On Bicycle Tire Tracks Geometry, Hatchet Planimeter, Menzin's Conjecture, and Oscillation of Unicycle Tracks

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CONTENTS

The model of a bicycle is a unit segment AB that can move in the plane so that it remains tangent to the trajectory of the point A (the rear wheel is fixed to the bicycle frame). The same model describes the hatchet planimeter. The trajectory of the front wheel and the initial position of the bicycle uniquely determine its motion and its terminal position; the monodromy map sending the initial position to the final position arises in this context.

According to a theorem of R. Foote, this mapping of a circle to a circle is a Möbius transformation. We extend this result to the multidimensional setting. Möbius transformations belong to one of three types: elliptic, parabolic, and hyperbolic. We prove the century-old Menzin conjecture: if the front wheel track is an oval with area at least π, then the respective monodromy is hyperbolic. We also study bicycle motions introduced by D. Finn in which the rear wheel follows the track of the front wheel. Such a "unicycle" track becomes more and more oscillatory in the forward direction. We prove that it cannot be infinitely extended backward and relate the problem to the geometry of the space of forward semi-infinite equilateral linkages.

1. INTRODUCTION

The geometry of bicycle tracks is a rich and fascinating subject. Here is a sample of questions that arise:

1. Given the tracks of the rear and front wheel, can you tell which way the bicycle has traveled?

2. The track of the front wheel is a smooth simple closed curve. Can one ride the bicycle so that the rear wheel's track is also a closed curve?

3. Can one ride a bicycle in such a way that the tracks of the rear and front wheels coincide (other than along a straight line)?

Our model of a bicycle is an oriented segment, say AB, of length ℓ that can move in the plane in such a way that the trajectory of point A always remains tangent to the segment. Point A represents the rear wheel, point B the front wheel; the rear wheel is fixed to the bicycle frame, whereas the front wheel can turn, and this explains the law of motion. (Usually we set $\ell = 1$, which can always be assumed by making a dilation, but sometimes we shall consider ℓ as a parameter and allow it to take very small or very large values.) Thus the endpoint of the oriented segment tangent to the trajectory of the rear wheel traces the trajectory of the front wheel; see [Finn 02, Konhauser et al. 96].

The same mathematical model describes another mechanical device, the Prytz or hatchet planimeter; see [Barnes 57, Crathorne 08, Foote 98]. Various kinds of planimeters were popular objects of study in the late nineteenth and early twentieth centuries.

The first of the above questions has the following answer: generically, one can determine the direction; but in some special cases one cannot, for example, for concentric circles of radii r and R satisfying $r^2 + \ell^2 = R^2$. Surprisingly, the problem of describing such "ambiguous" pairs of closed tracks is equivalent to Ulam's problem of describing (two-dimensional) bodies that float in equilibrium in all positions. See [Tabachnikov 06, Wegner 03, Wegner 06, Wegner 07] for a variety of results and references.

The content of the present paper has to do with the other two questions. In Section 2 we place the problem in the framework of contact geometry. We allow the trajectory of the rear wheel to be a wave front, that is, to have cusp singularities, but we show that the trajectory of the front wheel remains smooth. We deduce a useful differential equation relating the motions of the rear and front wheels.

Fixing a path Γ of the front wheel gives rise to a circle map: the initial direction of the segment, characterized by a point on the circle, determines its final direction; see Figure 1. We will refer to this map of the circle to itself (the two circles are identified by parallel translation) as the *monodromy map*.[1] It is a beautiful theorem of R. Foote [Foote 98] (see also [Levi and Weckesser 02]) that for every trajectory of the front wheel, the monodromy map is a Möbius transformation. In Section 3 we provide another proof of this theorem and extend it to bicycle motion in Euclidean space of any dimension.
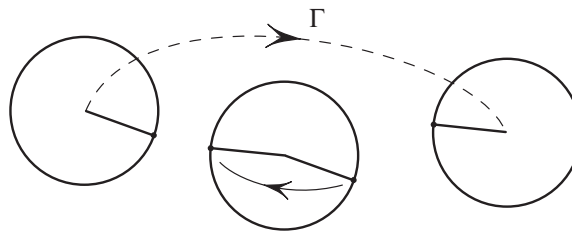
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[1] T. Tokieda has suggested the term "opisthodromy."

