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Preferential Attachment Random Graphs with General Weight Function

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Abstract. Start with graph $G_0 \equiv \{V_1, V_2\}$ with one edge connecting the two vertices V_1, V_2. Now create a new vertex V_3 and attach it (i.e., add an edge) to V_1 or V_2 with equal probability. Set $G_1 \equiv \{V_1, V_2, V_3\}$. Let $G_n \equiv \{V_1, V_2, \ldots, V_{n+2}\}$ be the graph after n steps, $n \geq 0$. For each i, $1 \leq i \leq n+2$, let $d_n(i)$ be the number of vertices in G_n to which V_i is connected. Now create a new vertex V_{n+3} and attach it to V_i in G_n with probability proportional to $w(d_n(i))$, $1 \leq i \leq n+2$, where $w(\cdot)$ is a function from $N \equiv \{1, 2, 3, \ldots\}$ to $(0, \infty)$. In this paper, some results on behavior of the degree sequence $\{d_n(i)\}_{n\geq 1, i\geq 1}$ and the empirical distribution $\{\pi_n(j) \equiv \frac{1}{n}\sum_{i=1}^{n} I(d_n(i) = j)\}_{n\geq 1}$ are derived. Our results indicate that the much discussed power-law growth of $d_n(i)$ and power law decay of $\pi(j) \equiv \lim_{n\to\infty} \pi_n(j)$ hold essentially only when the weight function $w(\cdot)$ is asymptotically linear. For example, if $w(x) = cx^2$ then for $i \geq 1$, $\lim_n d_n(i)$ exists and is finite with probability (w.p.) 1 and $\pi(j) \equiv \delta_{j1}$, and if $w(x) = cx^p$, $1/2 < p < 1$ then for $i \geq 1$, $d_n(i)$ grows like $(\log n)^q$ where $q = (1-p)^{-1}$. The main tool used in this paper is an embedding in continuous time of pure birth Markov chains.

1. Introduction

The following random graph sequence has been suggested as a model for many real-world networks such as the Internet.

Start with graph $G_0 \equiv \{V_1, V_2\}$ with two vertices V_1 and V_2 and one edge connecting them. Now create a new vertex V_3 and connect it to one of V_1 or V_2 with equal probability. Set $G_1 \equiv \{V_1, V_2, V_3\}$. Let $\mathbf{d}_1 = \{d_1(i), 1 \leq i \leq 3\}$ be the vector of degrees in G_1, i.e., $d_1(i)$ is the number of edges in G_1 that connect to V_i, $i = 1, 2, 3$. Now add a new vertex V_4 and connect it to V_i in G_1 with probability

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proportional to $w(d_1(i))$, $i = 1, 2, 3$, where $w(\cdot)$ is a function from $\mathbb{N} \equiv \{1, 2, \ldots\}$ to $(0, \infty)$. Set $G_2 \equiv \{V_i, 1 \leq i \leq 4\}$ and let $\mathbf{d}_2 \equiv \{d_2(i), 1 \leq i \leq 4\}$ be the vector of degrees in G_2.

Continuing this let $G_n \equiv \{V_i : 1 \leq i \leq n+2\}$ be the graph at step n with degree vector

$$\mathbf{d}_n \equiv \{d_n(i) : 1 \leq i \leq n+2\}, \tag{1.1}$$

where $d_n(i)$ is the number of vertices in G_n to which V_i is connected. Now add vertex V_{n+3} and connect it to vertex V_i of G_n with probability

$$\frac{w(d_n(i))}{\sum_{j=1}^{n+2} w(d_n(j))}, \; 1 \leq i \leq n+2.$$

Set $G_{n+1} \equiv \{V_i : 1 \leq i \leq n+3\}$, and so on. The object of the study in this paper is the limiting behavior as $n \to \infty$ of the *degree vector sequence* $\mathbf{d}_n$ defined in (1.1) and the *empirical distribution of the degrees* $\pi_n \equiv \{\pi_n(j)\}_{j \geq 1}$, where

$$\pi_n(j) \equiv \frac{1}{(n+2)} \sum_{i=1}^{n+2} I(d_n(i) = j), \; j \geq 1. \tag{1.2}$$

Typically, the weight function $w(\cdot)$ is nondecreasing and hence this model is referred to as a *preferential attachment model* in the literature. Albert and Barabasi considered the special case $w(x) \equiv x$ and claimed that this simple model explains the empirically observed features in large networks such as the power law decay of the degree distributions, small diameter, etc. (see [Albert and Barabasi 02, Barabasi and Albert 99]). These were established rigorously for this special case $w(x) \equiv x$ in the works of Bollobás, Riordan, Spencer, and Tardos and others (see [Bollobás et al. 01, Bollobás and Riordan 03, Cooper and Frieze 03]).

There is now an extensive literature on the preferential attachment model. The recent paper of Oliveira and Spencer [Oliveira and Spencer 05] and the books of Durrett [Durrett 06] and Chung and Lu [Chung and Lu 06] have extensive bibliographies on this subject.

More recently Athreya et al. considered the general linear case $w(x) \equiv \alpha x + \beta$, $\alpha > 0$, $\beta > 0$, allowing at step n a random number X_n of connections of the new vertex V_{n+3} to the chosen vertex V_i in $G_n \equiv \{V_i : 1 \leq i \leq n+2\}$, where $\{X_n\}_{n \geq 0}$ are independent and identically-distributed random variables, and established a number of results similar to those of Theorem 2.3 of the present paper [Athreya et al. 08].

