial \log(\tilde{\omega}^n) = \mathrm{Ric}(\omega) + \sqrt{-1}\,\partial\bar{\partial}\psi_\omega, \tag{1.5}$$

where we set $\mathrm{Ric}(\omega) := \sqrt{-1}\,\bar{\partial}\partial \log(\omega^n)$. For each nonnegative real number ν, let $\mathcal{K}_X^{(\nu)}$ denote the set of all $\omega \in \mathcal{K}_X$ such that

$$\mathrm{Ric}^\sigma(\omega) \geq \nu\omega,$$

i.e., $\mathrm{Ric}^\sigma(\omega) - \nu\omega$ is a positive semi-definite $(1,1)$-form on M. Now for $\varphi \in \mathcal{H}_X$, we set $\mathrm{Osc}(\varphi) := \max_M \varphi - \min_M \varphi$. Consider the set $\mathcal{S}^\sigma$ of all ω in $\mathcal{K}_X$ such that

$$\mathrm{Ric}^\sigma(\omega) = t\omega + (1-t)\omega_0 \quad \text{for some } t \in [0,1].$$

Let $\mathcal{I}^\sigma - \mathcal{J}^\sigma$ be the analogue of Aubin's functional as in Appendix 1. The main purpose of this paper is to prove the following theorems (see Sections 3, 4 and 5):

THEOREM A. (a) *If $\dot{\sigma} \leq 0 \leq \ddot{\sigma}$ on I_X, then for each $\nu > 0$, we have positive real constants C_0, C_1, C_1', C_1'', C_2 independent of the choice of the pair (ω_φ, ν) such that*

$$\mathrm{Osc}(\varphi) \leq C_0(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \omega_\varphi) + \frac{C(\nu)}{\nu} \tag{1.6}$$

for all ω_φ *in* $\mathcal{K}_X^{(\nu)} \cap \mathcal{S}^\sigma$, *where* $C(\nu) := C_1 + C_1'\nu + C_1''e^{C_2/\nu}$.

(b) *If* σ *is strictly convex, then for each* $\nu > 0$, *there exist positive real constants* C_0, C_1, C_1' *independent of the choice of the pair* (ω_φ, ν) *such that, by setting* $C(\nu) := C_1 + C_1'\nu$, *we have the inequality* (1.6) *for all* ω_φ *in* $\mathcal{K}_X^{(\nu)}$.

THEOREM B. *Let* $\nu > 0$ *and* $\omega \in \mathcal{K}_X^{(\nu)}$. *Furthermore, let* (X, σ) *be of Hamiltonian type* (*cf. Definition* 4.1), *where* σ *is weakly convex. Let* p *be an arbitrary point in* $\mathrm{zero}(X)$ *or in* M, *according as* (4.1.1) *or* (4.1.2) *holds* (*cf. Section* 4). *Put* $c := \sup_{s \in I_X} |\sigma(s)|$. *Then*

$$\mathrm{dist}_\omega(p, q) \leq \pi\{(2n - 1 + 4c)/\nu\}^{1/2} \quad \textit{for all } q \in M,$$

where $\mathrm{dist}_\omega(p, q)$ *denotes the distance between* p *and* q *on the complete Kähler manifold* (M, ω). *Hence, the diameter* $\mathrm{Diam}(M, \omega)$ *of the complete Kähler manifold* (M, ω) *satisfies*

$$\mathrm{Diam}(M, \omega) \leq 2^\delta \pi\{(2n - 1 + 4c)/\nu\}^{1/2}, \tag{1.7}$$

where δ *denotes* 1 *or* 0, *according as* (4.1.1) *or* (4.1.2) *holds. In particular, if* $|\psi_\omega|$ *is bounded from above on* M, *then* M *is compact and* $\pi_1(M)$ *is finite.*

Let $\mathcal{E}_X^\sigma$ be the set of all $\omega \in \mathcal{K}_X$ such that $\mathrm{Ric}^\sigma(\omega) = \omega$. We also consider the subgroup $Z(X)$ of G consisting of all $g \in G$ such that $\mathrm{Ad}(g)X = X$, and let $Z^0(X)$ denote the identity component of $Z(X)$. Then in Section 5, we apply Theorems A and B (Theorem B will be implicitly used) to showing that $\mathcal{E}_X^\sigma$ consists of a single $Z^0(X)$-orbit[†] under the assumption of convexity of σ.

THEOREM C. *Assume that* σ *is convex. Then* $\mathcal{E}_X^\sigma$ *consists of a single* $Z^0(X)$*-orbit, whenever* $\mathcal{E}_X^\sigma$ *is nonempty.*

This work is mainly motivated by the study of "Kähler-Einstein metrics" (cf. [M1]) which are closely related to the case where $\sigma(s) = -\log(s + C)$ (cf. [M5]). Parts of this work were done during my stay in International Centre for Mathematical Sciences (ICMS), Edinburgh in 1997. I thank especially Professor Michael Singer who invited me to give lectures at ICMS on various subjects of Kähler-Einstein metrics.

[†]For a similar result on Kähler-Ricci solitons, see [TZ1]. For "Kähler-Einstein metrics" in the sense of [M1], the arguments in Section 5 were given at the meeting in 1997 at ICMS, though at that time a crucial gap in a priori C^0 estimates was pointed out by G. Tian. Theorems A and B above solve this gap.

§2. Notation, convention and preliminaries

To each $\omega \in \mathcal{K}_X$ as in the introduction, we associate a multiplier Hermitian metric $\tilde{\omega}$ in (1.4) and an operator $\tilde{\Box}_\omega$ in (1.3). For complex-valued functions u, $v \in C^\infty(M)_\mathbb{C}$ on M, we put (cf. [L2], [K1], [Mat], [F1])

$$\langle\!\langle u, v \rangle\!\rangle_{\tilde{\omega}} := \int_M u\bar{v}e^{-\psi_\omega}\omega^n = \int_M u\bar{v}\tilde{\omega}^n.$$

In the arguments in [F1, p. 41], we replace the function F by ψ. Then $\tilde{\Box}_\omega$ is easily shown to be self-adjoint with respect to the above Hermitian inner product as follows:

Lemma 2.1.

$$\langle\!\langle u, \tilde{\Box}_\omega v \rangle\!\rangle_{\tilde{\omega}} = -\int_M (\bar{\partial}u, \bar{\partial}v)_\omega \tilde{\omega}^n = \langle\!\langle \tilde{\Box}_\omega u, v \rangle\!\rangle_{\tilde{\omega}}, \quad u, v \in C^\infty(M)_\mathbb{C}.$$

Proof. $\langle\!\langle u, \tilde{\Box}_\omega v \rangle\!\rangle_{\tilde{\omega}}$ is written as

$$\begin{aligned}
&\int_M u\{\overline{\Box_\omega v} - (\bar{\partial}\psi_\omega, \bar{\partial}v)_\omega\}\tilde{\omega}^n \\
&\quad = \int_M \{-(\bar{\partial}(ue^{-\psi_\omega}), \bar{\partial}v)_\omega - u(\bar{\partial}\psi_\omega, \bar{\partial}v)_\omega e^{-\psi_\omega}\}\omega^n \\
&\quad = -\int_M (\bar{\partial}u, \bar{\partial}v)_\omega \tilde{\omega}^n,
\end{aligned}$$

while $\langle\!\langle \tilde{\Box}_\omega u, v \rangle\!\rangle_{\tilde{\omega}}$ is just

$$\begin{aligned}
&\int_M \{\Box_\omega u - (\bar{\partial}u, \bar{\partial}\psi_\omega)_\omega\}v\tilde{\omega}^n \\
&\quad = \int_M \{-(\bar{\partial}u, \bar{\partial}(e^{-\psi_\omega}v))_\omega - v(\bar{\partial}u, \bar{\partial}\psi_\omega)_\omega e^{-\psi_\omega}\}\omega^n \\
&\quad = -\int_M (\bar{\partial}u, \bar{\partial}v)_\omega \tilde{\omega}^n.
\end{aligned}$$

Hence Lemma 2.1 is immediate. □

To an arbitrary smooth path $\phi = \{\varphi_t\,;\, a \le t \le b\}$ in $\mathcal{H}_X$, we associate a one-parameter family of Kähler forms $\omega(t)$, $a \le t \le b$, in $\mathcal{K}_X$ by

$$\omega(t) := \omega_{\varphi_t} = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t, \quad a \le t \le b. \tag{2.2}$$

Let $\dot\varphi_t$ denote the partial derivative $\partial\varphi_t/\partial t$ of φ_t with respect to t. Next, by the notation (1.4) in the introduction, we consider the Hermitian form $\tilde\omega(t)$ on M defined by

$$\tilde\omega(t) := \omega(t)\exp\{-\psi_{\omega(t)}/n\}. \tag{2.3}$$

LEMMA 2.4. (a) $(\partial/\partial t)\tilde\omega(t)^n = (\tilde\Box_{\omega(t)}\dot\varphi_t)\tilde\omega(t)^n$.

(b) $\int_M \tilde\omega^n = V_0$ *for all* $\omega \in \mathcal{K}_X$, *where* $V_0 := \int_M \tilde\omega_0^n > 0$.

Proof. (a) Recall that $u_{\omega(t)}$ is expressible as $u_{\omega_0} + \sqrt{-1}\,X\varphi_t$ (cf. [FM]). On the other hand, by $\varphi_t \in \mathcal{H}_X$, we see that $X_{\mathbb{R}}\varphi_t = 0$. Hence,

$$u_{\omega(t)} = u_{\omega_0} - \sqrt{-1}\,\bar X\varphi_t. \tag{2.5}$$

Then we obtain the required equality as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\tilde\omega(t)^n &= \frac{\partial}{\partial t}\{e^{-\psi_{\omega(t)}}\omega(t)^n\}\\ &= \Big\{\Box_{\omega(t)}\dot\varphi_t - \dot\sigma(u_{\omega(t)})\frac{\partial}{\partial t}u_{\omega(t)}\Big\}e^{-\psi_{\omega(t)}}\omega(t)^n\\ &= \{\Box_{\omega(t)}\dot\varphi_t + \sqrt{-1}\,\dot\sigma(u_{\omega(t)})\bar X\dot\varphi_t\}e^{-\psi_{\omega(t)}}\omega(t)^n\\ &= (\tilde\Box_{\omega(t)}\dot\varphi_t)\tilde\omega(t)^n.\end{aligned}$$

(b) In (a) above, we have $(\partial/\partial t)\int_M \tilde\omega(t)^n = \int_M (\tilde\Box_{\omega(t)}\dot\varphi_t)\tilde\omega(t)^n = \langle\langle\tilde\Box_\omega\dot\varphi_t, 1\rangle\rangle_{\tilde\omega} = 0$ and hence the function $V : \mathcal{K}_X \to \mathbb{R}$ defined by

$$V(\omega) := \int_M \tilde\omega^n, \quad \omega \in \mathcal{K}_X,$$

is constant along any smooth path in $\mathcal{K}_X$. Since every $\omega \in \mathcal{K}_X$ and ω_0 are joined by the smooth path $t\omega_0 + (1-t)\omega$, $0 \le t \le 1$, in $\mathcal{K}_X$, we now conclude that V is constant on $\mathcal{K}_X$, as required. □

By $\langle\langle u, \tilde\Box_\omega u\rangle\rangle_{\tilde\omega} = -\int_M (\bar\partial u, \bar\partial u)_\omega\tilde\omega^n \le 0$, all eigenvalues of $-\tilde\Box_\omega$ are non-negative real numbers. Let $\lambda_1 = \lambda_1(\tilde\omega) > 0$ be the first positive eigenvalue of $-\tilde\Box_\omega$, and assume

$$\mathcal{K}_X^{(\nu)} \neq \emptyset$$

for some $\nu > 0$. Then we have $c_1(M) > 0$, and by the Kodaira vanishing theorem, we see that $0 = h^{0,1}(M) = h^{1,0}(M)$. In particular, $G := \mathrm{Aut}^0(M)$

is a linear algebraic group. The corresponding Lie algebra $\mathfrak{g}$ is just the space $H^0(M, \mathcal{O}(TM))$ of holomorphic vector fields on M. We now have a $\mathbb{C}$-linear isomorphism of vector spaces

$$\mathfrak{g}^\omega \cong \mathfrak{g}, \quad u \leftrightarrow \operatorname{grad}^{\mathbb{C}}_\omega u, \tag{2.6}$$

where $\mathfrak{g}^\omega$ denotes the space of all $u \in C^\infty(M)_{\mathbb{C}}$, normalized by $\int_M u\tilde{\omega}^n = 0$, such that the condition $\operatorname{grad}^{\mathbb{C}}_\omega \varphi \in \mathfrak{g}$ is satisfied. Recall that

Fact 2.7. (see for instance [M3]) *For a real number $\nu > 0$, let $\omega \in \mathcal{K}^{(\nu)}_X$. Then*

(a) $\lambda_1(\tilde{\omega}) \geq \nu$.

(b) *If $\lambda_1(\tilde{\omega}) = \nu$, then $\{u \in C^\infty(M)_{\mathbb{C}}\,;\, \tilde{\square}_\omega u = -\lambda_1(\tilde{\omega})u\}$ is a subspace of $\mathfrak{g}^\omega$.*

Next, we consider the special case where the Kähler class of $\mathcal{K}_X$ is $2\pi c_1(M)_{\mathbb{R}}$. In this case, to each $\omega \in \mathcal{K}_X$, we can associate a unique function f_ω in $C^\infty(M)_{\mathbb{R}}$ satisfying $\int_M (e^{f_\omega} - 1)\omega^n = 0$ and $\operatorname{Ric}(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial} f_\omega$. Put $c_\omega := \int_M \tilde{\omega}^n / \int_M \omega^n = \int_M \tilde{\omega}_0^n / \int_M \omega_0^n$, which is independent of the choice of ω in $\mathcal{K}_X$. We now put

$$\tilde{f}_\omega := f_\omega + \psi_\omega + \log c_\omega = f_\omega + \sigma(u_\omega) + \log c_\omega. \tag{2.8}$$

Lemma 2.9. (a) $\operatorname{Ric}^\sigma(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial}\tilde{f}_\omega$.

(b) $\int_M (e^{\tilde{f}_\omega} - 1)\tilde{\omega}^n = 0$ *for all $\omega \in \mathcal{K}_X$.*

Proof. (a) follows immediately from (1.5), (2.8) and $\operatorname{Ric}(\omega) - \omega = \partial\bar{\partial} f_\omega$. As to (b), in view of (b) of Lemma 2.4, we obtain

$$\int_M e^{\tilde{f}_\omega}\tilde{\omega}^n = \left(\int_M e^{f_\omega} e^{\psi_\omega}\tilde{\omega}^n\right)\frac{\int_M \tilde{\omega}_0^n}{\int_M \omega_0^n} = \left(\int_M e^{f_\omega}\omega^n\right)\frac{\int_M \tilde{\omega}^n}{\int_M \omega^n} = \int_M \tilde{\omega}^n,$$

as required. □

§3. Proof of Theorem A

Let $\omega \in \mathcal{K}_X$. In the definition of $\tilde{\omega}$ in (1.4), replacing σ by 2σ, we consider volume forms $\operatorname{vol}_{\tilde{\omega}}$ and $\operatorname{vol}_{\tilde{\omega}_0}$ on M by setting

$$\operatorname{vol}_{\tilde{\omega}} := \omega^n \exp\{-2\sigma(u_\omega)\} \quad \text{and} \quad \operatorname{vol}_{\tilde{\omega}_0} := \omega_0^n \exp\{-2\sigma(u_{\omega_0})\}.$$

Put $V := \int_M \mathrm{vol}_{\tilde\omega} = \int_M \mathrm{vol}_{\tilde\omega_0}$. Replacing σ again by 2σ in the definition of $\tilde\Box_\omega$ in (1.3), we consider the operators D_ω and D_{ω_0} acting on $C^\infty(M)_{\mathbb{R}}$ by

$$D_\omega := \Box_\omega + 2\sqrt{-1}\,\dot\sigma(u_\omega)\bar X \quad\text{and}\quad D_{\omega_0} := \Box_{\omega_0} + 2\sqrt{-1}\,\dot\sigma(u_{\omega_0})\bar X. \tag{3.1}$$

Note that a smooth function on M is $X_{\mathbb{R}}$-invariant if and only if it is Q-invariant. Hence, we can write $\omega = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi$ for some Q-invariant function φ in $\mathcal{H}_X$. Then we obtain

$$-\Box_{\omega_0}\varphi < n \quad\text{and}\quad -\Box_\omega\varphi > -n. \tag{3.2}$$

Now by (2.5), we have $\sqrt{-1}\,\bar X\varphi = u_{\omega_0} - u_\omega$. On the other hand, $\min_M u_{\omega_0} = \min_M u_\omega = \alpha_X$ and $\max_M u_{\omega_0} = \max_M u_\omega = \beta_X$. In particular,

$$\max_M |\bar X\varphi| = \max_M |X\varphi| \le \max_M |u| + \max_M |u_0| \le 2C_3, \tag{3.3}$$

where $C_3 := \max\{|\alpha_X|, |\beta_X|\}$ is a positive constant independent of the choice of ω_0 and ω in $\mathcal{K}_X$. Put $C_4 := \max_{s\in I_X} |\dot\sigma(s)| > 0$. Then (3.1) and (3.2) above imply

$$-D_\omega\,\varphi = -\Box_\omega\,\varphi - 2\sqrt{-1}\,\dot\sigma(u_\omega)\bar X\varphi \ \ge -k' := -n - 4C_3C_4, \tag{3.4}$$

$$-D_{\omega_0}\varphi = -\Box_{\omega_0}\varphi - 2\sqrt{-1}\,\dot\sigma(u_{\omega_0})\bar X\varphi \le \ k'' := \ n + 4C_3C_4. \tag{3.5}$$

Let $\mathrm{Re}\,D_\omega := (D_\omega + \bar D_\omega)/2$ and $\mathrm{Re}\,D_{\omega_0} := (D_{\omega_0} + \bar D_{\omega_0})/2$ denote respectively the real part of D_ω and D_{ω_0}. Moreover, let $G_\omega(x,y)$ and $G_{\omega_0}(x,y)$ be the Green functions for the operators $\mathrm{Re}\,D_\omega$ and $\mathrm{Re}\,D_{\omega_0}$, respectively. More precisely,

$$\begin{cases} h(x) = V^{-1}\displaystyle\int_M h(y)\,\mathrm{vol}_{\tilde\omega}(y) + \int_M G_\omega(x,y)\{-(\mathrm{Re}\,D_\omega)(h)\}(y)\,\mathrm{vol}_{\tilde\omega}(y),\\ \displaystyle\int_M G_\omega(x,y)\,\mathrm{vol}_{\tilde\omega}(y) = 0, \end{cases}$$

hold for all $x \in M$ and $h \in C^\infty(M)_{\mathbb{R}}$, where equalities similar to the above hold also for the Green function $G_{\omega_0}(x,y)$ in terms of $\mathrm{vol}_{\tilde\omega_0}$ and $\mathrm{Re}\,D_{\omega_0}$.