FIGURE 1. The circle mapping generated by the curve Γ. According to Foote's theorem, this mapping is a Möbius transformation.

A (nontrivial) Möbius transformation is of one of three types: elliptic, parabolic, and hyperbolic. The first of these have no fixed points, while the last two have exactly two fixed points, one attracting and one repelling (parabolic transformations have a single neutral fixed point). Suppose the trajectory of the front wheel is a closed curve. Then up to conjugation, the respective monodromy, and therefore its type, does not depend on the initial point. In Section 3 we give a necessary and sufficient condition for the monodromy to be parabolic, namely that the trajectory of the rear wheel be a closed wave front with the total algebraic arc length equal to zero (the sign of the arc length changes in passing a cusp).

Still assuming that the trajectory of the front wheel is closed, a fixed point of the monodromy map corresponds to a closed trajectory of the rear wheel. Thus, in the hyperbolic case, for a given closed trajectory of the front wheel, there are exactly two bicycle motions such that the trajectory of the rear wheel is closed; each of these motions is hyperbolically attracting for one of the choices of the direction of motion; examples are shown in Figure 2, examples 1 and 4. In contrast, in the elliptic case, no trajectory of the rear wheel closes after one cycle. It is worth mentioning that for some trajectories of the front wheel, the monodromy is the identity: for every bicycle motion the trajectory of the rear wheel closes up.

A century-old conjecture by Menzin [Menzin 06] states, in our terminology, that if the trajectory of the front wheel is a closed convex curve bounding an area greater than $\pi\ell^2$, then the respective monodromy is of the hyperbolic type. In Menzin's words:

> [T]he tractrix will approach, asymptotically, a limiting closed curve. From purely empirical observations, it seems that this effect can be obtained so long as the length of arm does not exceed the radius of a circle of area equal to the area of the base curve.
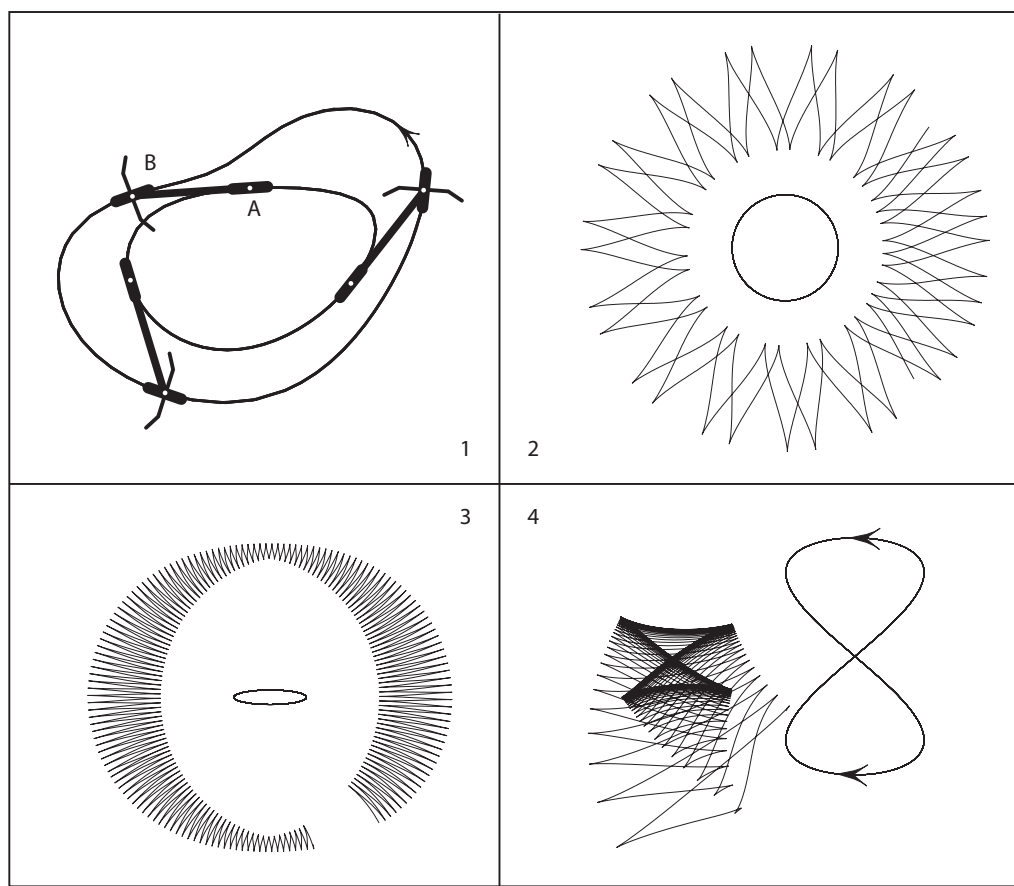
FIGURE 2. Examples 1 and 4 are hyperbolic; 2 and 3 are elliptic. The areas bounded by the two curves in 1 differ by $\pi\ell^2$.