The general model we propose above, i.e. with a general weight function $w(\cdot)$, has also been studied by Krapivsky and Redner [Krapivsky and Redner 01], Oliveira and Spencer [Oliveira and Spencer 05], Drinea, Enachescu, and Mitzenmacher [Drinea et al 01], and others. Rather than summarizing the results from all these papers, we focus on the latest one by Oliveria and Spencer [Oliveira and Spencer 05]. For the case $w(x) = (x+1)^p$, $p > 1$, with $1 + \frac{1}{k} < p < 1 + 1/(k-1)$ for some integer $k > 1$, they show that the eventual graph has the property that there is one distinguished vertex v that has an infinite number of descendants while all but a finite number of nodes have less than k descendants. They also establish a refinement of this.

The case $w(x) \sim cx^p$ with $p < 1$ has been treated by Rudas, who shows that the degree distribution decays like $\exp\big(-c\sum_{j=0}^{k-1}(w(j))^{-1}\big)$ for some $0 < c < \infty$ [Rudas 04]. A result related to this is Theorem 2.2 of this paper, which asserts that $d_n(i)$ grows like $(\log n)^q$, where $q = (1-p)^{-1}$.

The present paper treats the general case of the weight function $w(\cdot)$ in a unified manner. We have results for the three cases: $w(\cdot)$ asymptotically superlinear, linear, and sublinear. We wish to emphasize that we assume only that $w(\cdot)$ has the appropriate growth rate and do not assume an exact form for $w(\cdot)$ except in the linear case. Our method involves an embedding of the discrete sequence of graphs $\{G_n\}_{n\geq 0}$ in a continuous time setting involving a sequence of pure birth continuous time Markov chains and then using some recently established limit theorems for such processes (see [Athreya 08]).

Historically speaking, the technique of embedding a discrete sequence of random variables in continuous time processes has been known for at least forty years. The present author used this technique in his PhD thesis to prove limit theorems for the well-known Polya urn scheme and its generalized versions (see [Athreya 67, Athreya and Karlin 68, Athreya and Ney 04]). For applications of this embedding technique to clinical trials, see [Rosenberger 02].

Embedding methods similar to the one in the author's thesis [Athreya 67] have been used to study random graph sequence growth properties by a number of authors (see [Durrett 06, Chung and Lu 06]).

Our results indicate a natural trichotomy in the limiting behavior depending on whether $w(\cdot)$ is asymptotically superlinear, sublinear, or linear (see Theorems 2.1, 2.2, and 2.3).

Our results suggest that the much discussed power-law growth of the degree sequence $d_n(\cdot)$ and the power-law decay of the limiting distribution of the degree sequence occur essentially only when $w(\cdot)$ is asymptotically linear.

In the superlinear case, i.e. $\sum_{n=1}^{\infty} 1/w(n) < \infty$, we show (see Theorem 2.1) that there are only two possibilities:

1. either each vertex stops getting any new connections after some random time, i.e., for all i, $d_n(i)$ has a finite limit as $n \to \infty$,

2. or for each vertex V_i, there is positive probability that eventually all new vertices choose only V_i and hence for all $j \neq i$, $d_n(j)$ has a finite limit as $n \to \infty$, and further, for large j, V_j has no descendants.

And in either case the empirical degree distribution converges to the delta distribution at 1.

In the linear case (see Theorem 2.3) the results from [Athreya et al. 08] carry over.

In the sublinear case (see Theorem 2.2), when $w(x) \sim cx^p$ with $1/2 < p < 1$, for each i, $d_n(i)$ grows like $(\log n)^q$ where $q = (1-p)^{-1}$.

2. Main Results

Let $\{G_n, \mathbf{d}_n, \pi_n, w(\cdot)\}_{n \geq 0}$ be as in the previous section.

Theorem 2.1. (Superlinear Case.) *Let*

$$\sum_{n=1}^{\infty} \frac{1}{w(n)} < \infty. \tag{2.1}$$

(a) If, in addition to (2.1),

$$\sum_{n=1}^{\infty} \frac{n}{n + w(n)} = \infty, \tag{2.2}$$

then, $\forall\ i \geq 1$,

$$\lim_{n \to \infty} d_n(i) \equiv \xi_i < \infty \quad \textit{exists w.p. 1.} \tag{2.3}$$

(b) If, in addition to (2.1),

$$\sum_{n=1}^{\infty} \frac{n}{n + w(n)} < \infty, \tag{2.4}$$

then, $\forall\ i \geq 1$, $p_i \equiv P(A_i) > 0$ *where*

$$\begin{aligned} A_i \equiv \{\exists\ &\textit{a random } n_i < \infty \textit{ such that } \forall\ n \geq n_i, \textit{ the vertex in } G_n \\ &\textit{that gets connected to the new vertex } V_{n+3} \textit{ is } V_i\}. \end{aligned} \tag{2.5}$$

(c) Under (2.1),

$$\forall\ j \geq 1,\ \pi_n(j) \equiv \frac{1}{n} \sum_{i=1}^{n+2} I(d_n(i) = j) \to \delta_{1j} \tag{2.6}$$

w.p. 1 where $\delta_{11} = 1$ *and* $\delta_{1j} = 0$ *for* $j \neq 1$.

Corollary 2.2. *Let $w(n) \sim cn^p$ for some $c > 0$ and $p > 1$.*

(a) If $1 < p \leq 2$, then $\forall\ i \geq 1$

$$\lim_{n\to\infty} d_n(i) \equiv \xi_i < \infty \quad \text{exists w.p. 1.}$$

(b) If $p > 2$, then $\forall\ i \geq 1$, there is positive probability p_i that for some random $n_i < \infty$, and for all $n \geq n_i$,

$$d_n(i) = d_{n_i}(n_i) + (n - n_i).$$

Remark 2.3. Condition (2.1) suggests that $w(n)$ grows faster than at a linear rate and hence we say that $w(\cdot)$ is *superlinear*. If (2.1) and (2.2) hold, then (2.3) suggests that for large n, $d_n(i)$ does not grow at all, while if (2.1) and (2.4) hold, then (2.5) suggests that except possibly at one vertex, the degree $d_n(i)$ does not grow at all and for all but a finite number, the degree stays at one. Finally, (2.6) says that $\{\pi_j \equiv \lim_n \pi_n(j)\}$ is degenerate at 1.