Proof of Theorem A. Assuming $\omega \in \mathcal{K}_X^{(\nu)}$, let $\ddot\sigma \ge 0$ on I_X. We further assume that one of the following holds:

(a) $\dot\sigma \le 0$ on I_X and $\omega \in \mathcal{S}^\sigma$;

(b) or σ is strictly convex.

For the Q-action on M, take the averages $\tilde{G}_\omega(x,y)$, $\tilde{G}_{\omega_0}(x,y)$ of the functions $G_\omega(x,y)$, $G_{\omega_0}(x,y)$ respectively, i.e.,

$$\begin{cases} \tilde{G}_\omega(x,y) \;:= \displaystyle\int_Q G_\omega(q\cdot x, y)\, d\mu(q) \;= \int_Q G_\omega(x, q\cdot y)\, d\mu(q), \\ \tilde{G}_{\omega_0}(x,y) := \displaystyle\int_Q G_{\omega_0}(q\cdot x, y)\, d\mu(q) = \int_Q G_{\omega_0}(x, q\cdot y)\, d\mu(q), \end{cases}$$

where $d\mu = d\mu(q)$ denotes the Haar measure for the compact group Q of total volume 1. Let K_ω, K_{ω_0} be the positive real numbers defined by

$$-K_\omega = \inf_{x\neq y} \tilde{G}_\omega(x,y) \quad \text{and} \quad -K_{\omega_0} = \inf_{x\neq y} \tilde{G}_0(x,y),$$

where the infimums are taken over all $(x,y)\in M\times M$ such that $x\neq y$. By writing $\omega = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi$ for some Q-invarant function $\varphi\in C^\infty(M)_{\mathbb{R}}$ as above, we first of all see the equality $(\operatorname{Re} D_{\omega_0})(\varphi) = D_{\omega_0}\varphi$. Then by (3.5), we obtain

(3.6)
$$\begin{aligned}\varphi(x) &= V^{-1}\int_M \varphi\,\mathrm{vol}_{\tilde\omega_0} + \int_M \{\tilde{G}_{\omega_0}(x,y) + K_{\omega_0}\}\{-(\operatorname{Re} D_{\omega_0})(\varphi)\}(y)\,\mathrm{vol}_{\tilde\omega_0}(y)\\ &\le V^{-1}\int_M \varphi\,\mathrm{vol}_{\tilde\omega_0} + k'' V K_{\omega_0}.\end{aligned}$$

On the other hand, by $(\operatorname{Re} D_\omega)(\varphi) = D_\omega\varphi$ and (3.4), we also obtain

(3.7)
$$\begin{aligned}\varphi(x) &= V^{-1}\int_M \varphi\,\mathrm{vol}_{\tilde\omega} + \int_M \{\tilde{G}_{\omega}(x,y) + K_{\omega}\}\{-(\operatorname{Re} D_{\omega})(\varphi)\}(y)\,\mathrm{vol}_{\tilde\omega}(y)\\ &\ge V^{-1}\int_M \varphi\,\mathrm{vol}_{\tilde\omega} - k' V\, K_{\omega}.\end{aligned}$$

Now by (3.6) and (3.7), we see that (cf. (A.1.1) in Appendix 1)

$$\begin{aligned}\operatorname{Osc}(\varphi) &\le V^{-1}\int_M \varphi(\mathrm{vol}_{\tilde\omega_0} - \mathrm{vol}_{\tilde\omega}) + (k'' K_{\omega_0} + k' K_\omega)V\\ &\le V^{-1}\mathcal{I}^{2\sigma}(\omega_0,\omega) + (k'' K_{\omega_0} + k' K_\omega)V,\end{aligned} \tag{3.8}$$

where by [M3], there exist positive real constants C', C'' and C_2 independent of the choice of $\nu > 0$ and ω, such that

$$K_\omega \le \nu^{-1}(C' + C'' e^{C_2/\nu}) \tag{3.9}$$

under the assumption (a) above, while under the assumption (b) above, we also have (3.9) with $C'' = 0$. Now by Lemma A.1.5 and Proposition A.1 in Appendix 1, we have

$$\mathcal{I}^{2\sigma}(\omega_0, \omega) \le (m+2)(\mathcal{I}^{2\sigma} - \mathcal{J}^{2\sigma})(\omega_0, \omega) \le (m+2)e^c(\mathcal{I}^{\sigma} - \mathcal{J}^{\sigma})(\omega_0, \omega),$$

where $m := n - 1 + b_{2\sigma}$ by the notation in Lemma A.1.6 in Appendix 1, and we put $c := \max_{s\in I_X} |\sigma(s)| = \max\{|\alpha_X|, |\beta_X|\}$ as in the introduction. Hence in view of (3.8) and (3.9), by setting $C(\nu) := C_1 + C_1'\nu + C_1'' e^{C_2/\nu}$, we obtain

$$\mathrm{Osc}(\varphi) \le C_0(\mathcal{I}^{\sigma} - \mathcal{J}^{\sigma})(\omega_0, \omega) + \frac{C(\nu)}{\nu},$$

where $C_1 := k'C'V$, $C_1' := k''K_{\omega_0}V$, $C_1'' := k'C''V$ and $C_0 := V^{-1}(m+2)e^c$ are positive real constants depending neither on the choice of ω nor on $\nu > 0$, as required. □

§4. Proof of Theorem B

In this section, M is not necessarily compact, and we fix a nonconstant real-valued function $\sigma : I_X \to \mathbb{R}$ which is weakly convex, i.e., $\ddot{\sigma} \ge 0$ on I_X. Let zero(X) be the set of all points on M at which the nonzero holomorphic vector field $X = \mathrm{grad}^{\mathbb{C}}_{\omega} u_{\omega}$ vanishes.

Definition 4.1. Under the above assumption of weak convexity of σ, we say that (X, σ) is *of Hamiltonian type*, if one of the following two conditions is satisfied:

(4.1.1) $\qquad \mathrm{zero}(X) \ne \emptyset;$

(4.1.2) $\qquad \ddot{\sigma}(s) = 0 \quad \text{for all } s \in I_X.$

Remark 4.2. If M is compact, then the assumption $\mathcal{K}_X^{(\nu)} \ne \emptyset$ in Theorem A implies that $c_1(M) > 0$, and in particular G is a linear algebraic group. Hence, in this case (4.1.1) automatically holds.

Proof of Theorem B. The proof is divided into the following three steps:

Step 1. In this step, we apply the arguments in [Mil] to the Kähler manifold (M, ω). Let $\zeta : [0, \ell] \to M$ be an arclength-parametrized geodesic with $\zeta(0) = p$. Put $\zeta(\ell) = q$, and consider the set $\Omega(M; p, q)$ of all smooth paths $\gamma : [0,\ell] \to M$ such that $\gamma(0) = p$ and $\gamma(\ell) = q$. Recall that the energy functional $E : \Omega(M; p, q) \to \mathbb{R}$ is defined by

$$E(\gamma) := \int_0^\ell \|\gamma_*(\partial/\partial t)\|_\omega^2 \, dt, \quad \gamma \in \Omega(M; p, q).$$

Then ζ is a critical point of the functional E. Let $P_k = P_k(t)$, $k = 1, 2, \dots, 2n$, be parallel vector fields along ζ which are orthonormal everywhere along ζ. Consider the complex structure $J : TM_{\mathbb{R}} \to TM_{\mathbb{R}}$ of the complex manifold M, where $TM_{\mathbb{R}}$ denotes the real tangent bundle of M. Then by $\nabla J = 0$, we may assume that $P_1 = \zeta_*(\partial/\partial t)$ and $P_2 = JP_1$. Put $\hat{P}_k(t) = \sin(\pi t/\ell) P_k(t)$. Let $\mathrm{Hess}_\zeta E$ denote the Hessian of E at ζ. Then by setting $\hat{n} := 2n - 1$, we obtain

$$\frac{1}{2}\sum_{k=2}^{2n} (\mathrm{Hess}_\zeta E)(\hat{P}_k, \hat{P}_k) = \int_0^\ell \sin^2(\pi t/\ell) \left\{ \frac{\hat{n}\pi^2}{\ell^2} - S_\omega(P_1, P_1) \right\} dt, \tag{4.3.1}$$

where S_ω denotes the Ricci tensor of the Kähler metric ω, and is related to the Ricci form $\mathrm{Ric}(\omega)$ by $S_\omega(P_1, P_1) = \mathrm{Ric}(\omega)(P_1, JP_1)$.

Step 2. Fix an arbitrary $\tau \in [0, \ell]$. In a small open neighbourhood of $\zeta(\tau)$ in M, we choose a system $z = (z^1, z^2, \dots, z^n)$ of holomorphic local coordinates centered at $\zeta(\tau)$ such that

$$P_1(\tau) = \partial/\partial x^1 \quad \text{and} \quad JP_1(\tau) = \partial/\partial y^1,$$

where we write each z^α as a sum $x^\alpha + \sqrt{-1}\, y^\alpha$ of the real part and the imaginary part, and the vector fields $\partial/\partial x^\alpha$, $\partial/\partial y^\alpha$ are taken in terms of the coordinates system $(x^1, \dots, x^n, y^1, \dots, y^n)$. Since

$$\partial/\partial z^\alpha = (\partial/\partial x^\alpha - \sqrt{-1}\, \partial/\partial y^\alpha)/2 \quad \text{and} \quad \partial/\partial z^{\bar\beta} = (\partial/\partial x^\beta + \sqrt{-1}\, \partial/\partial y^\beta)/2,$$

we observe that the coordinates system $z = (z^1, z^2, \dots, z^n)$ can be chosen in such a way that $g_{\alpha\bar\beta}$ in the local expression of ω (cf. Section 1) satisfies

$$g_{\alpha\bar\beta}(\zeta(\tau)) = \frac{1}{2}\delta_{\alpha\beta} \quad \text{and} \quad dg_{\alpha\bar\beta}(\zeta(\tau)) = 0. \tag{4.3.2}$$

Let $\exp_{\zeta(\tau)} : (TM_{\mathbb{R}})_{\zeta(\tau)} \to M$ denotes the exponential map at the point $\zeta(\tau)$ of the Kähler manifold (M, ω), and put $\xi(s) := \exp_{\zeta(\tau)}(sJP_1)$, $-\varepsilon \le s \le \varepsilon$, with a sufficiently small positive real number ε. Then in a neighbourhood of $\zeta(\tau)$,

$$\begin{cases} P_1(t) = \zeta_*(\partial/\partial t) = \partial/\partial x^1 + O(|t-\tau|^2), \\ \xi_*(\partial/\partial s) = \partial/\partial y^1 + O(|s|^2), \end{cases} \tag{4.3.3}$$

where $O(w)$ denotes a function which is bounded by some constant times w. Now by our assumption, $X = \mathrm{grad}^{\mathbb{C}}_{\omega} u_{\omega}$ is a holomorphic vector field on M. Hence by the equality $\bar{\partial} X = 0$ and (4.3.2), we obtain $(\partial/\partial z^{\bar{1}})^2(u_{\omega})_{|\zeta(\tau)} = 0$ at the point $\zeta(\tau)$, and hence

$$\begin{cases} (\partial/\partial x^1)^2(u_{\omega})_{|\zeta(\tau)} = (\partial/\partial y^1)^2(u_{\omega})_{|\zeta(\tau)}\,, \\ (\partial^2/\partial x^1 \partial y^1)(u_{\omega})_{|\zeta(\tau)} = 0. \end{cases} \tag{4.3.4}$$

We now define a C^{∞} map $F : [-\varepsilon, \varepsilon] \times [0, \ell] \to M$ by sending each $(s,t) \in [-\varepsilon, \varepsilon] \times [0, \ell]$ to $F(s,t) := \exp_{\zeta(t)}(sJP_1) \in M$. Put $\tilde{u} := F^* u_{\omega}$ and $\tilde{\psi} := F^* \psi_{\omega}$ which are functions on $[-\varepsilon, \varepsilon] \times [0, \ell]$. Then by (1.2), we have $\tilde{\psi} = \sigma(\tilde{u})$. Next by (4.3.3),

$$\begin{cases} (\partial/\partial t)(\tilde{u})_{|s=0} = \zeta^*\{(\partial/\partial x^1)(u_{\omega})\} + O(|t-\tau|^2), \\ (\partial/\partial s)(\tilde{u})_{|t=\tau} = \xi^*\{(\partial/\partial y^1)(u_{\omega})\} + O(|s|^2), \end{cases} \tag{4.3.5}$$

in a neighbourhood of $(s,t) = (0, \tau)$. In view of (4.3.3), we differentiate the first line of (4.3.5) with respect to t at $t = \tau$, while we differentiate the second line of (4.3.5) with respect to s at $s = 0$. Then, since $\tau \in [0, \ell]$ is arbitrary, the first line of (4.3.4) yields

$$(\partial/\partial t)^2(\tilde{u}) = (\partial/\partial s)^2(\tilde{u}), \tag{4.3.6}$$

when restricted to $\{0\} \times [0, \ell]$. Recall that ∇ is the natural Hermitian connection associated to the Kähler metric ω (see Section 1). Since $P_2 = JP_1$ is parallel along the geodesic ζ, and since ξ is a geodesic, we obtain

$$\left(\nabla_{\partial/\partial t}\partial/\partial s\right)_{|(s,t)=(0,\tau)} = \left(\nabla_{\partial/\partial s}\partial/\partial s\right)_{|(s,t)=(0,\tau)} = 0,$$

where the pullback $F^*\nabla$ is denoted also by ∇ for simplicity. By combining this with (4.3.2) and $F_*\partial/\partial s_{|(s,t)=(0,\tau)} = \partial/\partial y^1$, we obtain

$$F_*(\partial/\partial s) = \partial/\partial y^1 + O(|s|^2 + |t-\tau|^2) \quad \text{for } |s|^2 + |t-\tau|^2 \ll 1$$

in a small neighbourhood of $\zeta(\tau) = F(0,\tau)$ in the image of F. Hence, together with the first line of (4.3.3), the second line of (4.3.4) implies

$$(\partial^2/\partial t\partial s)(\tilde{u}) = 0, \tag{4.3.7}$$

when restricted to $\{0\} \times [0,\ell]$. For the time being, until the end of Step 2, we assume that (4.1.1) above holds. Then by $p = \zeta(0) \in \mathrm{Zero}(X)$, the function u_ω on M has a critical value at p. In particular, $(\partial\tilde{u}/\partial s)(0,0) = 0$. On the other hand, (4.3.7) shows that $\partial\tilde{u}/\partial s$ is constant along $\{0\} \times [0,\ell]$. Therefore,

$$(\partial\tilde{u}/\partial s)(0,t) = 0 \quad \text{for all } t \in [0,\ell], \text{ if (4.1.1) holds.} \tag{4.3.8}$$

Step 3. Let σ be as in Definition 4.1, so that either (4.1.1) or (4.1.2) holds. Consider the function $\psi_\omega = \sigma(u_\omega)$. In view of (4.3.3), we see for all $\tau \in [0,\ell]$ the following:

$$\begin{aligned} &2\sqrt{-1}\,(\partial\bar{\partial}\psi_\omega)(P_1, JP_1)_{|\zeta(\tau)} \\ &\quad = 2\sqrt{-1}\,(\partial\bar{\partial}\psi_\omega)(\zeta_*(\partial/\partial t), \xi_*(\partial/\partial s))_{|\zeta(\tau)} \\ &\quad = \left\{(\partial/\partial x^1)^2(\psi_\omega) + (\partial/\partial y^1)^2(\psi_\omega)\right\}_{|\zeta(\tau)} \\ &\quad = \frac{\partial^2\tilde{\psi}}{\partial t^2}(0,\tau) + \frac{\partial^2\tilde{\psi}}{\partial s^2}(0,\tau). \end{aligned} \tag{4.3.9}$$

Consider the vector fields $Z_1 := (P_1 - \sqrt{-1}\,JP_1)/2$ and $\bar{Z}_1 := (P_1 + \sqrt{-1}\,JP_1)/2$ along the geodesic ζ. Since $(2/\sqrt{-1}\,)\,\theta(Z_1,\bar{Z}_1)$ equals $\theta(P_1, JP_1)$ along the geodesic for every 2-form θ on M, and since $\mathrm{Ric}(\omega) + \sqrt{-1}\,\partial\bar{\partial}\psi_\omega = \mathrm{Ric}^\sigma(\omega) \geq \nu\omega$, it now follows that

$$\begin{aligned} \mathrm{Ric}(\omega)(P_1, JP_1) + \sqrt{-1}\,(\partial\bar{\partial}\psi_\omega)(P_1, JP_1) &= \mathrm{Ric}^\sigma(\omega)(P_1, JP_1) \\ \geq \nu\omega(P_1, JP_1) = (2\nu/\sqrt{-1}\,)\,\omega(Z_1,\bar{Z}_1) &= \nu. \end{aligned}$$

By plugging the expression (4.3.9) of $2\sqrt{-1}\,(\partial\bar{\partial}\psi_\omega)(P_1, JP_1)_{|\zeta(\tau)}$ into the inequality just above, we see that the following inequality holds for all $\tau \in [0,\ell]$:

$$\mathrm{Ric}(\omega)(P_1, JP_1)_{|\zeta(\tau)} \geq \nu - \frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial t^2}(0,\tau) - \frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial s^2}(0,\tau).$$

By this together with (4.3.1), we obtain

$$\begin{aligned}\frac{1}{2}\sum_{k=2}^{2n}&(\mathrm{Hess}_\zeta\, E)(\hat{P}_k, \hat{P}_k)\\ &\leq \int_0^\ell \sin^2(\pi t/\ell)\left\{\frac{\hat{n}\pi^2}{\ell^2} - \nu + \frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial t^2}(0,t) + \frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial s^2}(0,t)\right\} dt.\end{aligned}$$

