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FIEDLER-ANDO THEOREM FOR ANDO-LI-MATHIAS MEAN OF POSITIVE OPERATORS

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Dedicated to Professor Kichi-Suke Saito in commemoration of his retirement

Abstract. In this paper, we show several operator inequalities involving the Hadamard product and the Ando-Li-Mathias mean of n positive operators on a Hilbert space, which are regarded as n-variable versions of the Fiedler-Ando theorem. As an application, we show an n-variable version of Fiedler type inequality via the Ando-Li-Mathias mean.

1. Introduction

Let $\{e_j\}$ be an orthonormal basis of a separable Hilbert space H and $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ be the tensor product of operators $A_1, A_2, \ldots, A_n$ on H regarding to $\{e_j\}$. Let $U_n : H \mapsto H \otimes H \otimes \cdots \otimes H$ be the isometry such that $U_n e_j = e_j \otimes e_j \otimes \cdots \otimes e_j$. Following after [8, 5], the Hadamard product $A_1 \circ A_2 \circ \cdots \circ A_n$ regarding to $\{e_j\}$ is expressed as

$$A_1 \circ A_2 \circ \cdots \circ A_n = U_n^*(A_1 \otimes A_2 \otimes \cdots \otimes A_n)U_n. \tag{1}$$

Fiedler [4] showed that if A is a positive definite matrix, then

$$A \circ A^{-1} \geq I. \tag{2}$$

As a generalization of the Fiedler inequality (2), Ando [1, Theorem 13] showed better estimates for below for the Hadamard product of positive definite matrices A, B by using the geometric mean of A and B:

$$A \circ B \geq (A \sharp B) \circ (A \sharp B), \tag{3}$$

where the geometric mean $A \sharp B$ for $A, B > 0$ is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

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In fact, put $B = A^{-1}$ in (3), then (3) implies the Fiedler inequality (2). Thus, we call (3) the Fiedler-Ando inequality. Afterwords, Aujla and Vasudeva [3] extended the Fiedler-Ando inequality (3): If A, B, C and D are positive definite, then

$$(A \circ B) \sharp (C \circ D) \geq (A \sharp C) \circ (B \sharp D). \tag{4}$$

We refer the reader to [7] for operator versions of the inequalities on the Hadamard product mentioned above.

By virtue of the Ando-Li-Mathias mean, we try to consider an n-variable version of (3) and (4). We will review the notion of the Ando-Li-Mathias mean of n positive operators on a Hilbert space. We simply call it the ALM mean. For any positive integer $n \geq 2$, the Ando-Li-Mathias mean $G(A_1, A_2, \ldots, A_n) = G_{\mathrm{ALM}}(A_1, A_2, \ldots, A_n)$ of any n-tuple of positive invertible operators $A_1, A_2, \ldots, A_n$ on a Hilbert space H is defined by induction as follows:

(i) $G_{\mathrm{ALM}}(A_1, A_2) = A_1 \sharp A_2$.

(ii) Assume that the geometric mean of any $(n-1)$-tuple of operators is defined. Let

$$G_{\mathrm{ALM}}((A_j)_{j \neq i}) = G_{\mathrm{ALM}}(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$$

and let sequences $\{A_i^{(r)}\}_{r=1}^{\infty}$ be $A_i^{(1)} = A_i$ and $A_i^{(r+1)} = G_{\mathrm{ALM}}((A_j^{(r)})_{j \neq i})$. Then there exists $\lim_{r \to \infty} A_i^{(r)}$ uniformly and it does not depend on i. Hence the geometric mean of n operators is defined by

$$G_{\mathrm{ALM}}(A_1, A_2, \ldots, A_n) = \lim_{r \to \infty} A_i^{(r)} \qquad \text{for } i = 1, 2, \ldots, n.$$

We list some properties of the ALM mean which we need later:

(P1) Consistency with scalars : $G_{\mathrm{ALM}}(A_1, \ldots, A_n) = (A_1 A_2 \cdots A_n)^{\frac{1}{n}}$ if the A_i's commute;

(P2) Joint homogeneity : $G_{\mathrm{ALM}}(a_1 A_1, \ldots, a_n A_n) = (a_1 a_2 \cdots a_n)^{\frac{1}{n}} G_{\mathrm{ALM}}(A_1, \ldots, A_n)$;

(P3) Permutation invariance : $G_{\mathrm{ALM}}(\pi(A_1, \ldots, A_n)) = G_{\mathrm{ALM}}(A_1, \ldots, A_n)$ for any permutation $\pi(A_1, \ldots, A_n)$ of $(A_1, \ldots, A_n)$;

(P4) Transformer inequality : $T^* G_{\mathrm{ALM}}(A_1, \ldots, A_n) T \leq G_{\mathrm{ALM}}(T^* A_1 T, \ldots, T^* A_n T)$ for every operator T;