Theorem 2.4. (Sublinear Case.) *Let $w(\cdot)$ satisfy*

$$\lim_{n\to\infty} \frac{w(n)}{cn^p} = 1 \quad \text{for some} \quad c > 0, \quad \frac{1}{2} < p < 1.$$

Then, there exist a nonrandom sequence $\{c(n)\}_{n\geq 1}$ and a constant $0 < \alpha < \infty$ such that

(a)

$$\forall\ i \geq 1, \ \frac{d_n(i)}{(c(n))^q} \to \alpha \quad \text{w.p. 1, where} \quad q = (1-p)^{-1}.$$

(b)

$$0 < c_1 \equiv \underline{\lim} \frac{c(n)}{\log n} \leq \overline{\lim} \frac{c(n)}{\log n} = c_2 < \infty.$$

Remark 2.5. This result suggests that if $w(\cdot)$ grows at a *sublinear* rate then $d_n(i)$ grows like $(\log n)^q$ and hence there is no power-law growth in this case also.

Theorem 2.6. *Let $w(n) = cn + \beta$, $c > 0$, $c > -\beta$. Let $d_n(i)$ and $\pi_j(n)$ be as in (1.1) and (1.2), respectively. Then,*

(a) $\exists$ *independent absolutely continuous positive random variables* $\{\xi_i\}_{i\geq 1}$ *and* V *such that* $\forall\ i \geq 1$,

$$\frac{d_n(i)}{n^\theta} \to \xi_i V \quad \textit{w.p. 1 as} \quad n \to \infty,$$

where $\theta = c/(2c+\beta)$;

(b) *if*

$$M_n \equiv \max\{d_n(i) : 1 \leq i \leq (n+2\}$$

and I_n *is an index such that* $d_n(I_n) = M_n$, *then*

$$\lim_{n\to\infty} \frac{M_n}{n^\theta} \equiv \left(\max_{1\leq i<\infty} \xi_i\right) V < \infty \quad \textit{w.p. 1}$$

and

$$\lim_{n\to\infty} I_n \equiv I < \infty \quad \textit{exists w.p. 1};$$

(c) *let*

$$p_j(y) \equiv P(Z(y) = j),\ \ j \geq 1,$$

where $\{Z(y) : y \geq 0\}$ *is a pure birth Markov process with* $Z(0) = 1$ *and birth rates* $\lambda_i \equiv ci + \beta$. *Then,* $\forall\ j \geq 1$, *as* $n \to \infty$,

$$\pi_n(j) \longrightarrow \pi_j \equiv \alpha \int_0^\infty p_j(y) e^{-\alpha y} dy \quad \textit{in probability,}$$

where $\alpha = (2c+\beta)^{-1}$, *and further*

$$\lim_{j\to\infty} j^{-(3+\beta/c)} \pi_j \equiv \gamma$$

exists and $0 < \gamma < \infty$.

Remark 2.7. Theorem 2.6 confirms for the linear weight function the power-law growth of the degrees $d_n(i)$ as well as that of the maximal degree and the power-law decay of $\{\pi_j\}$, phenomena observed empirically in some networks such as social networks and the Internet (see [Albert and Barabasi 02]). Further, part (b) says that the vertex that has the maximal degree freezes in time for large n.

Theorem 2.8. *Let* $w(n) \equiv c > 0$. *Then,*

(a)

$$\forall\ i \geq 1,\ \frac{d_n(i)}{(\log n)} \to 1 \quad \textit{w.p. 1.}$$

(b)

$$\forall\ j \geq 1,\ \pi_n(j) \equiv \frac{1}{n}\sum_{i=1}^{n+2} I(d_n(i) = j) \to \frac{1}{2^j} \equiv \pi_j \quad \textit{in probability.}$$

Remark 2.9. Note that in this case π_j decays geometrically fast. This case was treated by Erdős and Rényi in the 1950s (see [Durrett 06]).

These results are established by an embedding of the discrete time random graph sequence $\{G_n, d_n\}_{n\geq 0}$ in a sequence of continuous time pure birth Markov chains. This embedding is treated in the next section. The proofs of the main results (Theorems 2.1, 2.4, 2.6, and 2.8) are given in the last section.

3. The Embedding Theorem

Definition 3.1. A pure birth process with rate function $w(\cdot)$ is a continuous time Markov chain $\{Z(t) : t \geq 0\}$ with state space $N^+ \equiv \{0, 1, 2, \ldots\}$ and infinitesimal generator $A \equiv \big((a_{ij})\big)$ with $a_{ii} = -w(i)$, $a_{ij} = w(i)$ if $j = i+1$, and $a_{ij} = 0$ if $j \neq i$ or $i+1$. Assume that $w(i) > 0$ for all $i \geq 0$. It is constructed as follows. Let $Z(0) = i_0$. Let $\{L_j\}_{j\geq 0}$ be independent exponential random variables with $EL_j = \big(w(i_0 + j)\big)^{-1}$, $j \geq 0$. Let $T_0 = 0$, $T_j = \sum_{i=0}^{j-1} L_i$, $j \geq 1$. Now set

$$Z(t) = \begin{cases} i_0, & T_0 = 0 \leq t < T_1, \\ i_0 + 1, & T_1 \leq t < T_2, \\ i_0 + j, & T_j \leq t < T_{j+1}, \\ \vdots & \end{cases} \tag{3.1}$$