In Section 4 we prove this conjecture. The main tool is the classical Wirtinger inequality. Earlier, Foote [Foote 98] proved Menzin's conjecture for parallelograms.

Section 5 concerns Finn's construction of bicycle motion leaving a single track [Finn 02]. Consider a "seed" curve, tangent to the x-axis at points 0 and 1 with all derivatives and oriented to the right; see Figure 3 (the seed curve is also the "fat" curve in Figure 4). This curve is the initial trajectory of the rear wheel; drawing the tangent segments of length 1 to it yields the next curve, which is tangent to the x-axis at points 1 and 2 with all derivatives. Iterating this process, one obtains a bicycle motion that leaves a unicycle track, i.e., a curve that both wheels follow.

Numerical study shows that unless the seed curve is horizontal, the resulting unicycle track becomes more and more oscillatory; see Figures 3 and 4. We prove that the number of intersections with the x-axis and the number of extrema of the height function increase at least by one with every iteration of this construction. As a consequence, the seed curve with finitely many intersections
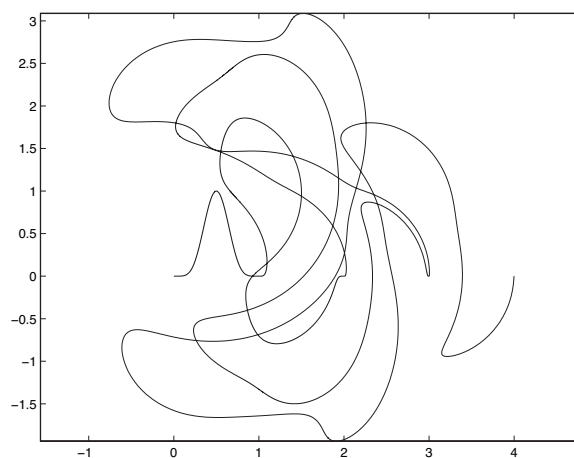

FIGURE 3. The figure shows first four iterates of the initial seed curve $y = 4^6 x^6 (1-x)^6$. Since this curve has only a finite order of contact with the x-axis, only finitely many iterations are defined.