(P5) Self-duality : $G_{\mathrm{ALM}}(A_1^{-1}, \ldots, A_n^{-1})^{-1} = G_{\mathrm{ALM}}(A_1, \ldots, A_n)$;

(P6) Information monotonicity : $\Phi(G_{\mathrm{ALM}}(A_1, \ldots, A_n)) \leq G_{\mathrm{ALM}}(\Phi(A_1), \ldots, \Phi(A_n))$ for any unital positive linear map Φ;

(P7) AGH inequality :

$$\left(\frac{1}{n} \sum_{i=1}^{n} A_i^{-1} \right)^{-1} \leq G_{\mathrm{ALM}}(A_1, \ldots, A_n) \leq \frac{1}{n} \sum_{i=1}^{n} A_i.$$

We refer the reader to [2, 6] for more information on the ALM mean.

In this paper, we show several operator inequalities involving the Hadamard product and the Ando-Li-Mathias mean of n positive operators on the Hilbert space, which are regarded as n-variable versions of the Fiedler-Ando theorem (3). As an application, we show an n-variable version of Fielder type inequality via the ALM mean.

2. Main results

First of all, we start with the following lemma which involves the Hadamard product and the ALM mean. We denote by $G(A_1, A_2, \ldots, A_n)$ the ALM mean for simplicity.

Lemma 2.1. *For any integer $n, k \geq 2$, let A_{ij} be positive operators for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Then*

$$\begin{aligned} &G(A_{11} \otimes A_{21} \otimes \cdots \otimes A_{n1}, A_{12} \otimes A_{22} \otimes \cdots \otimes A_{n2}, \ldots, A_{1k} \otimes A_{2k} \otimes \cdots \otimes A_{nk}) \\ &\quad = G(A_{11}, A_{12}, \ldots, A_{1k}) \otimes G(A_{21}, A_{22}, \ldots, A_{2k}) \otimes \cdots \otimes G(A_{n1}, A_{n2}, \ldots, A_{nk}). \end{aligned}$$

Proof. We may assume that A_{ij} are invertible for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. By definition, it follows that $G(A_{11} \otimes A_{21}, A_{12} \otimes A_{22}) = G(A_{11}, A_{12}) \otimes G(A_{21}, A_{22})$. Firstly we show the case of $k = 3$ and $n = 2$: A_{11}, A_{21}, A_{31} and A_{21}, A_{22}, A_{23}. In fact, we have $(A_{11} \otimes A_{21})^{(1)} = A_{11} \otimes A_{21}$ and

$$\begin{aligned} (A_{11} \otimes A_{21})^{(2)} &= G((A_{12} \otimes A_{22})^{(1)}, (A_{13} \otimes A_{23})^{(1)}) = G(A_{12} \otimes A_{22}, A_{13} \otimes A_{23}) \\ &= G(A_{12}, A_{13}) \otimes G(A_{22}, A_{23}) = A_{11}^{(2)} \otimes A_{21}^{(2)}. \end{aligned}$$

By induction on r, it follows that

$$\begin{aligned} (A_{11} \otimes A_{21})^{(r+1)} &= G((A_{12} \otimes A_{22})^{(r)}, (A_{13} \otimes A_{23})^{(r)}) = G(A_{12}^{(r)} \otimes A_{22}^{(r)}, A_{13}^{(r)} \otimes A_{23}^{(r)}) \\ &= G(A_{12}^{(r)}, A_{13}^{(r)}) \otimes G(A_{22}^{(r)}, A_{23}^{(r)}) = A_{11}^{(r+1)} \otimes A_{21}^{(r+1)} \end{aligned}$$

and as $r \to \infty$ it follows from the definition of the ALM mean that

$$G(A_{11} \otimes A_{21}, A_{12} \otimes A_{22}, A_{13} \otimes A_{23}) = G(A_{11}, A_{12}, A_{13}) \otimes G(A_{21}, A_{22}, A_{23}).$$

Similarly, by induction on r, we show the general case of any $k \geq 2$ and $n \geq 2$. In fact, we have $(A_{1j} \otimes A_{2j} \otimes \cdots \otimes A_{nj})^{(1)} = A_{1j} \otimes A_{2j} \otimes \cdots \otimes A_{nj}$ and

$$\begin{aligned} (A_{1j} \otimes A_{2j} \otimes \cdots \otimes A_{nj})^{(r+1)} &= G(((A_{1l} \otimes A_{2l} \otimes \cdots \otimes A_{nl})^{(r)})_{l \neq j}) \\ &= G((A_{1l}^{(r)} \otimes A_{2l}^{(r)} \otimes \cdots \otimes A_{nl}^{(r)})_{l \neq j}) \\ &= G((A_{1l}^{(r)})_{l \neq j}) \otimes G((A_{2l}^{(r)})_{l \neq j}) \otimes \cdots \otimes G((A_{nl}^{(r)})_{l \neq j}) \\ &= A_{1j}^{(r+1)} \otimes A_{2j}^{(r+1)} \otimes \cdots \otimes A_{nj}^{(r+1)} \end{aligned}$$