The sequences $\{T_j\}_{j\geq 0}$ are called the *jump or birth times* of $\{Z(t) : t \geq 0\}$. Let $T_\infty \equiv \lim_{n\to\infty} T_n$. Then for any $\lambda \geq 0$

$$E\left(e^{-\lambda T_\infty}\right) = \prod_{j=0}^{\infty} \frac{w(i_0 + j)}{\lambda + w(i_0 + j)}. \tag{3.2}$$

Thus, if $\sum_{i=1}^{\infty} 1/w(i) = \infty$ then $E\left(e^{-\lambda T_\infty}\right) = 0\ \forall\ \lambda > 0$ and hence $P(T_\infty = \infty) = 1$. On the other hand, if $\sum_{i=1}^{\infty} 1/w(i) < \infty$ then $E\left(e^{-\lambda T_\infty}\right) > 0$ for $\forall\ \lambda > 0$ and $\lim_{\lambda\downarrow 0} E\left(e^{-\lambda T_\infty}\right) = 1$ and hence $P(T_\infty < \infty) = 1$. Summarizing this we get the following well-known nonexplosion criterion.

Proposition 3.2. *Let $\{Z(t) : t \geq 0\}$ be as in (3.1). Then, $P(T_\infty = \infty) = 0$ or 1 accordingly as $\sum_1^{\infty} 1/w(i) = \infty$ or $< \infty$.*

Now let $\{Z_i(t) : t \geq 0\}_{i \geq 1}$ be independent and identically-distributed copies of $\{Z(t) : t \geq 0\}$ as in (3.1) with $Z(0) = 1$. Let, $\forall\, i \geq 1$, $\{T_{ij}\}_{j \geq 0}$ be the jump times of $\{Z_i(t) : t \geq 0\}$. Now define a new sequence of random times $\{\tau_n\}_{n \geq 0}$ as follows. Let

$$\begin{aligned}\tau_0 &\equiv 0,\\ \tau_1 &\equiv \min\{T_{11}, T_{21}\},\end{aligned}$$

the first time a birth takes place in either of the two processes $\{Z_i(t) : t \geq 0\}$, $i = 1, 2$. Now "start" the process $\{Z_3(t) : t \geq 0\}$ at time τ_1.

Let τ_2 be the first time after τ_1 that a birth takes place in any of the three processes $\{Z_1(t) : t \geq 0\}$, $\{Z_2(t) : t \geq 0\}$, and $\{Z_3(t - \tau_1) : t \geq \tau_1\}$. Now "start" the process $\{Z_4(t) : t \geq 0\}$ at time τ_2. Let τ_3 be the first time after τ_2 that a birth takes place in any of the four processes $\{Z_1(t) : t \geq 0\}$, $\{Z_2(t) : t \geq 0\}$, $\{Z_3(t - \tau_1) : t \geq \tau_1\}$, and $\{Z_4(t - \tau_2) : t \geq \tau_2\}$, and so on. It can be checked that $\{\tau_i\}_{i \geq 1}$ satisfy the following recurrence relation: let $\tau_{-1} = 0 = \tau_0$ and

$$\begin{aligned}\tilde{T}_{ij} &= \tau_{i-2} + T_{ij},\ j \geq 0,\ i \geq 1,\\ \tau_1 &= \min\{\tilde{T}_{ij}, \tilde{T}_{2j} : \tilde{T}_{ij} > \tau_0,\ j \geq 1\},\\ \tau_2 &= \min\{\tilde{T}_{ij} : \tilde{T}_{ij} > \tau_1,\ i = 1, 2, 3,\ j \geq 1\},\end{aligned}$$

and for $n \geq 1$

$$\tau_n = \min\{\tilde{T}_{ij} : j \geq 1,\ 1 \leq i \leq n+1,\ \tilde{T}_{ij} > \tau_{n-1}\}. \tag{3.3}$$

Theorem 3.3. (The Embedding Theorem.) *Let $\{Z_i(t) : t \geq 0\}_{i \geq 0}$ and $\{\tau_n\}_{n \geq 0}$ be as defined above and in (3.3). Let*

$$\tilde{d}_n(i) \equiv Z_i(\tau_n - \tau_{i-2}),\ 1 \leq i \leq n+2,$$

and

$$\tilde{\mathbf{d}}_n \equiv (\tilde{d}_n(i),\ 1 \leq i \leq n+2),\ n \geq 0.$$

Let $\mathbf{d}_n$, $n \geq 0$ be the degree vector sequence *as defined in (1.1) for the random graph sequence $\{G_n\}_{n \geq 0}$ in Section 1. Then, the two sequences of random vectors $\{\mathbf{d}_n : n \geq 0\}$ and $\{\tilde{\mathbf{d}}_n : n \geq 0\}$ have the same distribution.*

Proof. By construction, $\{\mathbf{d}_n : n \geq 0\}$ has the Markov property. Next, by the strong Markov property of the $\{Z(t) : t \geq 0\}$, the sequence $(\tilde{\mathbf{d}}_n)_{n \geq 0}$ also has the Markov property. Since $\mathbf{d}_0 = (1, 1) = \tilde{\mathbf{d}}_0$ w.p. 1, it suffices to show that the transition probability mechanism at stage n is the same for both sequences for all $n \geq 0$. Let $N \equiv \{1, 2, 3, \ldots\}$. Then, for each n, $\mathbf{d}_n$ and $\tilde{\mathbf{d}}_n \in N^{n+2}$. Consider the distribution of $\mathbf{d}_{n+1}$ given $\mathbf{d}_n = \mathbf{x}_n \equiv (x_1, x_2, \ldots, x_{n+2})$. From the model description in Section 1, it follows that

$$P\Big(\mathbf{d}_{n+1} = (x_1, x_2, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_{n+2}, 1) \mid \mathbf{d}_n = \mathbf{x}_n\Big) = \frac{w(x_i)}{\sum_{j=1}^{n} w(x_j)} \tag{3.4}$$

for $i = 1, 2, \ldots, n+2$. Similarly, given all the information unto time τ_n, the "birth" at time τ_{n+1} occurs in the process $\{Z_i(t) : t \geq 0\}$ with probability

$$\frac{w(Z_i(\tau_n - \tau_{i-2}))}{\sum_{j=1}^{n+2} w(Z_j(\tau_n - \tau_{j-2}))} \quad \text{for} \quad i = 1, 2, \ldots, n+2.$$