If (4.1.1) holds, then by (4.3.6) and (4.3.8), we see from $\tilde{\psi} = \sigma(\tilde{u})$ that

$$\begin{aligned}\frac{\partial^2\tilde{\psi}}{\partial s^2}(0,t) &= \left\{\dot{\sigma}(\tilde{u})\frac{\partial^2\tilde{u}}{\partial s^2} + \ddot{\sigma}(\tilde{u})\Big(\frac{\partial\tilde{u}}{\partial s}\Big)^2\right\}_{|(0,t)} = \left\{\dot{\sigma}(\tilde{u})\frac{\partial^2\tilde{u}}{\partial t^2}\right\}_{|(0,t)}\\ &\leq \left\{\dot{\sigma}(\tilde{u})\frac{\partial^2\tilde{u}}{\partial t^2} + \ddot{\sigma}(\tilde{u})\Big(\frac{\partial\tilde{u}}{\partial t}\Big)^2\right\}_{|(0,t)} = \frac{\partial^2\tilde{\psi}}{\partial t^2}(0,t),\end{aligned}$$

where the inequality just above follows from the weak convexity of σ. On the other hand, if (4.1.2) holds, then again by (4.3.6)

$$\frac{\partial^2\tilde{\psi}}{\partial s^2}(0,t) = \dot{\sigma}(\tilde{u})\frac{\partial^2\tilde{u}}{\partial s^2}(0,t) = \dot{\sigma}(\tilde{u})\frac{\partial^2\tilde{u}}{\partial t^2}(0,t) = \frac{\partial^2\tilde{\psi}}{\partial t^2}(0,t).$$

In both cases, we obtain

$$\frac{1}{2}\sum_{k=2}^{2n}(\mathrm{Hess}_\zeta\, E)(\hat{P}_k, \hat{P}_k) \leq \int_0^\ell \sin^2(\pi t/\ell)\left\{\frac{\hat{n}\pi^2}{\ell^2} - \nu + \frac{\partial^2\tilde{\psi}}{\partial t^2}(0,t)\right\} dt.$$

Let R.H.S. denote the right-hand side of this inequality. Then by taking integral by parts over and over again, we see that

$$\begin{aligned}\text{R.H.S.} &= \int_0^\ell \left\{\Big(\frac{\hat{n}\pi^2}{\ell^2} - \nu\Big)\sin^2(\pi t/\ell) - \frac{\pi}{\ell}\frac{\partial\tilde{\psi}}{\partial t}(0,t)\sin(2\pi t/\ell)\right\} dt\\ &= \int_0^\ell \left\{\Big(\frac{\hat{n}\pi^2}{\ell^2} - \nu\Big)\sin^2(\pi t/\ell) + \frac{2\pi^2}{\ell^2}\tilde{\psi}(0,t)\cos(2\pi t/\ell)\right\} dt\\ &\leq \frac{2\pi^2 c}{\ell} + \int_0^\ell \Big(\frac{\hat{n}\pi^2}{\ell^2} - \nu\Big)\sin^2(\pi t/\ell)\, dt = \frac{(\hat{n}+4c)\pi^2}{2\ell} - \frac{\ell\nu}{2}.\end{aligned}$$

Therefore, if $\ell > \pi\{(\hat{n}+4c)/\nu\}^{1/2}$, then R.H.S. < 0, and hence

$$\sum_{k=2}^{2n}(\mathrm{Hess}_\zeta\, E)(\hat{P}_k, \hat{P}_k) < 0,$$

which shows that $\zeta : [0,\ell] \to M$ is not an arclength-minimizing geodesic. Thus, we obtain $\mathrm{dist}_\omega(p,q) \leq \pi\{(\hat{n}+4c)/\nu\}^{1/2}$ for every $q \in M$, as required.

□

§5. Proof of Theorem C

Fix $0 < \alpha < 1$. Let $\mathcal{H}^{2,\alpha}_{X,0}$ denote the set of all $X_{\mathbb{R}}$-invariant function $\varphi \in C^{2,\alpha}(M)_{\mathbb{R}}$ such that $\int_M \varphi\tilde{\omega}_0^n = 0$ and that ω_φ is positive definite on M. Put

$$A(\varphi) := \tilde{\omega}_\varphi^n/\tilde{\omega}_0^n, \quad \varphi \in \mathcal{H}^{2,\alpha}_{X,0}. \tag{5.1.1}$$

For each $0 \le k \in \mathbb{Z}$, we consider the space $C^{k,\alpha}_{X,0}(M)_{\mathbb{R}}$ of all $X_{\mathbb{R}}$-invariant functions φ in $C^{k,\alpha}(M)_{\mathbb{R}}$ such that $\int_M \varphi\tilde{\omega}_0^n = 0$. Define $\Gamma : \mathcal{H}^{2,\alpha}_{X,0} \times \mathbb{R} \to C^{0,\alpha}_{X,0}(M)_{\mathbb{R}}$ by setting (cf. [BM], [S1])

$$\Gamma(\varphi,t) := A(\varphi) - \left\{\frac{1}{V_0}\int_M \exp(-t\varphi + \tilde{f}_{\omega_0})\tilde{\omega}_0^n\right\}^{-1} \exp(-t\varphi + \tilde{f}_{\omega_0}), \tag{5.1.2}$$

for all $(\varphi,t) \in \mathcal{H}^{2,\alpha}_{X,0} \times \mathbb{R}$, where V_0 is as in (b) of Lemma 2.4. Let T be the set of all $t \in [0,1)$ for which *the generalized Aubin's equation*

$$\Gamma(\varphi,t) = 0 \tag{5.1.3}$$

admits a solution $\varphi = \varphi_t$ in $\mathcal{H}^{2,\alpha}_{X,0}$. Note that φ automatically belongs to $\mathcal{H}_X$. For such a solution φ_t, we set $\omega(t) := \omega_{\varphi_t} = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t$ as in (A.2.2) in Appendix 2. Then

$$\mathrm{Ric}^\sigma(\omega(t)) = \omega_0 + t\sqrt{-1}\,\partial\bar{\partial}\varphi_t = t\omega(t) + (1-t)\omega_0, \tag{5.1.4}$$

where $\tilde{\omega}(t)$ is as in (2.3). In particular, $\omega(t)$ sits in $\mathcal{K}^{(t')}_X$ for some t' which exceeds t. Suppose that $\Gamma(\hat{\varphi},\hat{t}) = 0$ for some $(\hat{\varphi},\hat{t}) \in \mathcal{H}^{2,\alpha}_{X,0} \times [0,1)$. Then the Fréchet derivative $D_\varphi\Gamma : C^{2,\alpha}_{X,0}(M)_{\mathbb{R}} \to C^{0,\alpha}_{X,0}(M)_{\mathbb{R}}$ of Γ at $(\hat{\varphi},\hat{t})$ with respect to the factor φ is given by

$$\{D_\varphi\Gamma_{|(\varphi,t)=(\hat{\varphi},\hat{t})}\}(\eta) := A(\hat{\varphi})(\tilde{\Box}_{\hat{\varphi}} + \hat{t})(\eta - C_{\eta,\hat{\varphi}}), \quad \eta \in C^{2,\alpha}_{X,0}(M)_{\mathbb{R}}, \tag{5.1.5}$$

where $C_{\eta,\hat{\varphi}} := V_0^{-1}\int_M \eta\tilde{\omega}_{\hat{\varphi}}^n$ and $\tilde{\Box}_{\hat{\varphi}} := \tilde{\Box}_{\omega_{\hat{\varphi}}}$. By (5.1.4) and Fact 2.7, $\hat{t}$ is less than the first positive eigenvalue of $-\tilde{\Box}_{\hat{\varphi}}$. Hence, $D_\varphi\Gamma_{|(\varphi,t)}$ is invertible. Then by the implicit function theorem, we obtain

THEOREM 5.1. *If* $(\hat{\varphi},\hat{t}) \in \mathcal{H}^{2,\alpha}_{X,0} \times [0,1)$ *satisfies* $\Gamma(\hat{\varphi},\hat{t}) = 0$, *then there exist* $0 < \varepsilon \ll 1$ *and a smooth one-parameter family of functions* $\{\varphi_t\,;\,\hat{t}-\varepsilon < t < \hat{t}+\varepsilon\}$ *in* $\mathcal{H}^{2,\alpha}_{X,0}$ *satisfying* $\varphi_{\hat{t}} = \hat{\varphi}$ *such that* $\varphi = \varphi_t$ *is the unique solution of* (5.1.3) *for each* t *under the condition* $\|\varphi - \hat{\varphi}\|_{C^{2,\alpha}} \le \varepsilon$. *In particular,* T *is an open subset of* $[0,1)$.

Let $0 \leq a < b \leq 1$, and let φ_t, $a < t \leq b$, be a smooth one-parameter family of functions in $\mathcal{H}^{2,\alpha}_{X,0}$ such that, for all $a < t \leq b$, we have

$$\Gamma(\varphi_t, t) = 0. \tag{5.2.1}$$

Then each φ_t automatically belongs to $\mathcal{H}_X$. By setting $\omega(t) := \omega_{\varphi_t}$ as in the above, we obtain (5.1.4). We further put $\psi_t := \psi_{\omega(t)}$ and $\tilde{f}_t := \tilde{f}_{\omega(t)}$, where on the right-hand sides, we use the notation in the introduction and (2.8). Since $\mathrm{Ric}^\sigma(\omega(t)) = \omega(t) + \sqrt{-1}\,\partial\bar{\partial}\tilde{f}_t$, and since $\omega(t) = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t$, the identity (5.1.4) implies

$$\tilde{f}_t = -(1-t)\varphi_t + C_t, \tag{5.2.2}$$

where C_t is a real constant depending on t. By (5.1.1) and (a) of Lemma 2.4, we have $\partial A(\varphi_t)/\partial t = \{\tilde{\Box}_{\omega(t)}\dot{\varphi}_t\}A(\varphi_t)$. By differentiating (5.2.1) with respect to t, we obtain

$$\tilde{\Box}_{\omega(t)}\dot{\varphi}_t + t\dot{\varphi}_t + \varphi_t = \hat{C}_t, \tag{5.2.3}$$

for some real constant $\hat{C}_t$ depending on t. By (A.1.1) in Appendix 1 and by (b) of Proposition A.2 in Appendix 2, we see from (5.2.2) and (5.2.3) the following:

$$\begin{aligned}\frac{d}{dt}\mu^\sigma(\omega(t)) &= \int_M (\bar{\partial}\tilde{f}_t, \bar{\partial}\dot{\varphi}_t)_{\omega(t)}\tilde{\omega}(t)^n = -(1-t)\int_M (\bar{\partial}\varphi_t, \bar{\partial}\dot{\varphi}_t)_{\omega(t)}\tilde{\omega}(t)^n \\ &= -(1-t)\frac{d}{dt}(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \omega(t)) = (1-t)\int_M \varphi_t\{\tilde{\Box}_{\omega(t)}\dot{\varphi}_t\}\tilde{\omega}(t)^n \\ &= -(1-t)\int_M \{\tilde{\Box}_{\omega(t)}\dot{\varphi}_t + t\dot{\varphi}_t\}\{\tilde{\Box}_{\omega(t)}\dot{\varphi}_t\}\tilde{\omega}(t)^n \leq 0,\end{aligned}$$

where in the last inequality, we apply (a) of Fact 2.7 to $\omega(t) \in \mathcal{K}^{(t)}_X$. Thus, for any $0 \leq a < b \leq 1$, we obtain

THEOREM 5.2. *Along any smooth one-parameter family φ_t, $a < t \leq b$, of solutions in $\mathcal{H}_X$ of (5.2.1), the corresponding $\omega(t) := \omega_{\varphi_t} = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t$ satisfies*

$$\frac{d}{dt}\mu^\sigma(\omega(t)) = -(1-t)\frac{d}{dt}(\mathcal{I}^\sigma - \mathcal{J}_\sigma)(\omega_0, \omega(t)) \leq 0, \quad a < t \leq b.$$

Given an element $\theta \in \mathcal{E}_X^\sigma$, we consider the set T_θ of all $\tau \in [0,1]$ such that there exists a smooth one-parameter family of solutions

$$\varphi_t \in \mathcal{H}_{X,0}^{2,\alpha}, \quad \tau \le t \le 1, \tag{5.3.1}$$

of (5.2.1) satisfying $\omega_{\varphi_1} = \theta$. Put $\tau_\infty := \inf T_\theta$. Later in Theorem 5.6, we see that a slight perturbation of ω_0 allows us to assume $\tau_\infty < 1$. Under this assumption, we obtain

LEMMA 5.3.2. *Suppose that σ is convex. Then we have the following*:
(a) $\tau_\infty = 0$.
(b) *If σ is furthermore strictly convex, then* 0 *belongs to* T_θ.

Proof. Take a sequence $\mathcal{S} := \{\tau_j\}_{j=1}^\infty$ of points in the open interval $(\tau_\infty, 1]$ such that τ_j converges to τ_∞ as $j \to \infty$. Let

$$\varphi_{\tau_j} \in \mathcal{H}_{X,0}^{2,\alpha}, \quad j = 1, 2, \dots,$$

be the corresponding solutions of (5.2.1) at $t = \tau_j$. For simplicity, φ_{τ_j} is denoted by φ_j, and we put $\omega^{(j)} := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_j$. In view of Theorem 5.1, the proof is reduced to showing that some subsequence of $\mathcal{S}$ is convergent in $C^{2,\alpha}(M)_\mathbb{R}$ assuming that either τ_∞ is positive or σ is strictly convex. By Theorem 5.2,

$$(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \omega^{(j)}) \le C_3, \quad \text{for all } j = 1, 2, \dots, \tag{5.3.3}$$

where $C_3 := (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \theta)$. Since $\omega^{(j)}$ belongs to $\mathcal{K}_X^{(\tau_j)}$, and since $\tau_j \le 1$ for all j, the combination of (1.6) and (5.3.3) implies

$$\begin{aligned} |\tau_j \operatorname{Osc} \varphi_j| &\le \tau_j C_0 (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \omega^{(j)}) + C(\tau_j) \\ &\le C_0 C_3 \tau_j + C(\tau_j) = C_0 C_3 \tau_j + C_1 + C_1' \tau_j + C_1'' e^{C_2/\tau_j} \\ &\le C_0 C_3 + C_1 + C_1' + C_1'' e^{C_2/\tau_j}, \end{aligned}$$

where if σ is strictly convex, we can set $C_1'' = 0$ by Theorem A. Note that the constant C_0, C_1, C_1', C_1'', C_2, C_3 are independent of the choice of j, and that $|\tau_j \operatorname{Osc} \varphi_j|$, $j = 1, 2, \dots$, are bounded from above by $C_0 C_3 + C_1 + C_1' + C_1'' e^{C_2/\tau_\infty}$ or $C_0 C_3 + C_1 + C_1'$ according as τ_∞ is positive or σ is strictly convex. Hence, in both of these cases, we have a positive constant C_4 independent of j such that

$$\|\tau_j \varphi_j\|_{C^0(M)} \le C_4,$$

since we have $\varphi_j(p_j) = 0$ at some point $p_j \in M$ in view of the identity $\int_M \varphi_j \tilde{\omega}_0^n = 0$. Moreover, for all j,

$$\begin{aligned}\omega_{\varphi_j}^n &= A(\varphi_j)\exp\{\psi_{\omega^{(j)}} - \psi_{\omega_0}\}\omega_0^n \\ &= \left(\frac{1}{V_0}\int_M \exp(-\tau_j\varphi_j + \tilde{f}_{\omega_0})\tilde{\omega}_0^n\right)^{-1}\exp\{-\tau_j\varphi_j + \tilde{f}_{\omega_0} + \psi_{\omega^{(j)}} - \psi_{\omega_0}\}\omega_0^n,\end{aligned}$$

where $|\psi_{\omega^{(j)}}|$, $j = 1, 2, \dots$, on M are bounded from above by

$$c := \max_{s\in[\ell_0,\ell_1]} |\sigma(s)|.$$

Therefore, we have a positive constant C_5 independent of j such that

$$\|\varphi_j\|_{C^0(M)} \leq C_5, \quad \text{for all } j.$$

Then by standard arguments for complex Monge-Ampère equations (see for instance [M4]), $\mathcal{S}$ is uniformly bounded in $C^{k,\alpha}(M)_{\mathbb{R}}$ for all $0 \leq k \in \mathbb{Z}$, and consequently some subsequence of $\mathcal{S}$ is convergent in $C^{2,\alpha}(M)_{\mathbb{R}}$, as required. □