and as $r \to \infty$ we have

$$
\begin{aligned}
&G(A_{11} \otimes A_{21} \otimes \cdots \otimes A_{n1}, A_{12} \otimes A_{22} \otimes \cdots \otimes A_{n2}, \ldots, A_{1k} \otimes A_{2k} \otimes \cdots \otimes A_{nk}) \\
&= G(A_{11}, A_{12}, \ldots, A_{1k}) \otimes G(A_{21}, A_{22}, \ldots, A_{2k}) \otimes \cdots \otimes G(A_{n1}, A_{n2}, \ldots, A_{nk})
\end{aligned}
$$

and so the proof is complete. □

We shall use, for convenience, the notation

$$\prod_{i=1}^{n} \circ A_i = A_1 \circ A_2 \circ \cdots \circ A_n \qquad \text{and} \qquad \prod_{i=1}^{n} \circ A = A \circ A \circ \cdots \circ A \quad (n \text{ times}).$$

By Lemma 2.1, we show an n-variable version of the Fiedler-Ando theorem (3) by using the ALM mean:

Theorem 2.1. *Let $A_1, A_2, \ldots, A_n$ be positive operators for $n \geq 2$. Then*

$$\prod_{i=1}^{n} \circ\, A_i \geq \prod_{i=1}^{n} \circ\, G(A_1, A_2, \ldots, A_n).$$

Proof. It follows that

$$
\begin{aligned}
\prod_{i=1}^{n} \circ\, A_i &= G(A_1 \ \circ \cdots \circ \ A_n, A_1 \ \circ \cdots \circ \ A_n, \ldots, A_1 \ \circ \cdots \circ \ A_n) \\
&= G(A_1 \circ A_2 \circ \cdots \circ A_n, A_2 \circ A_3 \circ \cdots \circ A_n \circ A_1, \ldots, A_n \circ A_1 \circ \cdots \circ A_{n-1}) \\
&\qquad \text{by commutativity of Hadamard multiplication} \\
&= G(U_n^*(A_1 \otimes \cdots \otimes A_n)U_n, U_n^*(A_2 \otimes \cdots \otimes A_n \otimes A_1)U_n, \\
&\qquad \cdots, U_n^*(A_n \otimes A_1 \otimes \cdots \otimes A_{n-1})U_n) \\
&\geq U_n^* G(A_1 \otimes \cdots \otimes A_n, A_2 \otimes \cdots \otimes A_n \otimes A_1, \cdots, A_n \otimes \cdots \otimes A_{n-1})U_n \\
&\qquad \text{by transformer inequality (P4) for the ALM mean} \\
&= U_n^* \left[G(A_1, A_2, \ldots, A_n) \otimes G(A_2, \ldots, A_n, A_1) \otimes \cdots \otimes G(A_n, A_1, \ldots, A_{n-1})\right] U_n \\
&\qquad \text{by Lemma 2.1} \\
&= G(A_1, \ldots, A_n) \circ G(A_2, \ldots, A_n, A_1) \circ \cdots \circ G(A_n, \ldots, A_{n-1}) \\
&= \prod_{i=1}^{n} \circ\, G(A_1, \ldots, A_n) \quad \text{by permutation invariance (P3) for the ALM mean.}
\end{aligned}
$$

□

By Lemma 2.1, we show an n-variable version of Aujla-Vasudeva theorem (4) by using the ALM mean:

Theorem 2.2. *For any integer $n, k \geq 2$, let A_{ij} be positive operators for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Then*

$$\begin{aligned}
&G(A_{11} \circ A_{21} \circ \cdots \circ A_{n1}, A_{12} \circ A_{22} \circ \cdots \circ A_{n2}, \ldots, A_{1k} \circ A_{2k} \circ \cdots \circ A_{nk}) \\
&\quad \geq G(A_{11}, A_{12}, \ldots, A_{1k}) \circ G(A_{21}, A_{22}, \ldots, A_{2k}) \circ \cdots \circ G(A_{n1}, A_{n2}, \ldots, A_{nk})
\end{aligned}$$