This is due to the fact that if $Y_1, Y_2, \ldots, Y_k$ are independent exponential random variables with means $\{\lambda_i^{-1}\}_{i=1}^n$,then $Y = \min(Y_1, Y_2, \ldots, Y_k)$ is also exponentially distributed with mean

$$\left(\sum_1^k \lambda_i\right)^{-1} \text{ and } P(Y = Y_i) = \frac{\lambda_i}{(\sum_1^k \lambda_i)}, \; i = 1, 2, \ldots, k.$$

Thus, for any $\mathbf{x}_n \equiv (x_1, x_2, \ldots, x_{n+2})$,

$$P\left(\tilde{\mathbf{d}}_n = (x_1, x_2, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_{n+2}) \mid \tilde{\mathbf{d}}_n = \mathbf{x}_n\right) = \frac{w(x_i)}{\sum_{j=1}^{n+2} w(x_j)}. \tag{3.5}$$

Now (3.4) and (3.5) show that the conditional distribution of $\mathbf{d}_{n+1}$ given $\mathbf{d}_n = \mathbf{x}_n$ is the same as the conditional distribution of $\tilde{\mathbf{d}}_{n+1}$ given $\tilde{\mathbf{d}}_n = \mathbf{x}_n$ for any $\mathbf{x}_n \in N^{n+2}$. This completes the proof. $\square$

Next we establish a few key results on the random sequences $\{T_{i\infty} \equiv \lim_n T_{in}\}_{i \geq 1}$ and $\tau_\infty \equiv \lim_{n\to\infty} \tau_n$.

Theorem 3.4. *Let*

$$\sum_{n=1}^{\infty} \frac{1}{w(n)} < \infty.$$

Then,

(a) $\forall\, i,\; T_{i\infty} < \infty$ *w.p. 1,*

(b) $\forall\, i < j,\; P(\tau_{i-2} + T_{i\infty} = \tau_{j-2} + T_{j\infty}) = 0.$

Proof. Part (a) follows from Proposition 3.2. Alternately,

$$ET_{i\infty} = \lim_{n\to\infty} ET_{in} = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{w(j)} = \sum_{j=1}^{\infty} \frac{1}{w(j)} < \infty,$$

and hence $P(T_{i\infty} < \infty) = 1$.

From the embedding and the definition of $\{\tau_n\}$ in (3.3), it follows that $\forall\ i$, n,

$$\tau_n \le \tau_{i-2} + T_{\infty},$$

and hence $\tau_\infty \le \tau_{i-2} + T_{i\infty}$.

For any $i < j$

$$P(\tau_{i-2} + T_{i\infty} = \tau_{j-2} + T_{j\infty}) = E\Big(P(T_{j\infty} + \tau_{j-2} = \tau_{i-2} + T_{i\infty} \mid \mathcal{F}_j)\Big),$$

where $\mathcal{F}_j$ is the σ-algebra generated by

$$\{Z_r(t - \tau_{r-2}),\ t \ge \tau_{r-2},\ 1 \le r \le j-1,\ \tau_{j-2}\}.$$

Since $T_{j\infty}$ is independent of $\mathcal{F}_j$ and has a continuous distribution and since $\tau_{i-2} - \tau_{j-2} + T_{i\infty}$ is $\mathcal{F}_j$ measurable,

$$P(T_{j\infty} = \tau_{i-2} - \tau_{j-2} + T_{i\infty} \mid \mathcal{F}_j) = 0 \quad \text{w.p. } 1.$$

Thus part (b) follows. □

Theorem 3.5. *Let*

$$\sum_{n=1}^{\infty} \frac{1}{w(n)} < \infty. \tag{3.6}$$

(a) Then the event

$$A_i \equiv \{\tau_\infty = \tau_{i-2} + T_{i\infty}\}$$

coincides with the event

$$\tilde{A}_i \equiv \Big\{\exists\ n_i < \infty \text{ random such that } \tau_n \in \{\tau_{i-2}+T_{ij}, j \ge 1\} \text{ for all } n \ge n_i\Big\}.$$

(b) If, in addition to (3.6),

$$\sum_{n=1}^{\infty} \frac{n}{n + w(n)} = \infty, \tag{3.7}$$

then $\forall\ i$, $\tau_\infty < \tau_{i-2} + T_{i\infty}$ w.p. 1.

(c) *If, in addition to (3.6),*

$$\sum_{n=1}^{\infty} \frac{n}{n+w(n)} < \infty, \tag{3.8}$$

then $\forall\ i,\ p_i \equiv P(A_i) > 0$.

Proof.

(a) If there are two subsequences, one each from

$$\{\tau_{i-2} + T_{ij}, j \geq 1\} \quad \text{and} \quad \{\tau_{\ell-2} + T_{\ell j}, j \geq 1\}$$

for some (i, ℓ), $i \neq \ell$ such that τ_n belongs to each of them infinitely often, then letting $n \to \infty$ would yield $\tau_\infty = \tau_{i-2} + T_{i\infty} = \tau_{\ell-2} + T_{\ell\infty}$. But, by Theorem 3.4(b), this event has probability zero. Thus, on the event A_i, $\{\tau_\infty < \tau_{\ell-2} + T_{\ell\infty}\}$ for every $\ell \neq i$ w.p. 1 and hence w.p. 1 on A_i, $\tau_n \in \{\tau_{i-2} + T_{ij}, j \geq 1\}$ for all large n. Conversely, on $\tilde{A}_i$, $\tau_\infty \equiv \lim \tau_n = \tau_{i-2} + T_{i\infty}$. Thus $\forall\ i$, $\tilde{A}_i = A_i$ w.p. 1 proving (a).