with the x-axis (or a finite number of extrema) has at most finitely many preimages under Finn's construction.
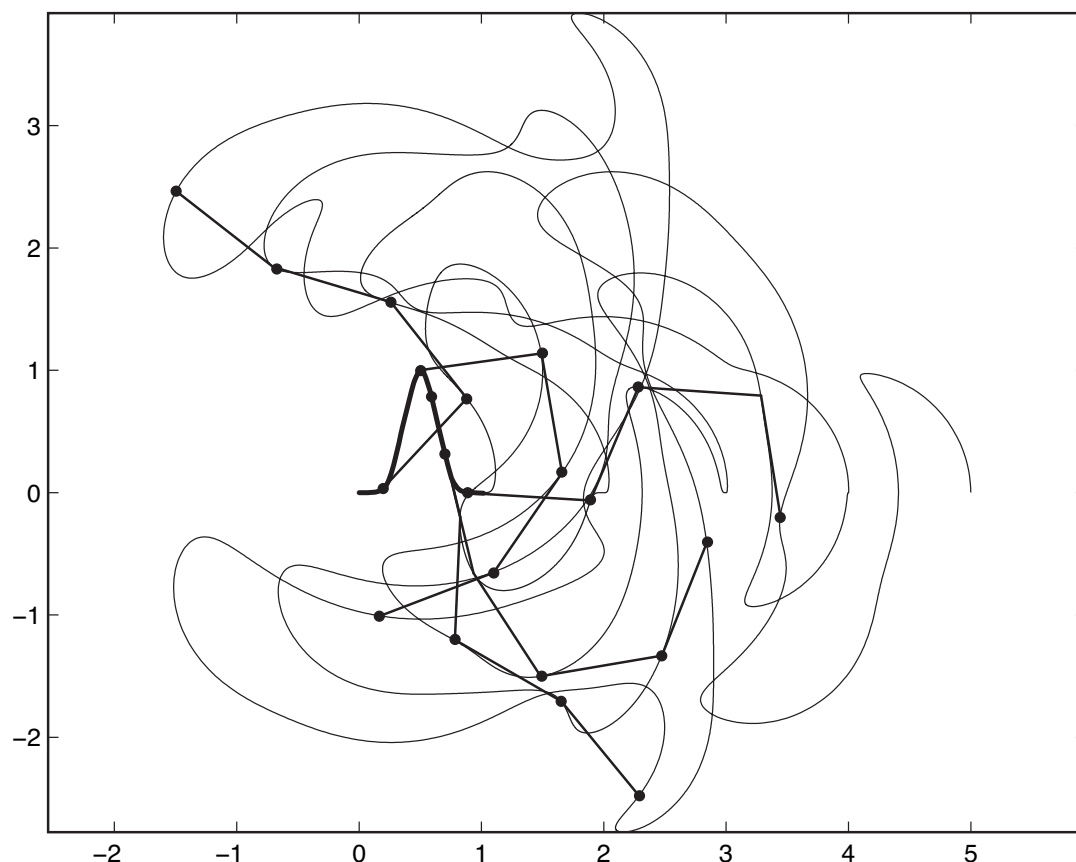
FIGURE 4. The rear wheel follows the track of the front wheel. The "seed" curve is shown in a heavier stroke. Several consecutive positions of the associated moving linkage are shown. The shape of the linkage is very sensitive to the position of the starting point on the seed curve.

This means that the corresponding unicycle track cannot extend back indefinitely. We also make a number of conjectures on the Finn construction that are strongly supported by numerical evidence.

A unicycle track can be viewed as an integral curve of a direction field in a certain infinite-dimensional space. Specifically, we consider the configuration space of equilateral forward infinite linkages in the plane. We constrain the velocity of the ith vertex to the direction of the ith link (heuristically, the ith link is the position of the bike on the $(i-1)$st step of Finn's construction). This constraint defines a field of directions. Now, a forward bicycle motion generating a single track corresponds to a particular integral curve of this field of directions. This field does not satisfy the uniqueness property: through every point there pass infinitely many smooth integral curves. We also generalize Finn's construction for an arbitrary initial equilateral forward infinite linkage in which the adjacent links are not perpendicular (the Finn construction corresponds to a linkage aligned along a line).

2. PRELIMINARIES: CONTACT GEOMETRIC POINT OF VIEW

We use the notation from Section 1. Denote the trajectory of the rear wheel A by γ and that of the front wheel B by Γ. We allow γ to have cusp singularities as in Figure 5. A proper perspective is provided by contact geometry; see [Arnold and Givental 90] or [Geiges 06].