Remark 5.3.4. In (b) of Lemma 5.3.2, even if σ is not strictly convex, we obtain $0 \in T_\theta$ just by the convexity of σ. This can be seen as follows: For each $r \in \mathbb{R}$, we put

$$\sigma_r(s) := \sigma(s) - r\log(s - \alpha_X + 1), \quad s \in I_X,$$

where α_X and I_X are as in the introduction. If r is positive, then $\ddot{\sigma}_r(s) > 0$ for all $s \in I_X$, and σ_r is strictly convex. In the arguments above, replacing σ by σ_r, we put $\psi_\omega^{[r]} := \sigma_r(u_\omega)$ and $\tilde{\omega}^{[r]} := \omega\exp(-\psi_\omega^{[r]}/n)$ for all $\omega \in \mathcal{K}_X$. For each $\varphi \in \mathcal{H}_{X,0}^{2,\alpha}$, we put

$$\begin{cases} A^{[r]}(\varphi) = \dfrac{(\tilde{\omega}_\varphi^{[r]})^n}{(\tilde{\omega}_0^{[r]})^n} = \dfrac{\omega_\varphi^n \exp(-\psi_{\omega_\varphi}^{[r]})}{\omega_0^n\exp(-\psi_{\omega_0}^{[r]})}, \\ \varphi^{[r]} = \varphi - \dfrac{1}{V_r}\displaystyle\int_M \varphi(\tilde{\omega}_0^{[r]})^n, \end{cases}$$

where $V_r := \int_M (\tilde{\omega}_0^{[r]})^n$. Put $\tilde{f}_\omega^{[r]} := f_\omega + \psi_\omega^{[r]} + \log\{\int_M (\tilde{\omega}_0^{[r]})^n / \int_M \omega_0^n\}$ for all $\omega \in \mathcal{K}_X$. Let us define a mapping $\tilde{\Gamma} : \mathcal{H}_{X,0}^{2,\alpha} \times \mathbb{R}^2 \to C_0^{0,\alpha}(M)_{\mathbb{R}}$ by

$$\begin{aligned}\tilde{\Gamma}(\varphi, t, r) := \frac{(\tilde{\omega}_0^{[r]})^n}{\tilde{\omega}_0^n}\Big\{&A^{[r]}(\varphi) \\ &- \Big(\frac{1}{V_r}\int_M \exp(-t\varphi^{[r]} + \tilde{f}_{\omega_0}^{[r]})(\tilde{\omega}_0^{[r]})^n\Big)^{-1}\exp(-t\varphi^{[r]} + \tilde{f}_{\omega_0}^{[r]})\Big\},\end{aligned}$$

where $(\varphi, t, r) \in \mathcal{H}^{2,\alpha}_{X,0} \times \mathbb{R}^2$. Suppose that $\tilde{\Gamma}(\hat{\varphi}, \hat{t}, 0) = 0$ for some $(\hat{\varphi}, \hat{t}) \in \mathcal{H}^{2,\alpha}_{X,0} \times [0,1)$. Then $\Gamma(\hat{\varphi}, \hat{t}) = 0$, and the Fréchet derivative $D_\varphi \tilde{\Gamma} : C^{2,\alpha}_{X,0}(M)_{\mathbb{R}} \to C^{0,\alpha}_{X,0}(M)_{\mathbb{R}}$ of $\tilde{\Gamma}$ with respect to φ is written as

$$D_\varphi \tilde{\Gamma}_{|(\varphi,t,r)=(\hat{\varphi},\hat{t},0)} = D_\varphi \Gamma_{|(\varphi,t)=(\hat{\varphi},\hat{t})}, \tag{5.3.5}$$

which is invertible. Hence, in a neighbourhood U of $(\hat{t}, 0)$ in $\mathbb{R}^2$, the solution $\hat{\varphi}$ of $\tilde{\Gamma}(\varphi, t, r) = 0$ at $(t, r) = (\hat{t}, 0)$ extends uniquely to

$$\hat{\varphi}_{t,r} \in C^{2,\alpha}_{X,0}(M)_{\mathbb{R}}, \quad (t, r) \in U,$$

depending on (t, r) continuously and satisfying $\tilde{\Gamma}(\hat{\varphi}_{t,r}, t, r) = 0$ for all $(t, r) \in U$ with $\hat{\varphi}_{\hat{t},0} = \hat{\varphi}$. As in Theorem 5.6 proved later, a slight perturbation of ω_0 (see (5.5.3)) allows us to assume that, for a sufficiently small $\delta > 0$, a smooth two-parameter family of functions

$$\varphi_{t,r} \in C^{2,\alpha}_{X,0}(M)_{\mathbb{R}}, \quad (t, r) \in [1-\delta, 1] \times [0, \delta], \tag{5.3.6}$$

exists satisfying $\theta = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_{1,0}$ and $\tilde{\Gamma}(\varphi_{t,r}, t, r) = 0$ for all $(t, r) \in [1-\delta, 1] \times [0, \delta]$. Then by Lemma 5.3.2 and Theorem 5.1, we see that (5.3.6) uniquely extends to a continuous family, denoted by the same notation, of functions

$$\varphi_{t,r} \in C^{2,\alpha}_{X,0}(M)_{\mathbb{R}}, \quad (0,0) \neq (t, r) \in [0, 1] \times [0, \delta], \tag{5.3.7}$$

satisfying $\tilde{\Gamma}(\varphi_{t,r}, t, r) = 0$ for all $(0,0) \neq (t, r) \in [0,1] \times [0, \delta]$. On the other hand, by Appendix 4, there exists a unique element γ_r of $\mathcal{H}^{2,\alpha}_{X,0}$ such that

$$\mathrm{Ric}^{\sigma_r}(\omega_{\gamma_r}) = \omega_0.$$

Then for each $r \in [0, \delta]$, the equation $\tilde{\Gamma}(\varphi, 0, r) = 0$ in $\varphi \in \mathcal{H}^{2,\alpha}_{X,0}$ has a unique solution $\varphi = \gamma_r$. In view of (5.3.7) above, this implies

$$\varphi_{0,r} = \gamma_r, \quad 0 < r \leq \delta.$$

By (5.3.5) applied to $(\hat{\varphi}, \hat{t}) = (\gamma_0, 0)$, letting δ be smaller if necessary, we see from the inverse function theorem that the solution $\varphi = \gamma_r$ of the equation $\tilde{\Gamma}(\varphi, 0, r) = 0$ in $\varphi \in \mathcal{H}^{2,\alpha}_{X,0}$ for $0 \leq r \leq \delta$ uniquely extends to a continuous family of functions

$$\varphi'_{t,r} \in C^{2,\alpha}_{X,0}(M)_{\mathbb{R}}, \quad (t, r) \in [0, \delta] \times [0, \delta], \tag{5.3.8}$$

satisfying $\varphi'_{0,r} = \gamma_r$ for $0 \leq r \leq \delta$ and $\tilde{\Gamma}(\varphi'_{t,r}, t, r) = 0$ for all $(t,r) \in [0,\delta] \times [0,\delta]$. Comparing (5.3.7) and (5.3.8), we obtain $\varphi_{t,r} = \varphi'_{t,r}$ for all $(0,0) \neq (t,r) \in [0,\delta] \times [0,\delta]$. In particular, $\varphi_{t,0}$ $(= \varphi'_{t,0})$ converges to γ_0 $(= \varphi'_{0,0})$ in $C^{2,\alpha}$ as t tends to 0. Thus, $0 \in T_\theta$.

By combining Lemma 5.3.2 and Remark 5.3.4, we obtain

THEOREM 5.3. *If σ is convex, then by a slight perturbation of ω_0 as in* (5.5.3)*, we have the situation that* 0 *belongs to T_θ.*

Take an arbitrary $Z^0(X)$-orbit $\mathbf{O}$ in $\mathcal{E}^\sigma_X$, which is a connected component of $\mathcal{E}^\sigma_X$ by Proposition A.5 in Appendix 5. Define a nonnegative C^∞ function $\iota : \mathbf{O} \to \mathbb{R}$ by

$$\iota(\theta) := (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \theta), \quad \theta \in \mathbf{O}. \tag{5.4.1}$$

For $\tilde{\mathcal{E}}^\sigma_X := \{\lambda \in \mathcal{H}_X \,;\, A(\lambda) = \exp(-\lambda + \tilde{f}_0)\}$, we have a natural identification $\tilde{\mathcal{E}}^\sigma_X \simeq \mathcal{E}^\sigma_X$ by sending each $\lambda \in \tilde{\mathcal{E}}^\sigma_X$ to $\omega_\lambda \in \mathcal{E}^\sigma_X$. Then the preimage, denoted by $\tilde{\mathbf{O}}$, of $\mathbf{O}$ under the identification $\tilde{\mathcal{E}}^\sigma_X \simeq \mathcal{E}^\sigma_X$ is written as

$$\tilde{\mathbf{O}} = \{\lambda \in C^{2,\alpha}(M)_{\mathbb{R}} \,;\, A(\lambda) = \exp(-\lambda + \tilde{f}_0) \text{ and } \omega_\lambda \in \mathbf{O}\}. \tag{5.4.2}$$

Moreover, we put $\mathbf{O}^\Gamma := \{\lambda \in \mathcal{H}^{2,\alpha}_{X,0} \,;\, \Gamma(\lambda, 1) = 0 \text{ and } \omega_\lambda \in \mathbf{O}\}$. Then $\mathbf{O}^\Gamma$, $\mathbf{O}$ and $\tilde{\mathbf{O}}$ are identified by

$$\mathbf{O}^\Gamma \simeq \mathbf{O} \simeq \tilde{\mathbf{O}}, \quad \lambda \leftrightarrow \omega_\lambda \leftrightarrow \lambda + \log\left\{\frac{1}{V_0}\int_M \exp(-\lambda + \tilde{f}_{\omega_0})\tilde{\omega}_0^n\right\}. \tag{5.4.3}$$

THEOREM 5.4. (a) *Assume that σ is convex. Then the function $\iota : \mathbf{O} \to \mathbb{R}$ is a proper map, and hence its absolute minimum is always attained at some point of the orbit $\mathbf{O}$.*

(b) *Let $\mathfrak{k}^\theta$ be as in* (A.5.3) *of Appendix* 5. *By* (5.4.3)*, to each $\theta \in \mathbf{O}$, we associate a unique $\lambda_\theta \in \tilde{\mathbf{O}}$ such that $\theta = \omega_{\lambda_\theta}$. Then the following are equivalent*:

(i) *θ is a critical point for ι*;

(ii) *$\int_M \lambda_\theta v \tilde{\theta}^n = 0$ for all $v \in \mathfrak{k}^\theta$.*

Proof of (a). For each positive real number r, we put $\mathbf{O}^\Gamma_r := \{\lambda \in \mathbf{O}^\Gamma \,;\, \iota(\omega_\lambda) \leq r\}$. By the same argument as in the proof of Lemma 5.3.2

(see the arguments after (5.3.3)), there exists a constant $C_5 = C_5(r) > 0$ independent of the choice of λ in $\mathbf{O}_r^\Gamma$ such that

$$\|\varphi\|_{C^{2,\alpha}(M)} \leq C_5$$

holds for all $\varphi \in \mathbf{O}_r^\Gamma$, where in this proof we use the inequality $\iota(\omega_\varphi) \leq r$ in place of (5.3.3). Now, (a) is straightforward. □

Proof of (b). Let $\lambda = \lambda(t)$, $-\varepsilon < t < \varepsilon$, be a smooth one-parameter family in $\tilde{\mathbf{O}}$ such that $\lambda(0) = \lambda_\theta$. Then $\omega_{\lambda(0)} = \theta$. In view of (A.1.1) in Appendix 1,

$$\begin{aligned} \text{(5.4.4)} \qquad \left\{\frac{d}{dt}\iota(\omega(t))\right\}_{|t=0} &= \int_M (\bar\partial\lambda(0), \bar\partial\dot\lambda(0))_\theta \tilde\theta^n \\ &= -\int_M \lambda(0)\big(\tilde\Box_\theta \dot\lambda(0)\big)\tilde\theta^n = \int_M \lambda(0)\dot\lambda(0)\tilde\theta^n, \end{aligned}$$

where we have $\dot\lambda(0) \in \mathfrak{k}^\theta$ $(= T_\theta(\tilde{\mathcal{E}}_X^\sigma) = T_\theta(\tilde{\mathbf{O}}))$ by (A.5.6) and (b) of Proposition A.5 of Appendix 5. The equivalence of (i) and (ii) is now immediate. □

We now consider the Hessian of $\iota : \mathbf{O} \to \mathbb{R}$ at a critical point $\theta = \omega_{\lambda_\theta} \in \mathbf{O}$ of ι, where $\lambda_\theta \in \tilde{\mathbf{O}}$ is as in (b) of Theorem 5.4. Let $\varphi_{s,t}$, $(s,t) \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$, be a smooth two-parameter family of functions in $\tilde{\mathbf{O}}$ such that $\lambda_\theta = \varphi_{0,0}$. Put $\omega_{s,t} := \omega_{\varphi_{s,t}}$. Then

$$\varphi' := \frac{\partial\varphi_{s,t}}{\partial s}_{|(s,t)=(0,0)} \quad \text{and} \quad \varphi'' := \frac{\partial\varphi_{s,t}}{\partial t}_{|(s,t)=(0,0)}$$

are regarded as elements in $T_\theta(\mathbf{O})$ $(= T_\theta(\mathcal{E}_X^\sigma))$ by the isomorphism $T_\theta(\mathcal{E}_X^\sigma) \cong \mathfrak{k}^\theta$ in (A.5.6) of Appendix 5. By differentiating $A(\varphi_{s,t}) = \exp(-\varphi_{s,t} + \tilde f_{\omega_0})$ with respect to t, we obtain

$$\text{(5.5.1)} \qquad \tilde\Box_{s,t}\left(\frac{\partial\varphi_{s,t}}{\partial t}\right) = -\frac{\partial\varphi_{s,t}}{\partial t},$$

where we put $\psi_{s,t} := \psi_{\omega_{s,t}}$, $u_{s,t} := u_{\omega_{s,t}}$, $\Box_{s,t} := \Box_{\omega_{s,t}}$, $\tilde\Box_{s,t} := \tilde\Box_{\omega_{s,t}}$ for simplicity. Differentiating (5.5.1) with respect to s at the origin $(s,t) = (0,0)$, we obtain

$$\text{(5.5.2)} \qquad (\partial\bar\partial\varphi', \partial\bar\partial\varphi'')_\theta - \ddot\sigma(u_\theta)(\bar X\varphi')(\bar X\varphi'') = (\tilde\Box_\theta + 1)\partial_s\partial_t\varphi(0).$$

Here, we used the identities $\tilde{\square}_{s,t} = \square_{s,t} + \sqrt{-1}\,\dot{\sigma}(u_{s,t})\bar{X}$, $u_{s,t} = u_{\omega_0} - \sqrt{-1}\,\bar{X}\varphi_{s,t}$ (see (1.3) and (2.5)) and we put

$$\partial_s\partial_t\varphi(0) := \left(\frac{\partial^2\varphi_{s,t}}{\partial s\partial t}\right)_{|(s,t)=(0,0)}.$$

Since $\tilde{\square}_\theta\varphi' = -\varphi'$, by comparing the identity (5.5.2) with (A.3.1) in Appendix 3 applied to $(\omega,\zeta,v) = (\theta,\varphi',\varphi'')$, we obtain

$$(\tilde{\square}_\theta + 1)(\partial\varphi',\partial\varphi'')_\theta = (\tilde{\square}_\theta + 1)\partial_s\partial_t\varphi(0). \tag{5.5.3}$$

Next, we put $\iota_{s,t} := \iota(\omega_{s,t})$ for simplicity. Then by the same computation as in (5.4.4), we obtain the identity

$$\frac{\partial\iota_{s,t}}{\partial t} = \int_M \varphi_{s,t}\frac{\partial\varphi_{s,t}}{\partial t}\tilde{\omega}_{s,t}^{\,n}.$$

In view of $\lambda_\theta = \varphi_{0,0}$ and (a) of Lemma 2.4, we further differentiate this with respect to s at the origin $(s,t) = (0,0)$. Then the Hessian $(\mathrm{Hess}\,\iota)_\theta$ of ι at θ is given by

$$\begin{aligned}(\mathrm{Hess}\,\iota)_\theta(\varphi',\varphi'') &= \frac{\partial^2\iota_{s,t}}{\partial s\partial t}_{|(s,t)=(0,0)} \\ &= \int_M \{\varphi'\varphi'' + \lambda_\theta\partial_s\partial_t\varphi(0) + \lambda_\theta\varphi''(\tilde{\square}_\theta\varphi')\}\tilde{\theta}^n \\ &= \int_M \{\varphi'\varphi''(1-\lambda_\theta) + \lambda_\theta\partial_s\partial_t\varphi(0)\}\tilde{\theta}^n.\end{aligned} \tag{5.5.4}$$

By (b) of Theorem 5.4 together with (A.5.3) of Appendix 5, we have an $X_{\mathbb{R}}$-invariant function $\xi \in C^\infty(M)_{\mathbb{R}}$ such that $\lambda_\theta = (\tilde{\square}_\theta + 1)\xi$. As in [BM, (6.7)], (5.5.4) is rewritten as

$$\begin{aligned}(\mathrm{Hess}\,\iota)_\theta(\varphi',\varphi'') &= \int_M \{\varphi'\varphi''(1-\lambda_\theta) + \xi(\tilde{\square}_\theta + 1)\partial_s\partial_t\varphi(0)\}\tilde{\theta}^n \\ &= \int_M \{\varphi'\varphi''(1-\lambda_\theta) + \xi(\tilde{\square}_\theta + 1)(\partial\varphi',\partial\varphi'')_\theta\}\tilde{\theta}^n \quad \text{(cf. (5.5.3))} \\ &= \int_M \varphi'\varphi''\tilde{\theta}^n + \frac{1}{2}\int_M \lambda_\theta\{(\tilde{\square}_\theta\varphi')\varphi'' + \varphi'(\tilde{\square}_\theta\varphi'')\}\tilde{\theta}^n \\ &\quad + \int_M \lambda_\theta(\partial\varphi',\partial\varphi'')_\theta\tilde{\theta}^n\end{aligned} \tag{5.5.5}$$

$$= \int_M \varphi'\varphi''\tilde{\theta}^n + \frac{1}{2}\int_M \lambda_\theta \tilde{\Box}_\theta(\varphi'\varphi'')\tilde{\theta}^n$$
$$= \int_M \varphi'\varphi''\Big(1 + \frac{1}{2}\tilde{\Box}_\theta\lambda_\theta\Big)\tilde{\theta}^n.$$

We now follows the arguments in [BM, Section 7]. Let $0 < t \le 1$ and $0 < \alpha < 1$. For each nonnegative integer k, let $C^{k,\alpha}_X(M)_{\mathbb{R}}$ be the space of all $X_{\mathbb{R}}$-invariant functions in $C^{k,\alpha}(M)_{\mathbb{R}}$, and consider the set $\mathcal{H}^{2,\alpha}_X$ of all $\varphi \in C^{2,\alpha}_X(M)_{\mathbb{R}}$ such that $\omega_\varphi := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$ is a positive definite $C^{0,\alpha}$ form on M. Put

$$(\mathfrak{k}^\theta_k)^\perp := \Big\{ w \in C^{k,\alpha}_X(M)_{\mathbb{R}}\ ;\ \int_M wv\tilde{\theta}^n = 0 \text{ for all } v \in \mathfrak{k}^\theta \Big\}.$$