(b) Let A_{ik} be the event $\tau_n \in \{\tau_{i-2} + T_{ij}, j \geq 1\}$ for all $n \geq k$. Then $A_{ik} \to A_i$ as $k \to \infty$ and

$$P(A_{ik}) = E\left(\prod_{n=k}^{\infty} \frac{w(\tilde{d}_k(i)+n)}{\left(\sum_{\substack{j=1\\ j\neq i}}^{k} w(\tilde{d}_k(j)) + w(\tilde{d}_k(i)+n) + (n-k)w(1)\right)}\right).$$

Suppose (3.7) holds, then

$$\sum_{n=k}^{\infty} \frac{(n-k)w(1) + \sum_{\substack{j=1\\ j\neq i}}^{k} w(\tilde{d}_k(j))}{\left((n-k)w(i) + \sum_{\substack{j=1\\ j\neq i}}^{k} w(\tilde{d}_k(j)) + w(\tilde{d}_k(i_0)+n)\right)} = \infty.$$

Thus, $P(A_{ik}) = 0$. This being true $\forall\ k$, $P(A_i) = \lim_k P(A_{ik}) = 0$. This proves (b).

(c) Suppose (3.8) holds. Then, $\forall\ i,\ k$, $P(A_{ik}) > 0$. Since $A_{ik} \to A_i$ as $k \to \infty$, $P(A_i) > 0$. □

There is an open question: under (3.8) is $\sum_1^\infty P(A_i) = 1$?

Theorem 3.6. *Let*

$$\inf w(j) = \delta > 0. \tag{3.9}$$

(a) *Then, there is a nonrandom sequence* $\{c(n)\}_{n\geq 0}$ *such that* $\{\tau_n - c(n)\}_{n\geq 0}$ *is a* L^2 *bounded martingale and hence converges w.p. 1 and in mean square to a random variable* Y *with an absolutely continuous distribution.*

(b) *Suppose, in addition to (3.9),* $w(\cdot)$ *is* sublinear*, i.e., for some*

$$c > 0,\ |\beta| < \infty,\ w(n) \leq cn + \beta,\ n \geq 1. \tag{3.10}$$

Then,

(i)

$$\tau_n \to \infty \quad \text{w.p. 1.}$$

(ii)

$$\frac{1}{(2c+\beta)} \leq \varliminf_n \frac{c(n)}{\log n} \leq \varlimsup_n \frac{c(n)}{\log n} \leq \frac{1}{\delta}.$$

(iii) If $w(n) = cn + \beta$ *for all* $n \geq 1$*,*

$$\lim_n \frac{c(n)}{\log n} = \frac{1}{(2c+\beta)}.$$

Proof.

(a) By construction $\forall\ j \geq 1$, conditioned on $\mathcal{F}_j$ (defined in the proof of Theorem 3.4), $\tau_{j+1} - \tau_j$ has an exponential distribution with mean

$$\left(\sum_{k=1}^{(j+2)} w(d_j(k))\right)^{-1} \equiv b_j, \text{ say.}$$

Then, $\{\delta_j \equiv (\tau_{j+1} - \tau_j) - b_j, \mathcal{F}_j\}_{j\geq 0}$ is a martingale difference sequence such that $E(\delta_j^2) = b_j^2$. From (3.9), $b_j^2 \leq (j+2)^2\delta^2$, implying that

$$\sum_{j=1}^{\infty} b_j^2 \leq \frac{1}{\delta^2}\sum_{j=1}^{\infty}\frac{1}{j^2} < \infty.$$

Thus

$$\left\{\sum_{0}^{n-1} \delta_j \equiv \tau_n - c(n),\ n \geq 0,\ \mathcal{F}_n\right\}_{n\geq 0}$$

is a L^2 bounded martingale where $c(n) = \sum_{j=0}^{n-1} b_j,\ n \geq 1$. This implies that $\{\tau_n - c(n)\}_{n\geq 0}$ converges w.p. 1 and in mean square (see [Athreya and

Lahiri 06, Theorem 3.3.9]). If $Y \equiv \lim_n(\tau_n - c(n))$ and $Y_1 = Y - (\tau_1 - c(1))$, then $\tau_1 - c(1)$ and Y_1 are independent with $\tau_1 - c(1)$ having an absolutely continuous distribution and hence Y has an absolutely continuous distribution.

(b) Since (3.10) holds,

$$\begin{aligned} \sum_{j=0}^{n-1} b_j &\geq \frac{1}{(2c+\beta)} \sum_{j=1}^{n-1} \frac{1}{j} \\ \implies c(n) &\geq \frac{1}{(2c+\beta)} \sum_{j=1}^{n-1} \frac{1}{j} \\ \implies \frac{\lim}{n} \frac{c(n)}{\log n} &\geq \frac{1}{(2c+\beta)}. \end{aligned}$$

Also, since (3.9) holds,

$$c(n) = \sum_{j=0}^{n-1} b_j \leq \frac{1}{\delta} \sum_{j=1}^{n} \frac{1}{j}$$

and hence

$$\varlimsup_n \frac{c(n)}{\log n} \leq \frac{1}{\delta}$$

proving (b).