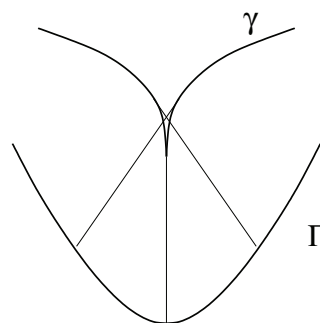**FIGURE 5**. Cusp of the curve γ.

The position of the segment AB is determined by its foot point $A(x, y)$ and by the angle θ between the x-axis and the segment. The infinitesimal motions in the configuration space $\{(x, y, \theta)\}$ are restricted by the non-skidding condition $(\dot{x}, \dot{y}) \parallel (\cos\theta, \sin\theta)$. This condition defines a field of tangent 2-planes in the configuration space. This field of planes is nonintegrable and is defined by the contact 1-form $\lambda = \sin\theta\, dx - \cos\theta\, dy$.

A smooth curve in a contact manifold is called *Legendrian* if its tangent line at every point lies in the contact plane. Denote by M the space of contact elements, that is, the configuration space of the segment. Let $p : M \to \mathbb{R}^2$ be the projection taking a contact element to its foot point. The image of a Legendrian curve is called a *wave front*; generically, it is a piecewise smooth curve with semicubical cusp singularities. The singularities occur at the points where the Legendrian curve is tangent to the fibers of the projection p. A wave front has a well-defined tangent line at every point and can be uniquely lifted to a Legendrian curve in the space of contact elements.

In this paper we consider the bicycle motions corresponding to smooth Legendrian curves in the space of contact elements. We shall see that the trajectory of the front wheel, unlike that of the rear one, is always a smooth curve.

The trajectory of the rear wheel uniquely determines the trajectory of the front wheel. Denote by T the correspondence $\gamma \to \Gamma$ that assigns to the point $x \in \gamma$ the endpoint of the unit tangent segment to γ at x. We assume that a continuous choice is made between the two orientations of the unit tangent segments at a point. This amounts to choosing a coorientation of γ: the frame formed by the coorienting vector and the chosen tangent vector is positive (recall that coorientation is a continuous choice of a normal direction to a curve). When the bicycle segment is not of unit length and has length ℓ, we denote by T_ℓ the respective transformation and by Γ_ℓ its image. Let us emphasize that T and T_ℓ are defined for a cooriented front γ.

The following two lemmas address the smoothness issue.

Lemma 2.1. *If γ is a regular C^k curve, $k \geq 1$, then Γ_ℓ is a regular C^{k-1} curve for all $\ell > 0$.*

Proof: Let γ be parameterized by its arc length s. By definition, $\Gamma(s) = \gamma(s) + \ell\gamma'(s)$, and it remains only to make sure that $\Gamma' = \gamma' + \ell\gamma'' \neq 0$. But the last two vectors are orthogonal, and the first has unit length. $\square$

Lemma 2.2. *Even if γ has cusps, the curve Γ_ℓ is smooth for all $\ell > 0$.*

Proof: Recall that a wave front is the plane projection of a smooth Legendrian curve in the space of contact elements. Let $p_1 : M \to \mathbb{R}^2$ take the segment AB to the point B. The correspondence T_ℓ is the composition of the Legendrian lifting of a wave front γ and the projection p_1. We claim that the fibers of p_1 are everywhere transverse to the contact distribution on M. This would imply the statement of the lemma, since the fibers of the projection are transverse to the Legendrian curve $p^{-1}(\gamma)$.

In terms of the coordinates in M, one has $p_1(x, y, \theta) = (x + \ell\cos\theta, y + \ell\sin\theta)$. The vector field $v = \partial_\theta + \ell\sin\theta\,\partial_x - \ell\cos\theta\,\partial_y$ is tangent to the fibers of p_1. One has $\lambda(v) = \ell$, and therefore v is everywhere transverse to the contact planes, and we are done. $\square$

Let γ be an oriented and cooriented closed wave front. The *Maslov index* $\mu(\gamma)$ is the algebraic number of cusps of γ; a cusp is *positive* if one traverses it along the coorientation and *negative* otherwise.