We here observe that $\mathfrak{z}^\theta(X) = \mathfrak{k}^\theta_{\mathbb{C}}$ by Proposition A.5 in Appendix 5. In order to solve the equation $\Gamma(\varphi, t) = 0$ in $\varphi \in \mathcal{H}^{2,\alpha}_{X,0}$, it suffices to solve the following equation in $\gamma \in \mathcal{H}^{2,\alpha}_X$:

$$A(\gamma) = \exp(-t\gamma + \tilde{f}_{\omega_0}). \tag{5.5.6}$$

Because any solution $\gamma \in \mathcal{H}^{k,\alpha}_X$ of (5.5.6) allows us to obtain a solution $\varphi \in \mathcal{H}^{k,\alpha}_{X,0}$ of the equation $\Gamma(\varphi, t) = 0$ by setting $\varphi := \gamma - (1/V_0)\int_M \gamma\tilde{\omega}^n_0$. Next, we see that (5.5.6) is further reduced to the equation

$$\Phi(t, \gamma) = 0, \tag{5.5.7}$$

where $\Phi(t,\gamma) := t\gamma - \tilde{f}_{\omega_0} + \log A(\gamma)$. Note that $(\mathfrak{k}^\theta_2)^\perp \subset (\mathfrak{k}^\theta_0)^\perp$. Let $P : C^{0,\alpha}_X(M)_{\mathbb{R}}\ (\cong \mathfrak{k}^\theta \oplus (\mathfrak{k}^\theta_0)^\perp) \to \mathfrak{k}^\theta$ be the projection to the first factor. For each $\gamma \in \mathcal{H}^{2,\alpha}_X$, write

$$\gamma = \lambda_\theta + x + y,$$

with $x := P(\gamma - \lambda_\theta) \in \mathfrak{k}^\theta$ and $y := (1 - P)(\gamma - \lambda_\theta) \in (\mathfrak{k}^\theta_2)^\perp$. Now, the equation (5.5.7) is written in the form

$$P\Phi(t, \lambda_\theta + x + y) = 0 \quad \text{and} \quad \Psi(t, x, y) = 0,$$

where $\Psi : \mathbb{R} \times \mathfrak{k}^\theta \times (\mathfrak{k}^\theta_2)^\perp \to (\mathfrak{k}^\theta_0)^\perp$ is the mapping defined by

$$\Psi(t, x, y) := (1 - P)\Phi(t, \lambda_\theta + x + y), \quad (t, x, y) \in \mathbb{R} \times \mathfrak{k}^\theta \times (\mathfrak{k}^\theta_2)^\perp.$$

Then $\Psi(1, 0, 0) = 0$ and the Fréchet derivative $D_y\Psi_{|(1,0,0)}$ of Ψ with respect to y at $(t, x, y) = (1, 0, 0)$ is

$$(\mathfrak{k}^\theta_2)^\perp \ni y' \longmapsto D_y\Psi_{|(1,0,0)}(y') = (\tilde{\Box}_\theta + 1)y' \in (\mathfrak{k}^\theta_0)^\perp,$$

which is invertible. Hence, the implicit function theorem enables us to obtain a smooth mapping $V \ni (t,x) \mapsto y_{t,x} \in (\mathfrak{k}_2^\theta)^\perp$ of a small neighbourhood V of $(1,0)$ in $\mathbb{R} \times \mathfrak{k}^\theta$ to the Banach space $(\mathfrak{k}_2^\theta)^\perp$ such that

i) $y_{1,0} = 0$,
ii) $\|y_{t,x}\|_{C^{2,\alpha}} \leq \delta$ on V for some $\delta > 0$, and
iii) $\Psi(t,x,y) = 0$ (where $\|y\|_{C^{2,\alpha}} \leq \delta$) is, as an equation in $y \in (\mathfrak{k}_2^\theta)^\perp$, uniquely solvable in the form $y = y_{t,x}$ on U.

The derivative $(\partial/\partial t)y_{t,x}$ is denoted by $\dot{y}_{t,x}$ for simplicity. Then by differentiating the identity $\Psi(t,x,y_{t,x}) = 0$ at $(t,x) = (1,0)$, we obtain

$$\tag{5.5.8} \begin{cases} (\tilde{\Box}_\theta + 1)(\dot{y}_{t,x|(1,0)}) = -\lambda_\theta, \\ (D_x y_{t,x})_{|(1,0)}(\varphi') \;\;=\;\; 0 \qquad \text{for all } \varphi' \in \mathfrak{k}^\theta, \end{cases}$$

where $(D_x y_{t,x})_{|(1,0)} : \mathfrak{k}^\theta \to (\mathfrak{k}_2^\theta)^\perp$ denotes the Fréchet derivative of the smooth mapping $V \ni (t,x) \mapsto y_{t,x} \in (\mathfrak{k}_2^\theta)^\perp$ with respect to x at $(t,x) = (1,0)$. Then the equation (5.5.7), on a small neighbourhood of $(t,\gamma) = (1,\lambda_\theta)$, reduces to

$$\Phi_0(t,x) = 0 \quad (\text{with } \gamma = \lambda_\theta + x + y_{t,x}),$$

where we put $\Phi_0(t,x) := P\Phi(t, \lambda_\theta + x + y_{t,x})$ for $(t,x) \in V$. Since $\Phi(1,x) = 0$ for all $x \in \tilde{\mathbf{O}}$, we have $\Phi_0 = 0$ on $\{t = 1\}$, and hence the mapping

$$V_{|\{t\neq 1\}} \ni (t,x) \longmapsto \Phi_1(t,x) := \Phi_0(t,x)/(t-1) \in \mathfrak{k}^\theta$$

naturally extends to a smooth map, denoted by the same Φ_1, of V to $\mathfrak{k}^\theta$. In view of the first identity of (5.5.8), we obtain

$$\Phi_1(1,0) = (\partial\Phi_0/\partial t)(1,0) = 0.$$

Then the Fréchet derivative $D_x\Phi_{1|(1,0)} : \mathfrak{k}^\theta \to \mathfrak{k}^\theta$ of Φ_1 with respect to x at $(t,x) = (1,0)$ is given by the following:

THEOREM 5.5. *By using the notation in Section 2 on the left-hand side, we have*

$$\langle\!\langle D_x\Phi_{1|(1,0)}(\varphi'), \varphi'' \rangle\!\rangle_{\tilde{\theta}} = (\operatorname{Hess} \iota)_\theta(\varphi', \varphi''), \quad \varphi', \varphi'' \in \mathfrak{k}^\theta.$$

Proof. Since $P(\tilde{\Box}_\theta + 1) = 0$ on $(\mathfrak{k}_2^\theta)^\perp$, the latter identity of (5.5.8) above together with (1.3) and (2.5) implies

$$\begin{aligned} D_x\Phi_{1|(1,0)}(\varphi') &= \{D_x(\partial\Phi_0/\partial t)\}_{|(1,0)}(\varphi') \\ &= \varphi' - P(\partial\bar{\partial}\dot{y}_{t,x|(1,0)}, \partial\bar{\partial}\varphi')_\theta + P\{\ddot{\sigma}(u_\theta)(\bar{X}\varphi')\bar{X}\dot{y}_{t,x|(1,0)}\}. \end{aligned}$$

Moreover, we observe the first identity of (5.5.8). Then by (A.3.2) in Appendix 3 applied to $(\omega, v_1, v_2, \zeta) = (\theta, \varphi'', \varphi', \dot{y}_{t,x|(1,0)})$, we obtain

$$\begin{aligned} &\langle\!\langle D_x\Phi_{1|(1,0)}(\varphi'), \varphi''\rangle\!\rangle_{\tilde{\theta}} \\ &\quad = \int_M \big(\varphi' - P(\partial\bar{\partial}\dot{y}_{t,x|(1,0)}, \partial\bar{\partial}\varphi')_\theta + P\{\ddot{\sigma}(u_\theta)(\bar{X}\varphi')\bar{X}\dot{y}_{t,x|(1,0)}\}\big)\varphi''\tilde{\theta}^n \\ &\quad = \int_M \big(\varphi'\varphi'' - \varphi''(\partial\bar{\partial}\dot{y}_{t,x|(1,0)}, \partial\bar{\partial}\varphi')_\theta + \varphi''\{\ddot{\sigma}(u_\theta)(\bar{X}\varphi')\bar{X}\dot{y}_{t,x|(1,0)}\}\big)\tilde{\theta}^n \\ &\quad = \int_M \{\varphi'\varphi'' - \varphi''\varphi'\lambda_\theta + (\partial\varphi'', \partial\varphi')_\theta\lambda_\theta\}\tilde{\theta}^n \\ &\quad = \int_M \{\varphi'\varphi''(1 - \lambda_\theta) + (\partial\varphi', \partial\varphi'')_\theta\lambda_\theta\}\tilde{\theta}^n. \end{aligned}$$

This together with the second equality of (5.5.5) implies the required identity. □

Regarding ω_0 as a function in ε, we write

(5.5.1) $$\omega_0 = \omega_0(\varepsilon), \quad \varepsilon \in [0,1].$$

Hence, the corresponding $\omega_\varphi := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$, $\tilde{f}_{\omega_0}$, ι, $A(\varphi)$, $\Gamma(t,\gamma)$, μ^σ and $\mathcal{H}^{2,\alpha}_{X,0}$ will be written respectively as $\omega_\varphi(\varepsilon)$, $\tilde{f}_{\omega_0(\varepsilon)}$, ι_ε, $A_\varepsilon(\varphi)$, $\Gamma_\varepsilon(t,\gamma)$, μ^σ_ε and $\mathcal{H}^{2,\alpha}_{X,0}(\varepsilon)$. For ι_ε at $\varepsilon = 0$, we see by (a) of Theorem 5.4 that the functional $\iota_0 : \mathbf{O} \to \mathbb{R}$ takes its absolute minimum at some point $\theta \in \mathbf{O}$. Then we have a unique function $\lambda_{\theta;0} \in C^\infty(M)_\mathbb{R}$ such that $\theta = \omega_{\lambda_{\theta;0}}(0)$ and that $A_0(\lambda_{\theta;0}) = \exp(-\lambda_{\theta;0} + \tilde{f}_{\omega_0(0)})$. Then by (b) of Theorem 5.4,

(5.5.2) $$\int_M \lambda_{\theta;0} v\tilde{\theta}^n = 0 \quad \text{for all } v \in \mathfrak{k}^\theta,$$

and the bilinear form $(\mathrm{Hess}\,\iota_0)_\theta : \mathfrak{k}^\theta \times \mathfrak{k}^\theta \to \mathbb{R}$ is positive semidefinite. Let us now perturb $\omega_0(0)$ by setting

(5.5.3) $$\omega_0(\varepsilon) := (1-\varepsilon)\omega_0(0) + \varepsilon\theta = \omega_0(0) + \sqrt{-1}\,\partial\bar{\partial}(\varepsilon\lambda_{\theta;0}), \quad 0 \le \varepsilon \le 1.$$

Let $\lambda_{\theta;\varepsilon} \in C^\infty(M)_{\mathbb{R}}$ be the unique function satisfying $\theta = \omega_{\lambda_{\theta;\varepsilon}}(\varepsilon)$ and $A_\varepsilon(\lambda_{\theta;\varepsilon}) = -\lambda_{\theta;\varepsilon} + \tilde{f}_{\omega_0(\varepsilon)}$. By $\omega_{\lambda_{\theta;0}}(0) = \theta = \omega_{\lambda_{\theta;\varepsilon}}(\varepsilon) = \omega_0(0) + \sqrt{-1}\,\partial\bar{\partial}(\varepsilon\lambda_{\theta;0}) + \sqrt{-1}\,\partial\bar{\partial}\lambda_{\theta;\varepsilon}$, we have

$$\lambda_{\theta;\varepsilon} = (1-\varepsilon)\lambda_{\theta;0} + C_\varepsilon \quad \text{for some } C_\varepsilon \in \mathbb{R}. \tag{5.5.4}$$

Since $\int_M v\tilde{\theta}^n = 0$ for all $v \in \mathfrak{k}^\theta$, (5.5.2) and (5.5.4) aboved imply $\int_M \lambda_{\theta;\varepsilon} v\tilde{\theta}^n = 0$ for all $v \in \mathfrak{k}^\theta$. Hence by (b) of Theorem 5.4, it follows that

$$\theta \text{ is a critical point for } \iota_\varepsilon : \mathbf{O} \to \mathbb{R}. \tag{5.5.5}$$

Let $0 < \varepsilon \ll 1$. For all $0 \neq v \in \mathfrak{k}^\theta$,

$$\begin{aligned}
(\operatorname{Hess}\iota_\varepsilon)_\theta(v,v) &= \int_M v^2\left(1 + \frac{1}{2}\tilde{\Box}_\theta\lambda_{\theta;\varepsilon}\right)\tilde{\theta}^n \qquad &\text{(cf. (5.5.5))}\\
&= (1-\varepsilon)\int_M v^2\left(1 + \frac{1}{2}\tilde{\Box}_\theta\lambda_{\theta;0}\right)\tilde{\theta}^n + \varepsilon\int_M v^2\tilde{\theta}^n \qquad &\text{(cf. (5.5.4))}\\
&= (1-\varepsilon)(\operatorname{Hess}\iota_0)_\theta(v,v) + \varepsilon\int_M v^2\tilde{\theta}^n > 0.
\end{aligned}$$

Then for such a $\omega_0 = \omega_0(\varepsilon)$ with ε fixed, Theorem 5.5 shows that $D_x\Phi_{1|(1,0)} : \mathfrak{k}^\theta \to \mathfrak{k}^\theta$ is invertible. Now by the implicit function theorem, the equation $\Phi_1(t,x) = 0$ in $x \in \mathfrak{k}^\theta$ is uniquely solvable in a small neighbourhood of $(t,x) = (1,0)$ to produce a smooth curve $x(t)$, $1-\delta \le t \le 1$, in $\mathfrak{k}^\theta$ for some $0 < \delta \ll 1$ such that

$$x(1) = 0 \quad \text{and} \quad \Phi_1(t, x(t)) = 0 \;\; (1-\delta \le t \le 1).$$

Replacing $\delta > 0$ by a smaller number if necessary, we obtain $\Phi(t, \lambda_{\theta;\varepsilon} + x(t) + y_{t,x(t)}) = 0$ for $1-\delta \le t \le 1$. In view of the reduction to (5.5.6) and (5.5.7), we obtain

THEOREM 5.6. *For each $Z^0(X)$-orbit $\mathbf{O}$ in $\mathcal{E}_X^\sigma$, let θ be a point on $\mathbf{O}$ at which ι in (5.4.1) takes its absolute minimum. Then replacing ω_0 by $(1-\varepsilon)\omega_0 + \varepsilon\theta$ for some $0 < \varepsilon \ll 1$, we have a $0 < \delta \ll 1$ such that there exists a smooth one-parameter family of functions $\{\varphi_t \,;\, 1-\delta \le t \le 1\}$ in $\mathcal{H}^{2,\alpha}_{X,0}$ satisfying $\omega_{\varphi_1} = \theta$ and $\Gamma(t,\varphi_t) = 0$ for all $t \in [1-\delta, 1]$.*

Proof of Theorem C. Let $\mathbf{O}'$ and $\mathbf{O}''$ be $Z^0(X)$-orbits in $\mathcal{E}^\sigma_X$. We consider the nonnegative function $\iota : \mathcal{K}_X \to \mathbb{R}$ defined by

$$\iota(\omega) := (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0, \omega), \quad \omega \in \mathcal{K}_X.$$

The restrictions of ι to $\mathbf{O}'$ and $\mathbf{O}''$ are denoted by $\iota' : \mathbf{O}' \to \mathbb{R}$ and $\iota'' : \mathbf{O}'' \to \mathbb{R}$, respectively. We follow the arguments in [BM, (8.2)]. The proof is divided into three steps.

Step 1. In view of Theorem 5.6, by perturbing ω_0 if necessary, we may assume that the function ι' is critical at some $\theta' \in \mathbf{O}'$ with positive definite Hessian. Next by (a) of Theorem 5.4, the function ι'' takes its absolute minimum at some point $\theta'' \in \mathbf{O}''$. For $0 < \varepsilon \ll 1$, we define a nonnegative function ι_ε on $\mathcal{K}_X$ by

$$\iota_\varepsilon(\omega) := (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega_0(\varepsilon), \omega), \quad \omega \in \mathcal{K}_X.$$

Let $\iota'_\varepsilon : \mathbf{O}' \to \mathbb{R}$ and $\iota''_\varepsilon : \mathbf{O}'' \to \mathbb{R}$ be the restrictions of the function ι_ε to $\mathbf{O}'$ and $\mathbf{O}''$, respectively. Put $\omega_0(\varepsilon) := (1-\varepsilon)\omega_0 + \varepsilon\theta''$. Then by (5.5.5), the function ι''_ε is critical at θ'' with positive definite Hessian. Moreover, by $\varepsilon \ll 1$, the restriction ι'_ε takes its local minimum with positive definite Hessian at some point θ'_ε of $\mathbf{O}'$ near θ'. Hence, replacing ω_0 by $\omega_0(\varepsilon)$, we may assume from the begining that both $\iota' : \mathbf{O}' \to \mathbb{R}$ and $\iota'' : \mathbf{O}'' \to \mathbb{R}$ have critical points with positive definite Hessian. Therefore by Theorem 5.6, for some $0 < \delta \ll 1$, we have smooth one-parameter families of functions $\{\varphi'_t\,;\, 1-\delta \le t \le 1\}$ and $\{\varphi''_t\,;\, 1-\delta \le t \le 1\}$ in $\mathcal{H}^{2,\alpha}_{X,0}$ satisfying the following conditions:

$$\Gamma(t, \varphi'_t) = \Gamma(t, \varphi''_t) = 0 \quad \text{for all } t \in [1-\delta, 1]; \tag{5.7.1}$$

$$\lim_{t\to 1} \omega_{\varphi'_t} = \omega_{\varphi'_1} \in \mathbf{O}' \quad \text{and} \quad \lim_{t\to 1} \omega_{\varphi''_t} = \omega_{\varphi''_1} \in \mathbf{O}''. \tag{5.7.2}$$

Then by Theorem 5.3, these extend to smooth one-parameter families of functions $\{\varphi'_t\,;\, 0 \le t \le 1\}$ and $\{\varphi''_t\,;\, 0 \le t \le 1\}$ in $\mathcal{H}^{2,\alpha}_{X,0}$ satisfying the equalities in (5.7.1) for all $t \in [0,1]$.