If $w(n) = cn + \beta$, $n \geq 1$, then $b_j^{-1} = (2c+\beta)(j+2)$, $\forall\, j \geq 0$

$$\implies c(n) = \frac{1}{(2c+\beta)} \sum_{j=0}^{n-1} \frac{1}{(j+2)} \implies \lim_n \frac{c(n)}{\log n} = \frac{1}{(2c+\beta)}. \qquad \square$$

4. Proofs of Main Results

The following results proved in [Athreya 08] will be needed in the proofs of Theorems 2.1, 2.4, 2.6, and 2.8.

Theorem 4.1. *Let $\{Z(t) : t \geq 0\}$ be a pure birth process as defined in Definition 3.1 with $Z(0) = 1$.*

(a) Let $\sum_{i=1}^{\infty} \frac{1}{w(i)} = \infty$, $\sum_{i=1}^{\infty} \frac{1}{w^2(i)} < \infty$. Then, for some $0 < c < \infty$,

$$\lim_{t\to\infty} Z(t)e^{-ct} \equiv \xi \text{ exists w.p. 1}$$

with $P(0 < \xi < \infty) = 1$ *iff*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{w(i)} - \frac{1}{ci} \right) \quad \textit{exists and is finite.}$$

Further, ξ *has an absolutely continuous distribution on* $(0, \infty)$.

(b) *Let* $\sum_{i=1}^{\infty} \frac{1}{w(i)} = \infty$ *and*

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{w^2(i)} \right) \Big/ \left(\sum_{i=1}^{n} \frac{1}{w(i)} \right)^2 = 0.$$

Then, for some $0 < c < \infty$, $0 < q < \infty$

$$\lim_{t\to\infty} \frac{Z(t)}{t^q} = c \quad \textit{in probability}$$

$$\textit{iff} \quad \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{w(i)} \right) n^{-p} = c^{-1/p}, \ p = \frac{1}{q}.$$

Proof of Theorem 2.1. By the embedding theorem (Theorem 3.3), to show (2.3) it suffices to show that $\forall\ i \geq 1$, $\lim_{n\to\infty} \tilde{d}_n(i) \equiv \tilde{\xi}_i < \infty$ exists w.p. 1 where $\tilde{d}_n(i) \equiv Z_i(\tau_n - \tau_{i-2})$ as defined in (3.2). Now from Theorem 3.4, $\forall\ i \geq 1$, $T_{i,\infty} < \infty$ w.p. 1. Further, from Theorem 3.5(b), $\forall\ i$, $\tau_\infty - \tau_{i-2} < T_{i,\infty}$ w.p. 1. Thus, $\tilde{d}_n(i) \uparrow Z_i(\tau_\infty - \tau_{i-2}) \equiv \tilde{\xi}_i < Z_i(T_{i,\infty} - \tau_{i-2}) < \infty$ w.p. 1. This proves (a).

From Theorem 3.5(c) and (a), $\forall\ i \geq 1$, $P(\tau_\infty - \tau_{i-2} = T_{i,\infty}) > 0$ and the event $\{\tau_\infty - \tau_{i-2} = T_{i,\infty}\}$ coincides with the event A_i.

Thus, $\forall\ i \geq 1$, $P(A_i) > 0$, proving (b).

By Theorem 3.4,

$$\tilde{d}_n(i) \uparrow \tilde{\xi}_i \equiv Z_i(\tau_\infty - \tau_{i-2}) \quad \text{w.p. } 1.$$

Also by Theorem 3.4, at most one event A_i happens. So w.p. 1 except possibly for one random index J, $\tau_\infty - \tau_{i-2} < T_{i,\infty}$ for all $i \neq J$. Hence, $\forall\ j \geq 1$, $i \geq 1$,

$$\begin{aligned}
&\left| I(\tilde{d}_n(i) = j) - I(\tilde{\xi}_i = j) \right| I(i \neq J) \\
&\quad \leq I\Big(|Z_i(\tau_n - \tau_{i-2}) - Z_i(\tau_\infty - \tau_{i-2})| \geq 1 \Big) I(J \neq i) \\
&\quad \leq \Big(I\Big(\sup_{0<u<v<\delta} |Z_i(u) - Z_i(v)| \geq 1 \Big) + I(\tau_n - \tau_{i-2} \geq \delta) \Big) I(J \neq i).
\end{aligned}$$

Thus

$$\Big|\frac{1}{n}\sum_{i=1}^{n}\Big(I(\tilde{d}_n(i)=j)-I(\tilde{\xi}_i=j)\Big)\Big|$$
$$\leq \frac{1}{n}+\frac{1}{n}\sum_{i=1}^{n}\Big(I\Big(\sup_{0<u<v<\delta}|Z_i(u)-Z_i(v)|\geq 1\Big)+I(\tau_n-\tau_{i-2}\geq\delta)\Big).$$

Now

$$\frac{1}{n}\sum_{i=1}^{n} I(\tau_n-\tau_{i-2}\geq\delta)\leq \epsilon+\frac{1}{n}\sum_{i>n\epsilon} I(\tau_n-\tau_{i-2}\geq\delta).$$

Since $\tau_n\uparrow\tau_\infty<\infty$, it follows that $\forall\ \delta>0,\ \epsilon>0$

$$\sup_{i>n\epsilon} I(\tau_n-\tau_{i-2}\geq\delta)\leq I(\tau_n-\tau_{n\epsilon-2}\geq\delta)\to 0,\ \text{w.p. 1 as } n\to\infty.$$

Also, by the strong law of large numbers (SLLN), w.p. 1,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} I\Big(\sup_{0<u<v<\delta}|Z_i(u)-Z_i(v)|\geq 1\Big)$$
$$=P\Big(\sup_{0<u<v<\delta}|Z_i(u)-Z_i(v)|\geq 1\Big)\equiv p_1(\delta),\quad \text{say.}$$