Let γ be an oriented and cooriented closed wave front. Denote by $\rho(\gamma)$ the rotation number, that is, the total (algebraic) number of turns made by its tangent direction. Let $\Gamma = T(\gamma)$.

Lemma 2.3. *One has $\rho(\Gamma) = \rho(\gamma) + \frac{1}{2}\mu(\gamma)$.*

Proof: Consider the one-parameter family of curves Γ_ℓ. By Lemma 2.2, this is a continuous family of smooth curves; hence the rotation number is the same for all ℓ. Consider the case of very small ℓ.

Along smooth arcs of γ, the curve Γ_ℓ is C^1-close to γ. At the cusps, smoothing occurs, and the rotation of Γ_ℓ differs from that of γ by $\pm\pi$. There are four cases, depending on the orientation and coorientation, depicted in Figure 6. When one traverses a cusp along the coorientation, the total rotation of Γ_ℓ gains π, and when a cusp is traversed against the coorientation, the total rotation of Γ_ℓ loses π. This implies the result. $\square$

We introduce the following notation. Let x be the arc-length parameter along the curve Γ. The position of the segment AB with $B = \Gamma(x)$ is determined by the angle made by the tangent vector $\Gamma'(x)$ and the vector BA. Let this angle be $\pi - \alpha(x)$. The function $\alpha(x)$ uniquely determines the curve γ, the locus of points A. Let $\kappa(x)$ be the curvature of $\Gamma(x)$. Denote by t the arc-length parameter on γ and by k the curvature of γ. Note that at cusps, $k = \infty$.
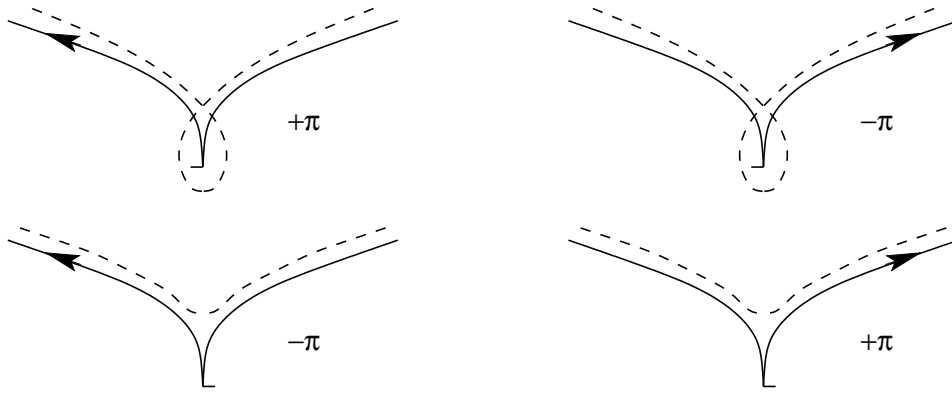
FIGURE 6. Cusps of the curve γ and their smoothings Γ_ℓ.
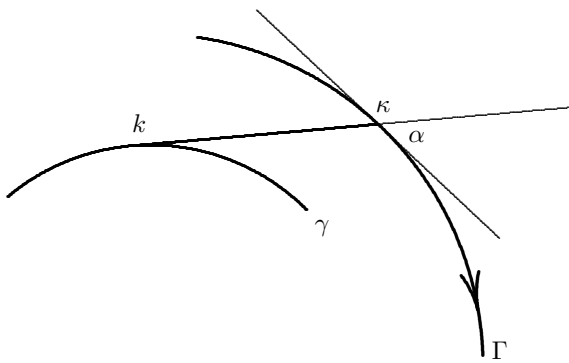

FIGURE 7. Notation for Proposition 2.4.

The next result is borrowed from [Tabachnikov 06]; see also [Finn 02].