Step 2. Appendix 4 shows that $\varphi_0 \in \mathcal{H}^{2,\alpha}_{X,0}$ satisfying the equation $\Gamma(\varphi_0, 0) = 0$ is unique. Hence, by Theorem 5.3 together with Step 1, the local uniqueness in Theorem 5.1 implies the uniqueness of a smooth one-parameter family of functions

$$\{\varphi_t\,;\, 0 \le t < 1\}$$

in $\mathcal{H}^{2,\alpha}_{X,0}$ satisfying $\Gamma(\varphi_t, t) = 0$ for all $0 \leq t < 1$. In particular, we obtain $\varphi'_t = \varphi''_t$ for all $0 \leq t < 1$. This together with (5.7.2) implies $\mathbf{O}' = \mathbf{O}''$, as required. □

§6. Corollaries of Theorem C

Throughout this section, we assume that σ is convex. Let $\mu^\sigma : \mathcal{K}_X \to \mathbb{R}$ be the function defined in Appendix 2. Then by the arguments in [BM] and [Ba], we obtain the following corollaries of Theorem C:

COROLLARY D. *If $\mathcal{E}^\sigma_X \neq \emptyset$, then the function $\mu^\sigma : \mathcal{K}_X \to \mathbb{R}$ takes its absolute minimum exactly on $\mathcal{E}^\sigma_X$.*

COROLLARY E. *If $\mathcal{E}^\sigma_X \neq \emptyset$, then for any, possibly non-connected, compact subgroup H of $Z(X)$, there exists an H-invariant metric ω in $\mathcal{E}^\sigma_X$.*

Proof of Corollary D. For an arbitrary element η of $\mathcal{K}_X$, we have a unique element η' of $\mathcal{K}_X$ such that $\eta = \mathrm{Ric}^\sigma(\eta')$ (see for instance [M4] and Appendix 4). Put

$$\omega_0(0) = \eta$$

by the notation in (5.5.1). Choosing a $Z^0(X)$-orbit $\mathbf{O}$ in $\mathcal{E}^\sigma_X$, let θ be a point at which $\iota : \mathbf{O} \to \mathbb{R}$ in (5.4.1) takes its absolute minimum. For $0 < \varepsilon \ll 1$, we perturb $\eta = \omega_0(0)$ by

$$\omega_0(\varepsilon) := (1-\varepsilon)\eta + \varepsilon\theta$$

as in (5.5.3). Then by Theorem 5.3 together with Theorem 5.6, we have a smooth one-parameter family of functions $\{\varphi_{t;\varepsilon}\ ;\ 0 \leq t \leq 1\}$ in $\mathcal{H}^{2,\alpha}_{X,0}(\varepsilon)$ satisfying

$$\omega(1;\varepsilon) = \theta \quad \text{and} \quad \Gamma_\varepsilon(t, \varphi_{t;\varepsilon}) = 0, \quad 0 \leq t \leq 1,$$

where Γ_ε and $\mathcal{H}^{2,\alpha}_{X,0}(\varepsilon)$ are as in the arguments immediately after (5.5.1), and for simplicity we put $\omega(t;\varepsilon) := \omega_{\varphi_{t,\varepsilon}}$ for all $0 \leq t \leq 1$. Now by Theorem 5.2,

$$M^\sigma(\omega(0;\varepsilon), \theta) \leq 0, \tag{6.1}$$

where M^σ is as in Appendix 2. We next observe that $\mathrm{Ric}^\sigma(\eta') = \eta = \omega_0(0)$, and that $\mathrm{Ric}^\sigma(\omega(0;\varepsilon)) = \omega_0(\varepsilon)$. Let $\varepsilon \to 0$. Since $\omega_0(\varepsilon) \to \omega_0(0)$ in $C^{0,\alpha}$, it follows that $\omega(0;\varepsilon) \to \eta'$ in $C^{2,\alpha}$. Hence, (6.1) implies

$$M^\sigma(\eta', \theta) \leq 0, \ \text{ i.e., } \ B_\sigma \leq \mu^\sigma(\eta') \quad \text{for all } \eta \in \mathcal{K}_X, \tag{6.2}$$

where we put $B_\sigma := \mu^\sigma(\theta)$. On the other hand, by Theorem C and (a) of Proposition A.2 in Appendix 2, the function μ^σ takes a constant value B_σ on $\mathcal{E}_X^\sigma$. Then by Lemma 6.3 below, we have the inequality $B_\sigma \le \mu^\sigma(\eta') \le \mu^\sigma(\eta)$, and the equality $B_\sigma = \mu^\sigma(\eta)$ holds if and only if $\eta \in \mathcal{E}_X^\sigma$, as required. □

LEMMA 6.3. (cf. [Ba] for Kähler-Einstein cases) *For each $\omega \in \mathcal{K}_X$, let ω' be the element of $\mathcal{K}_X$ such that $\mathrm{Ric}^\sigma(\omega') = \omega$. Then the inequality $\mu^\sigma(\omega') \le \mu^\sigma(\omega)$ holds, and the equality $\mu^\sigma(\omega') = \mu^\sigma(\omega)$ holds if and only if $\omega' = \omega$, i.e., $\omega \in \mathcal{E}_X^\sigma$.*

Proof. Put $\omega_0 := \omega$. For $c_t := \log V_0 - \log\{\int_M \exp(t\tilde{f}_{\omega_0})\tilde{\omega}_0^n\}$, let $\varphi_t \in \mathcal{H}_{X,0}^{2,\alpha}$ denote the solution (see for instance [M4]) of the equation:

$$A(\varphi_t) = \exp(t\tilde{f}_{\omega_0} + c_t), \quad 0 \le t \le 1. \tag{6.4}$$

For simplicity, we put $\omega(t) := \omega_{\varphi_t}$ and $\tilde{\Box}_t := \tilde{\Box}_{\omega(t)}$. Then $\omega(0) = \omega_0 = \omega$. Differentiating (6.4) with respect to t, we obtain $\tilde{\Box}_t\dot{\varphi}_t = \tilde{f}_{\omega_0} + \dot{c}_t$. Next by taking $\bar{\partial}\partial$ of both sides of (6.4), we see that $\mathrm{Ric}^\sigma(\omega(t)) - \omega(t) = \sqrt{-1}\,\partial\bar{\partial}\{(1-t)\tilde{f}_{\omega_0} - \varphi_t\}$. Therefore,

$$\begin{aligned}
\frac{d}{dt}\mu^\sigma(\omega(t)) &= -\int_M \dot{\varphi}_t\tilde{\Box}_t\{(1-t)\tilde{f}_{\omega_0} - \varphi_t\}\tilde{\omega}(t)^n \\
&= -(1-t)\int_M (\tilde{\Box}_t\dot{\varphi}_t)^2\tilde{\omega}(t)^n + \int_M \dot{\varphi}_t(\tilde{\Box}_t\varphi_t)\tilde{\omega}(t)^n \\
&\le -\frac{d}{dt}\{(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega(0), \omega(t))\},
\end{aligned}$$

where $\tilde{\omega}(t)$ is as in (2.3). Thus, by $\omega(0) = \omega$ and $\omega(1) = \omega'$ (cf. Appendix 4), we obtain $\mu^\sigma(\omega') - \mu^\sigma(\omega) \le -(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega, \omega') \le 0$, and $\mu^\sigma(\omega') = \mu^\sigma(\omega)$ if and only if $\omega' = \omega$. □

We consider an arbitrary smooth path $\Lambda = \{\omega_{\lambda_t}\,;\, a \le t \le b\}$ sitting in $\mathcal{E}_X^\sigma$, where $\{\lambda_t\,;\, a \le t \le b\}$ is the corresponding smooth path in $C^\infty(M)_{\mathbb{R}}$ such that $\int_M \dot{\lambda}_t\tilde{\omega}_{\lambda_t}^n = 0$ for all t. Then the length $\mathcal{L}(\Lambda)$ of the path Λ in $\mathcal{E}_X^\sigma$ is defined by

$$\mathcal{L}(\Lambda) := \int_a^b \left(\int_M \dot{\lambda}_t^2\tilde{\omega}_{\lambda_t}^n\right)^{1/2} dt.$$

This naturally defines a Riemannian metric on $\mathcal{E}_X^\sigma$. Let $\theta \in \mathcal{E}_X^\sigma$. Then by the notation in Appendix 5, the identity component $Z^0(X)$ of $Z(X)$ (see also Section 1) is nothing but the complexification $K_{\mathbb{C}}$ of K in G (cf. (a) of Proposition A.5). Then we have:

Proposition 6.5. *If $\mathcal{E}_X^\sigma \neq \emptyset$, then $Z(X)$ acts isometrically on $\mathcal{E}_X^\sigma$, and in particular, $\mathcal{E}_X^\sigma$ is isometric to the Riemannian symmetric space $K_{\mathbb{C}}/K$ endowed with a suitable metric.*

Proof. Note that $\mathcal{E}_X^\sigma \cong Z^0(X)/K = K^{\mathbb{C}}/K$ by Theorem C. Then it suffices to show that $Z(M)$ acts isometrically on $\mathcal{E}_X^\sigma$. Let $g \in Z(M)$, and we can write $g^*\omega_0 = \omega_{\varphi_g}$ for some $\varphi_g \in C^\infty(M)_{\mathbb{R}}$. For a smooth path Λ in $\mathcal{E}_X^\sigma$ as above, we have $g^*\omega_{\lambda_t} = \omega_{\xi_t}$ for all t, where $\xi_t := \varphi_g + g^*\lambda_t$. In view of $g^*\tilde{\omega}_{\lambda_t} = \tilde{\omega}_{\xi_t}$, we obtain

$$\mathcal{L}(g^*\Lambda) = \int_a^b \left(\int_M \dot{\xi}_t^2 \tilde{\omega}_{\xi_t}^n \right)^{1/2} dt = \int_a^b \left(\int_M g^*\dot{\lambda}_t^2 g^*\tilde{\omega}_{\lambda_t}^n \right)^{1/2} dt = \mathcal{L}(\Lambda),$$

as required. □

Proof of Corollary E. We follow the arguments in [BM]. By Proposition 6.5, $\mathcal{E}_X^\sigma$ is isometric to the Riemannian symmetric space $K^{\mathbb{C}}/K$ without compact factors. Hence, $\mathcal{E}_X^\sigma$ is a simply connected Riemannian manifold with nonpositive sectional curvature. Since the compact group H acts isometrically on $\mathcal{E}_X^\sigma$, the action has a fixed point in $\mathcal{E}_X^\sigma$, as required. □

Appendix 1. Inequalities between Aubin's functionals

For $\sigma \in C^\infty(I_X)_{\mathbb{R}}$ as in the introduction, the purpose of this appendix is to establish inequalities between multiplier Hermitian analogues $\mathcal{I}^\sigma : \mathcal{K}_X \times \mathcal{K}_X \to \mathbb{R}$ and $\mathcal{J}^\sigma : \mathcal{K}_X \times \mathcal{K}_X \to \mathbb{R}$ of Aubin's functionals (cf. [A1], [BM], [T1]). Let ω', $\omega'' \in \mathcal{K}_X$. In view of (1.1), we can write $\omega' := \omega_{\varphi'}$ and $\omega'' := \omega_{\varphi''}$ for some φ', $\varphi'' \in \mathcal{H}_X$. Then by using the notation in (1.4), we define $\mathcal{I}^\sigma$ and the difference $\mathcal{I}^\sigma - \mathcal{J}^\sigma$ by

$$\text{(A.1.1)} \qquad \begin{cases} \mathcal{I}^\sigma(\omega', \omega'') := \displaystyle\int_M (\varphi'' - \varphi')\{(\tilde{\omega}')^n - (\tilde{\omega}'')^n\}, \\ (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'') := \displaystyle\int_a^b \left\{ \int_M (\bar{\partial}\varphi_t, \bar{\partial}\dot{\varphi}_t)_{\omega(t)} \tilde{\omega}(t)^n \right\} dt, \end{cases}$$

where $\phi := \{\varphi_t \,;\, a \leq t \leq b\}$ is an arbitrary smooth path in $\mathcal{H}_X$ satisfying the equalities $\varphi_a = 0$, $\varphi_b = \varphi'' - \varphi'$ and $\omega(t) = \omega' + \sqrt{-1}\,\partial\bar{\partial}\varphi_t$ for all t with $a \leq t \leq b$.

Claim. *$(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'')$ defined in the second line of* (A.1.1) *depends only on (ω', ω''), and is independent of the choice of the path ϕ.*

Proof. In view of (a) of Lemma 2.4 and the first line of (A.1.1), by using the notation in (2.3), we obtain

$$\text{(A.1.2)} \quad \frac{d}{dt}\mathcal{I}^\sigma(\omega', \omega(t)) = \int_M \dot{\varphi}_t\{(\tilde{\omega}')^n - \tilde{\omega}(t)^n\} + \int_M (\bar{\partial}\varphi_t, \bar{\partial}\dot{\varphi}_t)_{\omega(t)}\tilde{\omega}(t)^n,$$

Hence, it suffices to show that the integral $\int_a^b \big(\int_M \dot{\varphi}_t\{(\tilde{\omega}')^n - \tilde{\omega}(t)^n\}\big)\, dt$ is independent of the choice of the path ϕ above. Let

$$[0,1] \times [a,b] \ni (s,t) \longmapsto \varphi_{s,t} \in C^\infty(M)_{\mathbb{R}}$$

be a smooth 2-parameter family of functions in $C^\infty(M)_{\mathbb{R}}$ such that $\omega_{\varphi_{s,t}} \in \mathcal{K}_X$ for all (s,t). For such a family $\varphi = \varphi_{s,t}$ of functions, we consider the 1-form

$$\Theta := \left(\int_M \frac{\partial \varphi}{\partial s}\{(\tilde{\omega}')^n - \tilde{\omega}_\varphi^n\}\right) ds + \left(\int_M \frac{\partial \varphi}{\partial t}\{(\tilde{\omega}')^n - \tilde{\omega}_\varphi^n\}\right) dt$$

on $[0,1] \times [a,b]$. In view of (2.2) and (2.5),

$$\begin{aligned} d\Theta &= ds \wedge dt \int_M \left\{ \frac{\partial \varphi}{\partial s}\frac{\partial}{\partial t}(\tilde{\omega}_\varphi^n) - \frac{\partial \varphi}{\partial t}\frac{\partial}{\partial s}(\tilde{\omega}_\varphi^n) \right\} \\ &= ds \wedge dt \int_M \left\{ \frac{\partial \varphi}{\partial s}\Big(\tilde{\square}_{\omega_\varphi}\frac{\partial \varphi}{\partial t}\Big) - \frac{\partial \varphi}{\partial t}\Big(\tilde{\square}_{\omega_\varphi}\frac{\partial \varphi}{\partial s}\Big) \right\}\tilde{\omega}_\varphi^n = 0, \end{aligned}$$

and this implies the required independence. ∎

Next, take the infinitesimal form of the second line of (A.1.1) with respect to t, and subtract it from (A.1.2). Then by integration,

$$\text{(A.1.3)} \qquad \mathcal{J}^\sigma(\omega', \omega'') = \int_a^b \left(\int_M \dot{\varphi}_t\{(\tilde{\omega}')^n - \tilde{\omega}(t)^n\}\right) dt$$

for $\omega(t)$ and ϕ as above. In (A.1.1) and (A.1.3), we choose ϕ such that $\varphi_t := t\hat{\varphi}$, $0 \le t \le 1$, where $a = 0$, $b = 1$ and $\hat{\varphi} := \varphi'' - \varphi'$. Then

$$\text{(A.1.4)} \quad \begin{cases} \mathcal{I}^\sigma(\omega', \omega'') = f(1), \quad \mathcal{J}^\sigma(\omega', \omega'') = \displaystyle\int_0^1 f(t)\, dt, \\ (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'') = \displaystyle\int_0^1 \{f(1) - f(t)\}\, dt, \\ (\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'') = \displaystyle\int_0^1 \left\{\int_M t(\bar{\partial}\hat{\varphi}, \bar{\partial}\hat{\varphi})_{\omega(t)}\tilde{\omega}(t)^n\right\} dt \ge 0, \end{cases}$$

where $f = f(t)$ is defined by

$$f(t) := \int_M \hat{\varphi}\{(\tilde{\omega}')^n - \tilde{\omega}(t)^n\} = t^{-1}\mathcal{I}^\sigma(\omega', \omega(t))$$
$$= t^{-1}\mathcal{I}^\sigma(\omega', \omega' + t(\omega'' - \omega')).$$

In the last inequality of (A.1.4), we easily see that $(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'') = 0$ if and only if ω' coincides with ω''. Let k be a nonnegative real number. Replacing $\sigma \in C^\infty(I_X)_{\mathbb{R}}$ by $k\sigma \in C^\infty(I_X)_{\mathbb{R}}$, we have functionals $\mathcal{J}^{k\sigma} : \mathcal{K}_X \times \mathcal{K}_X \to \mathbb{R}$ and $\mathcal{I}^{k\sigma} : \mathcal{K}_X \times \mathcal{K}_X \to \mathbb{R}$. For instance, if $k = 0$, then $\mathcal{I}^{k\sigma}$ and $\mathcal{J}^{k\sigma}$ are nothing but the restriction to $\mathcal{K}_X \times \mathcal{K}_X$ of the ordinary Aubin's functional $\mathcal{I}$ and $\mathcal{J}$. Put $c := \max_{s \in I_X} |\sigma(s)|$ as in the introduction. Then by the last line of (A.1.4), we can easily compare $\mathcal{I}^{k\sigma} - \mathcal{J}^{k\sigma}$ and $\mathcal{I}^\sigma - \mathcal{J}^\sigma$ as follows:

LEMMA A.1.5. *For all $\omega', \omega'' \in \mathcal{K}_X$, using the notation in* (1.2), *we have the inequalities $e^{-|k-1|c}(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'') \leq (\mathcal{I}^{k\sigma} - \mathcal{J}^{k\sigma})(\omega', \omega'') \leq e^{|k-1|c}(\mathcal{I}^\sigma - \mathcal{J}^\sigma)(\omega', \omega'')$.*

Put $b_\sigma := (\beta_X - \alpha_X) \max_{s \in I_X} |\dot{\sigma}(s)| > 0$. To each positive real number $m > 0$, we associate a function $q_m = q_m(t)$ on the closed interval $[0, 1]$ by setting

$$q_m(t) := 1 - (1-t)^{m+1}, \quad 0 \leq t \leq 1.$$

LEMMA A.1.6. *If $m := n - 1 + b_\sigma$, then $f(t) \leq f(1)q_m(t)$ for all $0 \leq t \leq 1$,*