Thus, w.p. 1, for any $\delta>0$, $\epsilon>0$,

$$\overline{\lim_{n\to\infty}}\Big|\frac{1}{n}\sum_{i=1}^{n}\Big(I(\tilde{d}_n(i)=j)-I(\tilde{\xi}_i=j)\Big)\Big|\leq p_1(\delta)+\epsilon.$$

Now as $\delta\downarrow 0$, $p_1(\delta)\downarrow 0$. So, w.p. 1

$$\overline{\lim_{n\to\infty}}\Big|\frac{1}{n}\sum_{i=1}^{n} I(\tilde{d}_n(i)=j)-\frac{1}{n}\sum_{i=1}^{n} I(\tilde{\xi}_i=j)\Big|=0. \tag{4.1}$$

Next $I(\tilde{\xi}_i=j)=I(Z_i(\tau_\infty-\tau_{i-2})=j)$, and as before, for $\delta>0$.

$$\Big|\frac{1}{n}\sum_{i=1}^{n}\big(I(\tilde{\xi}_i=j)-I(Z_i(0)=j)\big)\Big|$$
$$\leq\frac{1}{n}+\frac{1}{n}\sum_{i=1}^{n} I(Z_i(\delta)-Z_i(0)\geq 1)+\frac{1}{n}\sum_{i=1}^{n} I(\tau_\infty-\tau_{i-2}\geq\delta).$$

Again, by SLLN, w.p. 1,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} I(Z_i(\delta)-Z_i(0)\geq 1)\to P(|Z_i(\delta)-Z_i(0)|\geq 1)\equiv p_2(\delta),\ \text{say.}$$

Also, since $\tau_i \uparrow \tau_\infty < \infty$ w.p. 1

$$\frac{1}{n}\sum_{i=1}^{n} I(\tau_\infty - \tau_{i-2} \geq \delta) \to 0 \quad \text{w.p. } 1, \ \forall\, \delta > 0.$$

Now as $\delta \downarrow 0$, $p_2(\delta) \downarrow 0$. So w.p. 1

$$\varlimsup_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^{n} I(\tilde{d}_n(i) = j) - \frac{1}{n}\sum_{i=1}^{n} I(\tilde{\xi}_i = j)\right| = 0. \tag{4.2}$$

Now by hypothesis $Z_i(0) = 1\ \forall\, i \geq 1$. So

$$\frac{1}{n}\sum_{i=1}^{n} I(Z_i(0) = j) \to \delta_{1j} \equiv \begin{cases} 1 & \text{if} \quad j = 1, \\ 0 & \text{if} \quad j \neq 1. \end{cases} \tag{4.3}$$

From (4.1)–(4.3) it follows that, w.p. 1,

$$\tilde{\pi}_n(j) \equiv \frac{1}{n}\sum_{i=1}^{n} I(\tilde{d}_n(i) = j) \to \delta_{1j}.$$

Now by the embedding theorem (i.e., Theorem 3.3),

$$\pi_n(j) \equiv \frac{1}{n}\sum_{i=1}^{n} I(d_n(i) = j) \to \delta_{1j} \quad \text{w.p. } 1,$$

proving Theorem 2.1(c).

Thus Theorem 2.1 is fully proved. □

Proof of Theorem 2.4. By Theorem 4.1(b) there exist α, $0 < \alpha < \infty$, such that $\forall\, i \geq 1$,

$$\lim_{t\uparrow\infty} \frac{Z_i(t)}{t^q} = \alpha \quad \text{exists w.p. } 1, \ 0 < \alpha < \infty.$$

The sequence $w(n) \sim c_1 n^p$, $c > 0$, $\frac{1}{2} < p < 1$ implies that $w(\cdot)$ is sublinear. Indeed, for some $|\beta| < \infty$, $0 < c < \infty$, $w(n) \leq cn + \beta$ for all $n \geq 1$.

Also, by Theorem 3.6, there exists a sequence $\{c(n)\}_{n\geq 1}$ such that $\forall\, i \geq 1$

$$\{\tau_n - \tau_{i-2} - c(n)\}$$

converges w.p. 1 and in L^2 (and hence $(\tau_n - \tau_{i-2})/c(n) \to 1$) w.p. 1 and

$$\frac{1}{(2c+\beta)} < \varliminf \frac{c(n)}{\log n} \leq \varlimsup \frac{c(n)}{\log n} \leq \frac{1}{\delta}.$$

Thus, $\forall\ i \geq 1$,

$$\frac{\tilde{d}_n(i)}{(c(n))^q} \equiv \frac{Z_i(\tau_n - \tau_{i-2})}{(\tau_n - \tau_{i-2})^q}\Big(\frac{(\tau_n - \tau_{i-2})}{c(n)}\Big)^q.$$

As $n \to \infty$, the right side converges w.p. 1 to α. So, by the embedding theorem (Theorem 3.3), Theorem 2.4 follows. □

Proof of Theorem 2.6. This has been proved under the assumption $\beta > 0$ [Athreya et al. 08]. Now, using Theorem 4.1(a), the proofs in that reference can be extended to the present case where β need not be positive. □

Proof of Theorem 2.8. If $w(n) \equiv c > 0$ then $\tau_n - \frac{1}{c}\log n$ converges w.p. 1 and in mean square. By Theorem 4.1 (b), $\forall\ i \geq 1$

$$\frac{\tilde{d}_n(i)}{(\tau_n - \tau_{i-2})} = \frac{Z_i(\tau_n - \tau_{i-2})}{\tau_n - \tau_{i-2}} \to c \quad \text{w.p. } 1,$$

yielding $\tilde{d}_n(i)/\log n \to 1$ w.p. 1 proving Theorem 2.8(a).

The second part follows from the proof of Theorem 1.2 in [Athreya et al. 08] and noting that in the special case of a Poisson process with rate c, the expression for π_j reduces to $\frac{1}{2^j}$, $j \geq 1$. □

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