Proposition 2.4. *With the notation depicted in Figure 7, the condition $T_\ell(\gamma) = \Gamma$ is equivalent to the differential equation on the function $\alpha(x)$:*

$$\frac{d\alpha(x)}{dx} + \frac{\sin \alpha(x)}{\ell} = \kappa(x). \qquad (2\text{--}1)$$

One has

$$\left| \frac{dt}{dx} \right| = |\cos \alpha|, \quad k = \frac{\tan \alpha}{\ell}.$$

In particular, the cusps of γ correspond to the instances of $\alpha = \pm \pi/2$.

Proof: Let J denote the rotation of the plane through the angle $\pi/2$. Then the endpoint of the segment of length ℓ making the angle $\pi - \alpha(x)$ with $\Gamma'(x)$ is

$$\gamma(x) = \Gamma(x) - \ell\Gamma'(x)\cos\alpha(x) + \ell J(\Gamma'(x))\sin\alpha(x). \quad (2\text{--}2)$$

For $T_\ell(\gamma) = \Gamma$ to hold, the tangent direction $\gamma'(x)$ should be collinear with the respective segment, that is, be parallel to the vector

$$v(x) := -\Gamma'(x)\cos\alpha(x) + J(\Gamma'(x))\sin\alpha(x).$$

Differentiate (2–2), taking into account that $\Gamma''(x) = \kappa(x)J(\Gamma'(x))$, and equate the cross product with $v(x)$ to zero to obtain (2–1).

It is straightforward to calculate that $|d\gamma/dx| = |\cos\alpha|$, hence $|dt/dx| = |\cos\alpha|$. The computation of the curvature k is also straightforward. □

It is natural to adopt the following convention: the sign of the length element dt on γ changes at each cusp. This is consistent with Proposition 2.4, since cusps correspond to $\alpha = \pi/2$, that is, to sign changes of $\cos\alpha$. With this convention, we have $dt = \cos\alpha(x)\,dx$. In particular, the signed perimeter of γ is $\int_\Gamma \cos\alpha(x)\,dx$.

3. BICYCLE MONODROMY MAP

If $\gamma(t)$ is the arc-length parameterized trajectory of the rear bicycle wheel, then the trajectory of the front wheel is $\Gamma(t) = \gamma(t) \pm \gamma'(t)$ (the sign depends on the coorientation of γ and changes at its cusps). We extend this definition to bicycle rides in multidimensional space $\mathbb{R}^n$.

On the other hand, if Γ is given, then one can recover γ once the initial position of the bicycle is chosen. The set of all possible positions of the bicycle with a fixed position of the front wheel is a unit sphere S^{n-1}. Thus there arises the time-x monodromy map M_x, which assigns the time-x position of the bicycle with a prescribed front wheel trajectory to its initial position: $M_x : S^{n-1} \to S^{n-1}$.

Consider the hyperbolic space H^n realized as the pseudosphere $x_1^2 + \cdots + x_n^2 - x_0^2 = -1$ in the pseudo-Euclidean

space $\mathbb{R}^{n,1}$ with the metric $dx_0^2 - dx_1^2 - \cdots - dx_n^2$. The Möbius group $O(n,1)$ consists of linear transformations preserving the metric and acts on H^n by isometries. This action extends to the null cone $x_1^2 + \cdots + x_n^2 = x_0^2$ and to its spherization S^{n-1}, the sphere at infinity of the hyperbolic space. In particular, we obtain an action of the Lie algebra $o(n,1)$ on S^{n-1}.

The following result is a multidimensional generalization of Foote's theorem [Foote 98].[2] We identify all unit spheres S^{n-1} along a curve $\Gamma(x)$ by parallel translations.

Theorem 3.1. *For all x, one has $M_x \in O(n,1)$.*

Proof: Note first that the rear wheel's velocity is $\text{proj}_r \, v = (r \cdot v)r$, where $r = AB$. Since M_x is the map of the sphere centered at the front wheel, we consider the moving frame with the origin at the front wheel. This frame undergoes parallel translation as the wheel moves with speed v. In the moving frame, the rear wheel's velocity is $\omega(v) = (-v + (r \cdot v)r) \perp r$. We thus have a time-dependent vector field on the sphere, and our map M_x is the time-x map of this vector field. It suffices therefore to show that this vector field corresponds to an element of the Lie algebra $o(n,1)$.