Proof. It follows from Lemma 2.1 and transformer inequality (P4) that

$$\begin{aligned}
&G(A_{11} \circ A_{21} \circ \cdots \circ A_{n1}, A_{12} \circ A_{22} \circ \cdots \circ A_{n2}, \ldots, A_{1k} \circ A_{2k} \circ \cdots \circ A_{nk}) \\
&= G(U_n^*(A_{11} \otimes A_{21} \otimes \cdots \otimes A_{n1})U_n, U_n^*(A_{12} \otimes A_{22} \otimes \cdots \otimes A_{n2})U_n, \\
&\qquad \ldots, U_n^*(A_{1k} \otimes A_{2k} \otimes \cdots \otimes A_{nk})U_n) \\
&\geq U_n^* G(A_{11} \otimes A_{21} \otimes \cdots \otimes A_{n1}, A_{12} \otimes A_{22} \otimes \cdots \otimes A_{n2}, \ldots, A_{1k} \otimes A_{2k} \otimes \cdots \otimes A_{nk})U_n \\
&\qquad\qquad \text{by transformer inequality (P4) for the ALM mean} \\
&= U_n^* \left[G(A_{11}, A_{12}, \ldots, A_{1k}) \otimes G(A_{21}, A_{22}, \ldots, A_{2k}) \otimes \cdots \otimes G(A_{n1}, A_{n2}, \ldots, A_{nk})\right] U_n \\
&\qquad\qquad \text{by Lemma 2.1} \\
&= G(A_{11}, A_{12}, \ldots, A_{1k}) \circ G(A_{21}, A_{22}, \ldots, A_{2k}) \circ \cdots \circ G(A_{n1}, A_{n2}, \ldots, A_{nk}).
\end{aligned}$$

□

By Theorem 2.2, we show an n-variable version of the Fiedler theorem (2) via the ALM mean:

Corollary 2.1. *Let A be a positive invertible operator and $a_1, \ldots, a_n \in \mathbb{R}$ such as $\sum_{i=1}^n a_i = 0$. Then*

$$A^{a_1} \circ \cdots \circ A^{a_n} \geq I.$$

Proof. It follows from Theorem 2.2 and consistency with scalars (P1) for the ALM mean that

$$\begin{aligned}
\prod_{i=1}^{n} \circ\, A^{a_i} &= G(A^{a_1} \circ A^{a_2} \circ \ldots \circ A^{a_n}, A^{a_1} \circ A^{a_2} \circ \ldots \circ A^{a_n}, \ldots, A^{a_1} \circ A^{a_2} \circ \ldots \circ A^{a_n}) \\
&= G(A^{a_1} \circ A^{a_2} \circ \cdots \circ A^{a_n}, A^{a_2} \circ A^{a_3} \circ \cdots \circ A^{a_1}, \ldots, A^{a_n} \circ A^{a_1} \circ \cdots \circ A^{a_{n-1}}) \\
&\qquad\qquad \text{by commutativity of Hadamard multiplication} \\
&\geq G(A^{a_1}, A^{a_2}, \ldots, A^{a_n}) \circ G(A^{a_2}, A^{a_3}, \ldots, A^{a_1}) \circ \cdots \circ G(A^{a_n}, A^{a_1}, \ldots, A^{a_{n-1}}) \\
&\qquad\qquad \text{by Theorem 2.2} \\
&= A^{\sum a_i/n} \circ \cdots \circ A^{\sum a_i/n} \\
&\qquad\qquad \text{by consistency with scalars (P1) for the ALM mean} \\
&= I \circ \cdots \circ I = I \qquad \text{by } \textstyle\sum_{i=1}^n a_i = 0.
\end{aligned}$$

□

In [1, Corollary 16.1], Ando showed the estimate from above for the Hadamard product by diagonal matrices. Recall that $A \circ I$ is just the diagonal matrix formed from a matrix A. As an application of Theorem 2.2, we give a simple proof of its operator version:

Corollary 2.2. *Let $A_1, A_2, \ldots, A_n$ be positive operators for $n \geq 2$. Then*

$$\prod_{i=1}^{n} \circ A_i \leq \prod_{i=1}^{n} (A_i^n \circ I)^{\frac{1}{n}}.$$

Proof. By consistency with scalars (P1), we have $A_1 = G(A_1^n, I, \ldots, I), \ldots, A_n = G(I, \ldots, I, A_n^n)$. Hence it follows that

$$\begin{aligned}
\prod_{i=1}^{n} \circ A_i &= G(A_1^n, I, \ldots, I) \circ G(I, A_2^n, I, \ldots, I) \circ \cdots \circ G(I, \ldots, I, A_n^n) \\
&\leq G(A_1^n \circ I \circ \cdots I, I \circ A_2^n \circ I \circ \cdots \circ I, \ldots, I \circ \cdots \circ I \circ A_n^n) \\
&\qquad \text{by Theorem 2.2} \\
&= G(A_1^n \circ I, A_2^n \circ I, \ldots, A_n^n \circ I) \\
&= \prod_{i=1}^{n} (A_i^n \circ I)^{\frac{1}{n}} \qquad \text{by consistency with scalars (P1) of the ALM mean}
\end{aligned}$$

and so the proof is complete. $\square$

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