Proof. We may assume that $\hat{\varphi}$ is nonconstant. For $\omega(t) = \omega' + t\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}$, we write the function $\psi_{\omega(t)}$ just as $\psi(t)$ for simplicity. By differentiation, the definition of $f(t)$ yields

$$\dot{f}(t) = -\int_M \hat{\varphi}(\tilde{\Box}_{\omega(t)}\,\hat{\varphi})\tilde{\omega}(t)^n = \int_M (\bar{\partial}\hat{\varphi}, \bar{\partial}\hat{\varphi})_{\omega(t)}\tilde{\omega}(t)^n$$
$$= n\sqrt{-1}\int_M (\partial\hat{\varphi} \wedge \bar{\partial}\hat{\varphi})e^{-\psi(t)}\omega(t)^{n-1} > 0,$$

and by $f(0) = 0$, we have $f(t) > 0$ for all $0 < t \leq 1$. Differentiate the equality just above with respect to t. Then by $u_{\omega(t)} = u_{\omega'} + t\sqrt{-1}\,X\hat{\varphi}$ and $\dot{\psi}(t) = \sqrt{-1}\,\dot{\sigma}(u_\omega)X\hat{\varphi}$,

$$\ddot{f}(t) = n\sqrt{-1}\int_M \partial\hat{\varphi} \wedge \bar{\partial}\hat{\varphi}\,\{-\omega(t)\dot{\psi}(t) + (n-1)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}\}e^{-\psi(t)}\omega(t)^{n-2}$$

$$= n\sqrt{-1}\int_M \partial\hat{\varphi}\wedge\bar{\partial}\hat{\varphi}$$
$$\wedge\left\{-\sqrt{-1}\,\omega(t)\dot{\sigma}(u_{\omega(t)})X\hat{\varphi}+(n-1)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}\right\}e^{-\psi(t)}\omega(t)^{n-2}.$$

Now by $\max_M|X\hat{\varphi}|\leq\max_M|u_{\omega(1)}-u_{\omega(0)}|\leq\beta_X-\alpha_X$, we have

$$\max_M|\dot{\sigma}(u_{\omega(t)})X\hat{\varphi}|\leq b_\sigma$$

for all $0\leq t\leq 1$. By $(1-t)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}+\omega(t)=\omega''>0$, we further obtain

$$(1-t)\left\{-\sqrt{-1}\,\omega(t)\dot{\sigma}(u_{\omega(t)})X\hat{\varphi}+(n-1)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}\right\}+m\omega(t)>0$$

for all $0\leq t\leq 1$. Hence,

$$(1-t)\ddot{f}(t)+m\dot{f}(t)>0,\quad 0\leq t\leq 1.$$

This implies $(d/dt)(\log\dot{f}(t))>-m/(1-t)=(d/dt)(\log\dot{q}(t))$ for $0\leq t<1$, where we put $q(t):=f(1)q_m(t)$ for simplicity. Hence, $\dot{f}(t)/\dot{q}(t)$ is monotone increasing for $0\leq t<1$, while we have both $\dot{f}(1)>0=\dot{q}(1)$ and $f(1)=q(1)$. Therefore, if there were $t_0\in(0,1)$ such that $f(t_0)=q(t_0)$, then in view of the behaviour of the curve $\{(f(t),q(t))\,;\,0\leq t\leq 1\}$, it would follow that $\dot{f}(t_0)<\dot{q}(t_0)$ in contradiction to $f(0)=0=q(0)$. We now conclude that $f(t)\leq q(t)$ for all $0\leq t\leq 1$, as required. □

Remark A.1.7. If $\sigma(s)=-\log(s+C)$, $s\in I_X$, for some real constant $C>-\alpha_X$, then we obtain $f(t)\leq f(1)q_n(t)$ for all $0\leq t\leq 1$ as follows: For such a function σ, we have

$$e^{-\psi_{\omega(t)}}=u_{\omega'}+t\sqrt{-1}\,X\hat{\varphi}+C\quad\text{and}\quad -\dot{\sigma}(u_{\omega(t)})e^{-\psi_{\omega(t)}}=1,$$

and $-(1-t)\sqrt{-1}\,\dot{\sigma}(u_{\omega(t)})e^{-\psi_{\omega(t)}}X\hat{\varphi}+e^{-\psi_{\omega(t)}}=u_{\omega'}+\sqrt{-1}\,X\hat{\varphi}+C=e^{-\psi_{\omega''}}>0$ follows. Hence, in view of $(1-t)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}+\omega(t)=\omega''>0$, we obtain

$$(1-t)\left\{-\sqrt{-1}\,\omega(t)\dot{\sigma}(u_{\omega(t)})X\hat{\varphi}+(n-1)\sqrt{-1}\,\partial\bar{\partial}\hat{\varphi}\right\}+n\omega(t)>0.$$

Then $(1-t)\ddot{f}(t)+n\dot{f}(t)>0$ for all $0\leq t\leq 1$. Finally, the same argument as in the above proof of Lemma A.1.6 yields the required inequality.

In the definition of $f(t)$, since $\omega(1)=\omega''$, we obtain

$$f(1)-f(t)=\int_M(-\hat{\varphi})\{(\tilde{\omega}'')^n-\tilde{\omega}(t)^n\},$$

where $\omega(t) = \omega'' + (1-t)\,\partial\bar{\partial}\,(-\hat{\varphi})$. Replace $1-t$ by t. Then by (A.1.3), the right-hand side of the middle line of (A.1.4) is regarded as $\mathcal{J}^\sigma(\omega'',\omega')$. Hence,

$$\text{(A.1.8)}\quad \mathcal{J}^\sigma(\omega',\omega'')+\mathcal{J}^\sigma(\omega'',\omega') = \mathcal{I}^\sigma(\omega',\omega'') = \mathcal{I}^\sigma(\omega'',\omega'), \quad \omega',\omega''\in\mathcal{K}_X.$$

By Lemma A.1.6, we have $f(1)-f(t)\geq f(1)(1-q_m(t))$ for all $0\leq t\leq 1$. Integrating this inequality over [0,1], we see that

$$\begin{aligned}(\mathcal{I}^\sigma-\mathcal{J}^\sigma)(\omega',\omega'') &\geq f(1)\int_0^1 (1-q_m(t))\,dt\\ &= (m+2)^{-1}f(1) = (m+2)^{-1}\mathcal{I}^\sigma(\omega',\omega'').\end{aligned}$$

Hence, by (A.1.8), we obtain the following fundamental inequalities between the multiplier Hermitian analogues of Aubin's functionals:

Proposition A.1. *$0\leq\mathcal{I}^\sigma(\omega',\omega'')\leq(m+2)(\mathcal{I}^\sigma-\mathcal{J}^\sigma)(\omega',\omega'')\leq(m+1)\mathcal{I}^\sigma(\omega',\omega'')$ for all ω', $\omega''\in\mathcal{K}_X$, where $m:=n-1+b_\sigma$.*

Remark A.1.9. Suppose that $\sigma(s)=-\log(s+C)$, $s\in I_X$, for some real constant $C>-\alpha_X$. Then by Remark A.1.7, we can improve the estimate as follows:

$$0\leq\mathcal{I}^\sigma(\omega',\omega'')\leq(n+2)(\mathcal{I}^\sigma-\mathcal{J}^\sigma)(\omega',\omega'')\leq(n+1)\mathcal{I}^\sigma(\omega',\omega'').$$

Appendix 2. K-energy maps for multiplier Hermitian metrics

In this appendix, we shall define a multiplier Hermitian analogue $\mu^\sigma:\mathcal{K}_X\to\mathbb{R}$ of the K-energy map, where the Kähler class of $\mathcal{K}$ is assumed to be $2\pi c_1(M)_{\mathbb{R}}$. As in (2.8) in Section 2, we have functions $\tilde{f}_\omega\in\mathcal{K}_X$, $\omega\in\mathcal{K}_X$, such that

$$\text{(A.2.1)}\quad \begin{cases}\mathrm{Ric}^\sigma(\omega)-\omega=\sqrt{-1}\,\partial\bar{\partial}\tilde{f}_\omega;\\ \tilde{f}_\omega:=f_\omega+\psi_\omega+\log\Bigl(\frac{\int_M\tilde{\omega}_0^n}{\int_M\omega_0^n}\Bigr)=f_\omega+\sigma(u_\omega)+\log\Bigl(\frac{\int_M\tilde{\omega}_0^n}{\int_M\omega_0^n}\Bigr),\end{cases}$$

where f_ω is as in (2.8). For ω' and ω'' in $\mathcal{K}_X$, let $\{\varphi_t\,;\,a\leq t\leq b\}$ be an arbitrary smooth path in $\mathcal{H}_X$ such that $\omega(a)=\omega'$ and $\omega(b)=\omega''$, where we put

$$\text{(A.2.2)}\quad \omega(t):=\omega_{\varphi_t}=\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi_t,\quad a\leq t\leq b.$$

LEMMA A.2.3. *In the below, we use the notation* (1.4), *and in particular,* $\tilde{\omega}(t)$ *is as in* (2.3). *Then the integral* $M^\sigma(\omega', \omega'')$ *defined below depends only on the pair* (ω', ω''), *and is independent of the choice of the path* $\{\varphi_t\,;\, a \leq t \leq b\}$ *in* $\mathcal{H}_X$:

$$M^\sigma(\omega', \omega'') := \int_a^b \left\{ \int_M (\bar\partial \tilde f_{\omega(t)}, \bar\partial \dot\varphi_t)_{\omega(t)} \tilde\omega(t)^n \right\}$$
$$= -\int_a^b \left\{ \int_M \tilde f_{\eta_t} (\tilde\Box_{\omega(t)} \dot\varphi_t) \tilde\omega(t)^n \right\}.$$

Proof. Let $[0,1] \times [a,b] \ni (s,t) \mapsto \varphi_{s,t} \in \mathcal{H}_X$ be a smooth 2-parameter family of functions in $\mathcal{H}_X$. Then $\eta_{s,t} := \omega_{\varphi_{s,t}}$ sits in $\mathcal{K}_X$ for all (s,t). For simplicity, $f_{\eta_{s,t}}$, $\tilde f_{\eta_{s,t}}$, $\psi_{\eta_{s,t}}$, $u_{\eta_{s,t}}$, $\Box_{\eta_{s,t}}$, $\tilde\Box_{\eta_{s,t}}$ are denoted by $f_{s,t}$, $\tilde f_{s,t}$, $\psi_{s,t}$, $u_{s,t}$, $\Box_{s,t}$, $\tilde\Box_{s,t}$, respectively. We define

$$\Theta := \left\{ \int_M \tilde f_{s,t} (\tilde\Box_{s,t} \partial_s \varphi) \tilde\omega_{s,t}^n \right\} ds + \left\{ \int_M \tilde f_{s,t} (\tilde\Box_{s,t} \partial_t \varphi) \tilde\omega_{s,t}^n \right\} dt,$$

where $\partial_s\varphi := \partial\varphi_{s,t}/\partial s$ and $\partial_t\varphi := \partial\varphi_{s,t}/\partial t$. Then the proof is reduced to showing $d\Theta = 0$ on $[0,1] \times [a,b]$. By $\tilde\Box_{s,t} = \Box_{s,t} + \sqrt{-1}\,\dot\sigma(u_{s,t})\bar X$ and [M5, (2.6.1)],

$$\begin{aligned}
&\frac{\partial}{\partial t}(\tilde\Box_{s,t}\partial_s\varphi) - \frac{\partial}{\partial s}(\tilde\Box_{s,t}\partial_t\varphi)\\
&\quad = \sqrt{-1}\,\frac{\partial}{\partial t}\{\dot\sigma(u_{s,t})\bar X(\partial_s\varphi)\} - \sqrt{-1}\,\frac{\partial}{\partial s}\{\dot\sigma(u_{s,t})\bar X(\partial_t\varphi)\}\\
&\quad = \sqrt{-1}\,\ddot\sigma(u_{s,t})\frac{\partial u_{s,t}}{\partial t}\bar X(\partial_s\varphi) - \sqrt{-1}\,\ddot\sigma(u_{s,t})\frac{\partial u_{s,t}}{\partial s}\bar X(\partial_t\varphi)\\
&\quad = \ddot\sigma(u_{s,t})\bar X(\partial_t\varphi)\bar X(\partial_s\varphi) - \ddot\sigma(u_{s,t})\bar X(\partial_s\varphi)\bar X(\partial_t\varphi) = 0,
\end{aligned}$$

where we used the equality $u_{s,t} = u_{\omega_0} - \sqrt{-1}\,\bar X\varphi_{s,t}$ (see Section 2). Hence, by $(\partial/\partial t)(\tilde\omega_{s,t}^n) = (\tilde\Box_{s,t}\partial_t\varphi)\tilde\omega_{s,t}^n$ and $(\partial/\partial s)(\tilde\omega_{s,t}^n) = (\tilde\Box_{s,t}\partial_s\varphi)\tilde\omega_{s,t}^n$, we obtain

$$d\Theta = ds \wedge dt \int_M \left\{ -\frac{\partial \tilde f_{s,t}}{\partial t}(\tilde\Box_{s,t}\partial_s\varphi) + \frac{\partial \tilde f_{s,t}}{\partial s}(\tilde\Box_{s,t}\partial_t\varphi) \right\} \tilde\omega_{s,t}^n. \tag{A.2.4}$$

On the other hand,

$$\frac{\partial f_{s,t}}{\partial t} = -(\Box_{s,t} + 1)\partial_t\varphi + C'_{s,t} \quad \text{and} \quad \frac{\partial f_{s,t}}{\partial s} = -(\Box_{s,t} + 1)\partial_s\varphi + C''_{s,t}$$

for some real constants $C'_{s,t}$ and $C''_{s,t}$ depending only on s and t. Hence, by $\psi_{s,t} = \sigma(u_{s,t}) = \sigma(u_{\omega_0} - \sqrt{-1}\,\bar{X}\varphi_{s,t})$, we see that

(A.2.5)

$$\begin{cases} \dfrac{\partial \tilde{f}_{s,t}}{\partial t} = -(\Box_{s,t}+1)\partial_t\varphi + \dfrac{\partial \psi_{s,t}}{\partial t} + C'_{s,t} = -(\tilde{\Box}_{s,t}+1)\partial_t\varphi + C'_{s,t}, \\ \dfrac{\partial \tilde{f}_{s,t}}{\partial s} = -(\Box_{s,t}+1)\partial_s\varphi + \dfrac{\partial \psi_{s,t}}{\partial s} + C''_{s,t} = -(\tilde{\Box}_{s,t}+1)\partial_s\varphi + C''_{s,t}. \end{cases}$$

By (A.2.4) and (A.2.5), we finally obtain the following required identity:

$$d\Theta = ds \wedge dt \int_M \big\{\partial_t\varphi\big(\tilde{\Box}_{s,t}\partial_s\varphi\big) - \partial_s\varphi\big(\tilde{\Box}_{s,t}\partial_t\varphi\big)\big\}\tilde{\omega}^n_{s,t} = 0.$$

□

By Lemma A.2.3 above, for all ω, ω', $\omega'' \in \mathcal{K}_X$, it is easily seen that M^σ satisfies the 1-cocycle conditions

$$\begin{cases} M^\sigma(\omega,\omega') + M^\sigma(\omega',\omega) = 0, \\ M^\sigma(\omega,\omega') + M^\sigma(\omega',\omega'') + M^\sigma(\omega'',\omega) = 0. \end{cases}$$

As a multiplier Hermitian analogue of a K-energy map, we can now define $\mu^\sigma : \mathcal{K}_X \to \mathbb{R}$ by setting $\mu^\sigma(\omega) := M^\sigma(\omega_0,\omega)$ for all $\omega \in \mathcal{K}_X$. As in the introduction, let $\mathcal{E}^\sigma_X$ denote the set of all ω in $\mathcal{K}_X$ such that $\mathrm{Ric}^\sigma(\omega) = \omega$. Then by (A.2.1) and Lemma A.2.3 together with (b) of Lemma 2.9, we obtain

Proposition A.2. (a) *An element ω in $\mathcal{K}_X$ is a critical point of $\mu_\sigma : \mathcal{K}_X \to \mathbb{R}$ if and only if $\omega \in \mathcal{E}^\sigma_X$, i.e., the function $\tilde{f}_\omega$ defined in* (A.2.1) *is zero everywhere on M.*

(b) *For an arbitrary smooth path $\{\varphi_t\ ;\ a \le t \le b\}$ in $\mathcal{H}_X$, the one-parameter family of Kähler forms $\omega(t)$, $a \le t \le b$, in $\mathcal{K}_X$ defined by* (A.2.2) *satisfies*

$$\frac{d}{dt}\mu^\sigma(\omega(t)) = \int_M (\bar{\partial}\tilde{f}_{\omega(t)}, \bar{\partial}\dot{\varphi}_t)_{\omega(t)}\tilde{\omega}(t)^n, \quad a \le t \le b.$$

Appendix 3. Technical equalities related to the operator $\tilde{\square}_\omega$

In this appendix, related to the operator $\tilde{\square}_\omega$, some technical equalities analogous to those in [BM, Lemma 2.3] will be given. Note that, by the notation in (2.6) and Appendix 5, we have the inclusion $\mathrm{Ker}_{\mathbb{R}}(\tilde{\square}_\omega + 1) \subset \mathfrak{g}^\omega$ for all $\omega \in \mathcal{E}_X^\sigma$. Now, we have:

PROPOSITION A.3. *Let $\omega \in \mathcal{E}_X^\sigma$ and $\zeta \in C^\infty(M)_{\mathbb{R}}$. Then for all $v, v_1, v_2 \in \mathrm{Ker}_{\mathbb{R}}(\tilde{\square}_\omega + 1)$,*

$$\tilde{\square}_\omega(\partial\zeta, \partial v)_\omega = (\partial\bar{\partial}\zeta, \partial\bar{\partial}v)_\omega + (\partial(\tilde{\square}_\omega\zeta), \partial v)_\omega - \ddot{\sigma}(u_\omega)(\bar{X}\zeta)(\bar{X}v). \tag{A.3.1}$$

In particular, $(\tilde{\square}_\omega + 1)(\partial v_1, \partial v_2)_\omega = (\partial\bar{\partial}v_1, \partial\bar{\partial}v_2)_\omega - \ddot{\sigma}(u_\omega)(\bar{X}v_1)(\bar{X}v_2) = (\tilde{\square}_\omega + 1)(\partial v_2, \partial v_1)_\omega$, and

$$\begin{aligned}\int_M \{v_1 v_2 - (\partial v_1, \partial v_2)_\omega\}\{(\tilde{\square}_\omega + 1)\zeta\}\tilde{\omega}^n \\ = -\int_M v_1(\partial\bar{\partial}\zeta, \partial\bar{\partial}v_2)_\omega\tilde{\omega}^n + \int_M \ddot{\sigma}(u_\omega)v_1(\bar{X}\zeta)(\bar{X}v_2)\,\tilde{\omega}^n.\end{aligned} \tag{A.3.2}$$

Proof. (A.3.1) follows from (1.3) and [BM, (2.3.1)] in view of the following identities:

$$\begin{aligned}(\partial\{\sqrt{-1}\,\dot{\sigma}(u_\omega)\bar{X}\zeta\}, \partial v)_\omega - \sqrt{-1}\,\dot{\sigma}(u_\omega)\bar{X}(\partial\zeta, \partial v)_\omega \\ = (\bar{X}\zeta)\ddot{\sigma}(u_\omega)\sqrt{-1}\,(\partial u_\omega, \partial v)_\omega = \ddot{\sigma}(u_\omega)(\bar{X}\zeta)(\bar{X}v).\end{aligned}$$

For (A.3.2), put $\xi := (\tilde{\square}_\omega + 1)\zeta$. Then following [BM, p. 21], by (1.3) and (1.4), we obtain

$$\begin{aligned}\int_M \{v_1 v_2 - (\partial v_1, \partial v_2)_\omega\}\xi\tilde{\omega}^n &= -\int_M \{v_1(\tilde{\square}_\omega v_2) + (\partial v_1, \partial v_2)_\omega\}\xi\tilde{\omega}^n \\ &= -\sqrt{-1}\int_M (v_1\partial\bar{\partial}v_2 + \partial v_1 \wedge \bar{\partial}v_2)\xi \wedge ne^{-\psi_\omega}\omega^{n-1} \\ &\qquad + \int_M v_1(\partial\psi_\omega, \partial v_2)_\omega \xi e^{-\psi_\omega}\omega^n \\ &= -\sqrt{-1}\int_M \partial(v_1\bar{\partial}v_2)\xi \wedge ne^{-\psi_\omega}\omega^{n-1} \\ &\qquad + \sqrt{-1}\int_M v_1(\partial\psi_\omega \wedge \bar{\partial}v_2)\xi \wedge ne^{-\psi_\omega}\omega^{n-1} \\ &= \sqrt{-1}\int_M v_1\partial\xi \wedge \bar{\partial}v_2 \wedge ne^{-\psi_\omega}\omega^{n-1} = \int_M v_1(\partial\xi, \partial v_2)_\omega\tilde{\omega}^n\end{aligned}$$

$$= \int_M v_1(\partial(\tilde{\Box}_\omega \zeta), \partial v_2)_\omega \tilde{\omega}^n + \int_M v_1(\partial\zeta, \partial v_2)_\omega \tilde{\omega}^n.$$

This together with (A.3.1) above implies the required identity (A.3.2) as follows:

$$\begin{aligned}
&\int_M \{v_1 v_2 - (\partial v_1, \partial v_2)_\omega\}\xi\tilde{\omega}^n + \int_M (\partial\bar{\partial}\zeta, \partial\bar{\partial}v_2)_\omega v_1 \tilde{\omega}^n \\
&\quad = \int_M \{\tilde{\Box}_\omega(\partial\zeta, \partial v_2)_\omega + \ddot{\sigma}(u_\omega)(\bar{X}\zeta)(\bar{X}v_2)\} v_1 \tilde{\omega}^n + \int_M v_1(\partial\zeta, \partial v_2)_\omega \tilde{\omega}^n \\
&\quad = \int_M (\partial\zeta, \partial v_2)_\omega \overline{\{(\tilde{\Box}_\omega + 1)v_1\}}\tilde{\omega}^n + \int_M \ddot{\sigma}(u_\omega) v_1 (\bar{X}\zeta)(\bar{X}v_2)\tilde{\omega}^n \\
&\quad = \int_M \ddot{\sigma}(u_\omega) v_1 (\bar{X}\zeta)(\bar{X}v_2)\tilde{\omega}^n.
\end{aligned}$$

□

Appendix 4. Uniqueness of solutions for equations of Calabi-Yau's type

Fix $\omega_0 \in \mathcal{K}_X$ and $\sigma \in C^\infty(I_X)_\mathbb{R}$ as in the introduction, and let V_0 be as in Lemma 2.4. In this appendix, we discuss the following equation of Calabi-Yau's type:

$$\text{(A.4.1)} \qquad \mathrm{Ric}^\sigma(\omega) = \omega_0.$$

Here, any solution ω of (A.4.1) is required to belong to $\mathcal{K}_X$. The purpose of this appendix is to show the following uniqueness:

Proposition A.4. *The equation* (A.4.1) *has a unique solution* ω *in* $\mathcal{K}_X$.

Before getting into the proof, we give some remark. Let $0 < \alpha < 1$, and we consider the mapping $\Gamma : \mathcal{H}^{2,\alpha}_{X,0} \times \mathbb{R} \to C^{0,\alpha}_0(M)_\mathbb{R}$ defined in (5.1.2) by

$$\Gamma(\varphi, t) := A(\varphi) - \left\{\frac{1}{V_0}\int_M \exp(-t\varphi + \tilde{f}_{\omega_0})\tilde{\omega}_0^n\right\}^{-1} \exp(-t\varphi + \tilde{f}_{\omega_0}),$$

where $V_0 := \int_M \tilde{\omega}^n$ and $A(\varphi) := \tilde{\omega}_\varphi^n / \tilde{\omega}_0^n$. Note that, if $(\varphi, t) \in \mathcal{H}^{2,\alpha}_{X,0} \times \mathbb{R}$ satisfies $\Gamma(\varphi, t) = 0$, then φ automatically belongs to $C^\infty(M)_\mathbb{R}$. Hence, it is easily seen that the set of the solutions of (A.4.1) and the set of the solutions of $\Gamma(\varphi, 0) = 0$ are identified by

$$\text{(A.4.2)} \quad \{\varphi \in \mathcal{H}^{2,\alpha}_{X,0}\,;\, \Gamma(\varphi, 0) = 0\} \simeq \{\omega \in \mathcal{K}_X\,;\, \mathrm{Ric}^\sigma(\omega) = \omega_0\}, \quad \varphi \leftrightarrow \omega_\varphi.$$

Proof of Proposition A.4. By (A.4.2), it suffices to show that $\varphi \in \mathcal{H}^{2,\alpha}_{X,0}$ satisfying $\Gamma(\varphi, 0) = 0$ is unique. Suppose that φ', φ'' in $\mathcal{H}^{2,\alpha}_{X,0}$ satisfy

$$\Gamma(\varphi', 0) = 0 = \Gamma(\varphi'', 0).$$

Since the Fréchet derivatives $D_\varphi \Gamma_{|(\varphi',0)}$, $D_\varphi \Gamma_{|(\varphi'',0)}$ are invertible (cf. (5.1.5)), we have smooth one-parameter families $\{\varphi'_t\ ;\ -\varepsilon < t \leq 0\}$, $\{\varphi''_t\ ;\ -\varepsilon < t \leq 0\}$ (where $0 < \varepsilon \ll 1$) of functions in $\mathcal{H}^{k,\alpha}_{X,0}$ satisfying $\varphi'_0 = \varphi'$ and $\varphi''_0 = \varphi''$ such that $\Gamma(\varphi'_t, t) = 0 = \Gamma(\varphi''_t, t)$ for all t with $-\varepsilon < t \leq 0$. Put

$$e'_t := \frac{1}{V_0}\int_M \exp(-t\varphi'_t + \tilde{f}_{\omega_0})\tilde{\omega}_0^n \quad \text{and} \quad e''_t := \frac{1}{V_0}\int_M \exp(-t\varphi''_t + \tilde{f}_{\omega_0})\tilde{\omega}_0^n.$$

For $t = 0$, (b) of Lemma 2.9 yields $e'_0 = 1$ and $e''_0 = 1$, and hence we can find $c'_t, c''_t \in \mathbb{R}$, $-\varepsilon < t \leq 0$, depending on t continuously such that $e'_t = \exp(tc'_t)$ and $e''_t = \exp(tc''_t)$ for all t with $-\varepsilon < t \leq 0$. Then by setting $\xi'_t := \varphi'_t + c'_t$ and $\xi''_t := \varphi''_t + c''_t$, we have

$$\text{(A.4.3)} \qquad A(\xi'_t) = \exp(-t\xi'_t + \tilde{f}_{\omega_0}) \quad \text{and} \quad A(\xi''_t) = \exp(-t\xi''_t + \tilde{f}_{\omega_0}).$$

For simplicity, we put $\omega'_t := \omega_{\xi'_t}$ and $\omega''_t := \omega_{\xi''_t}$ $(-\varepsilon < t \leq 0)$. Note that, by (2.5), $\psi_{\omega'_t} = \sigma(u_{\omega'_t}) = \sigma(u_{\omega_0} - \sqrt{-1}\,\bar{X}\xi'_t)$ and $\psi_{\omega''_t} = \sigma(u_{\omega_0} - \sqrt{-1}\,\bar{X}\xi''_t) = \sigma(u_{\omega'_t} - \sqrt{-1}\,\bar{X}(\xi''_t - \xi'_t))$, while $A(\xi''_t)/A(\xi'_t) = \{e^{-\psi_{\omega''_t}}(\omega''_t)^n\}/\{e^{-\psi_{\omega'_t}}(\omega'_t)^n\}$. For each t with $-\varepsilon < t < 0$, let p_t be the point on M at which the function $\xi''_t - \xi'_t$ on M takes its maximum. Then by (A.4.3), the maximum principle shows that

$$1 \geq \{A(\xi''_t)/A(\xi'_t)\}(p_t) = \exp\{-t(\xi''_t - \xi'_t)(p_t)\}.$$

Then $(\xi''_t - \xi'_t)(p) \leq (\xi''_t - \xi'_t)(p_t) \leq 0$ for all $p \in M$, i.e., $\xi''_t \leq \xi'_t$ on M. By exactly the same argument, we have $\xi'_t \leq \xi''_t$ on M. Hence, $\xi''_t = \xi'_t$ on M for all t with $-\varepsilon < t < 0$. Let t tend to 0. By passing to the limit, we see that $\xi''_0 = \xi'_0$, i.e., $\varphi'' - \varphi'$ is a constant on M. Then by $\varphi', \varphi'' \in \mathcal{H}^{2,\alpha}_{X,0}$, we immediately obtain $\varphi'' = \varphi'$ on M, as required. □

Appendix 5. A multiplier Hermitian analogue of Matsushima's obstruction

In this appendix, Matsushima's obstruction [Mat] will be generalized for multiplier Hermitian metrics of type σ, where σ is an arbitrary real-valued function on I_X. Assuming $\mathcal{E}^\sigma_X \neq \emptyset$, let $\theta \in \mathcal{E}^\sigma_X$. Write

$$\theta = \sqrt{-1}\sum_{\alpha,\beta} g(\theta)_{\alpha\bar{\beta}}\, dz^\alpha \wedge dz^{\bar{\beta}},$$

in terms of a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. Since $\mathrm{Ric}^\sigma(\theta) = \theta$, the Kähler class of $\mathcal{K}_X$ is $2\pi c_1(M)_{\mathbb{R}}$. Then by (2.8) and (a) of Lemma 2.9,

$$f_\theta = -\psi_\theta + C_0 \tag{A.5.1}$$

for some real constant C_0. By [F1, p. 41], $\mathfrak{g}^\theta$ in (2.6) coincides with the kernel $\mathrm{Ker}_{\mathbb{C}}(\tilde{\Box}_\theta + 1)$ of the operator $\tilde{\Box}_\theta + 1$ on $C^\infty(M)_{\mathbb{C}}$, since by (A.5.1), $\tilde{\Box}_\theta$ is written in the form

$$\tilde{\Box}_\theta = \Box_\theta + \sum_{\alpha,\beta} g(\theta)^{\bar{\beta}\alpha} \frac{\partial f_\theta}{\partial z^\alpha} \frac{\partial}{\partial z^{\bar{\beta}}}.$$

LEMMA A.5.2. *The vector space $\mathfrak{g}^\theta$ in* (2.6) *forms a complex Lie algebra in terms of the Poisson bracket by θ, and in particular the $\mathbb{C}$-linear isomorphism $\mathfrak{g}^\theta \cong \mathfrak{g}$ in* (2.6) *is an isomorphism of complex Lie algebras.*

Proof. For each $v_1, v_2 \in C^\infty(M)_{\mathbb{C}}$, we consider their Poisson bracket $[v_1, v_2] \in C^\infty(M)_{\mathbb{C}}$ on the Kähler manifold (M, θ) as in [FM]. Let $u_1, u_2 \in \mathfrak{g}^\theta$. Then by $\mathrm{grad}^{\mathbb{C}}_\theta [u_1, u_2] = [\mathrm{grad}^{\mathbb{C}}_\theta u_1, \mathrm{grad}^{\mathbb{C}}_\theta u_2]$, we see that $[u_1, u_2] + k_0$ belongs to $\mathfrak{g}^\theta$ for some constant $k_0 \in \mathbb{C}$. Hence it suffices to show $k_0 = 0$, i.e.,

$$\int_M [u_1, u_2]\tilde{\theta}^n = 0.$$

Let $F : \mathfrak{g} \to \mathbb{C}$ be the Futaki character. Then by [FM, (2.1)] and [M1, Theorem 2.1], we see that $\int_M (1 - e^{f_\theta})[u_1, u_2]\theta^n = F([\mathrm{grad}^{\mathbb{C}}_\theta u_1, \mathrm{grad}^{\mathbb{C}}_\theta u_2]) = 0$. Therefore, in view of (A.5.1), we obtain

$$\int_M [u_1, u_2]\tilde{\theta}^n = \exp(-C_0) \int_M [u_1, u_2] e^{f_\theta} \theta^n = \exp(-C_0) \int_M [u_1, u_2]\theta^n = 0,$$

as required. □

For the centralizer $\mathfrak{z}(X)$ of X in $\mathfrak{g}$, the group $Z^0(X)$ in the introduction is exactly the complex Lie group generated by $\mathfrak{z}(X)$ in G. Consider the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{z}(X)$ associated to the group K of all isometries in $Z^0(X)$ on the Kähler manifold (M, θ). Let $\mathfrak{k}_{\mathbb{C}}$ be the complexification of $\mathfrak{k}$ in the complex Lie algebra $\mathfrak{g}$. Put

$$\begin{cases} \mathfrak{z}^\theta(X) := \{u \in \mathrm{Ker}_{\mathbb{C}}(\tilde{\Box}_\theta + 1) \,;\, X_{\mathbb{R}} u = 0\}, \\ \quad \mathfrak{k}^\theta \quad := \{u \in \mathrm{Ker}_{\mathbb{R}}(\tilde{\Box}_\theta + 1) \,;\, X_{\mathbb{R}} u = 0\}, \end{cases} \tag{A.5.3}$$

where $\mathrm{Ker}_{\mathbb{R}}(\tilde{\Box}_\theta+1)$ denotes the kernel of the operator $(\tilde{\Box}_\theta+1)$ on $C^\infty(M)_{\mathbb{R}}$. Put $\mathfrak{k}^\theta_{\mathbb{C}} := \mathfrak{k}^\theta + \sqrt{-1}\,\mathfrak{k}^\theta$ in $C^\infty(M)_{\mathbb{C}}$. Then by $\mathfrak{k}^\theta_{\mathbb{C}} \subset \mathfrak{z}^\theta(X) \subset \mathfrak{g}^\theta$ and $\mathfrak{g}^\theta \cong \mathfrak{g}$, we obtain

$$\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{z}(X). \tag{A.5.4}$$

Note that $Z(X)$ acts on $\mathcal{E}^\sigma_X$ by $Z(X) \times \mathcal{E}^\sigma_X \ni (g,\theta) \mapsto (g^{-1})^*\theta \in \mathcal{E}^\sigma_X$. Since the isotropy subgroup of $Z^0(X)$ at θ is K, we can write the $Z^0(X)$-orbit $\mathbf{O}$ through θ as

$$\mathbf{O} \cong Z^0(X)/K, \tag{A.5.5}$$

Let $T_\theta(\mathcal{E}^\sigma_X)$ and $T_\theta(\mathbf{O})$ denote the tangent spaces at θ of $\mathcal{E}^\sigma_X$ and $\mathbf{O}$, respectively. In view of the homeomorphism $\tilde{\mathcal{E}}^\sigma_X \simeq \mathcal{E}^\sigma_X$ immediately after (5.4.1) in Section 5, the differentiation of the equation $A(\varphi) = \exp(-\varphi + \tilde{f}_0)$ with respect to φ yields

$$\begin{array}{ccccc} T_\theta(\mathcal{E}^\sigma_X) & \cong & \mathfrak{k}_{\mathbb{C}}/\mathfrak{k} & \cong & \mathfrak{k}^\theta \quad (= T_\theta(\tilde{\mathcal{E}}^\sigma_X)) \\ \sqrt{-1}\,\partial\bar{\partial}v & \leftrightarrow & [\sqrt{-1}\ \mathrm{grad}^{\mathbb{C}}_\theta\, v/2] & \leftrightarrow & v, \end{array} \tag{A.5.6}$$

where for every γ in $\mathfrak{k}_{\mathbb{C}}$, we mean by $[\gamma]$ the natural image of γ under the projection of $\mathfrak{k}_{\mathbb{C}}$ onto $\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}$. On the other hand, by (A.5.5), we have the isomorphism

$$T_\theta(\mathbf{O}) \cong \mathfrak{z}(X)/\mathfrak{k}. \tag{A.5.7}$$

Since $\mathbf{O} \subset \mathcal{E}^\sigma_X$, we have $T_\theta(\mathbf{O}) \subset T_\theta(\mathcal{E}^\sigma_X)$. This together with (A.5.4), (A.5.6) and (A.5.7) implies that $\mathfrak{z}(X) = \mathfrak{k}_{\mathbb{C}}$, i.e., $T_\theta(\mathbf{O}) = T_\theta(\mathcal{E}^\sigma_X)$. Thus, we obtain

PROPOSITION A.5. (a) *If $\mathcal{E}^\sigma_X \neq \emptyset$, then $Z^0(X)$ is a reductive algebraic group. Actually for an arbitrary $\theta \in \mathcal{E}^\sigma_X$, we have $\mathfrak{z}(X) = \mathfrak{k}_{\mathbb{C}}$, i.e., $\mathfrak{z}^\theta(X) = \mathfrak{k}^\theta_{\mathbb{C}}$ by the above notation.*

(b) *If $\mathcal{E}^\sigma_X \neq \emptyset$, then each connected component of $\mathcal{E}^\sigma_X$ is a single $Z^0(X)$-orbit under the natural action of $Z^0(X)$ on $\mathcal{E}^\sigma_X$.*

Remark A.5.8. The above arguments are valid also for $X = 0$. If $X = 0$, then (a) of Proposition A.5 is nothing but Matsushima's theorem [Mat].

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