The Lie algebra $o(n,1)$ consists of the matrices

$$C(M,v) = \begin{pmatrix} M & v \\ v^* & 0 \end{pmatrix},$$

where $M \in o(n)$ is an $n \times n$ skew-symmetric matrix and v is an n-dimensional vector; it includes matrices of the special form $C(0,v) = C(v)$. We will show that these special matrices generate the vector field $\omega(v)$ mentioned above. (As a side remark, the Lie algebra $o(n,1)$ is generated by its n-dimensional subspace $C(0, \mathbb{R}^n)$.)

Let us compute the action of $C(v)$ on the unit sphere S^{n-1}. For a unit n-dimensional vector r, consider the point $(r,1)$ of the null cone at height 1. Then

$$(E + \varepsilon C(v)) \begin{pmatrix} r \\ 1 \end{pmatrix} = \begin{pmatrix} r + \varepsilon v \\ 1 + \varepsilon r \cdot v \end{pmatrix}$$
$$= k \begin{pmatrix} r - \varepsilon\omega(v) \\ 1 \end{pmatrix} + O(\varepsilon^2),$$

where $k = (1 + \varepsilon r \cdot v)$. Thus $C(v)$ corresponds to the vector field $\omega(v)$ on the sphere, and the result follows. $\square$

Remark 3.2. It is quite likely that an analogue of Theorem 3.1 holds if $\mathbb{R}^n$ is replaced by either spherical or hyperbolic space. We do not dwell on it here.[3]

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[2]Foote studies the Prytz planimeter.

[3]See [Howe et al. 09] where this is proved, along with versions of Menzin's conjecture in the elliptic and hyperbolic planes.

Remark 3.3. It is interesting to point out possible connection with the so-called snake-charmer algorithm [Hausmann and Rodriguez 07], in which the monodromy also takes values in the Möbius group.

Remark 3.4. In dimension 2, the monodromy is a projective transformation of S^1 identified with $\mathbb{RP}^1$ by the (stereographic) projection from a point of the circle. If α is an angular coordinate on the circle, then $y = \tan(\alpha/2)$ is a projective coordinate. Equation (2–1) can be rewritten as a Riccati equation:

$$y'(x) = -y(x) + \frac{1}{2}\left(y^2(x) + 1\right)\kappa(x).$$

The infinitesimal monodromy of the Riccati equation $y' = f(x) + g(x)y + h(x)y^2$ is generated by the vector fields $d/dy, y\,d/dy, y^2 d/dy$, which generate $\text{sl}(2,\mathbb{R}) = o(2,1)$.

Now we consider corollaries of Theorem 3.1 in the case $n = 2$. Recall the classification of orientation-preserving isometries of the hyperbolic plane: an elliptic isometry is a rotation about a point of H^2, and the corresponding map of the circle at infinity is conjugate to a rotation; a hyperbolic isometry has two fixed points at infinity, one exponentially attracting and another repelling; a parabolic isometry has a unique fixed point at infinity with derivative 1; see, e.g., [Beardon 83].

Let Γ, the trajectory of the front wheel, be closed. Then the monodromy map M along Γ is well defined, up to conjugation (that is, changing the starting point on Γ amounts to replacing M by a conjugate transformation); in particular, its type—elliptic, parabolic, hyperbolic—does not depend on the starting point.

The first corollary concerns the case of M hyperbolic.

Corollary 3.5. *Let the trajectory γ of the rear wheel be a generic closed cooriented wave front. Then the trajectory Γ of the front wheel is also closed, and there exists a unique additional closed trajectory of the rear wheel γ^* with the same front wheel trajectory Γ. The correspondence $\gamma \leftrightarrow \gamma^*$ is an involution on the space of cooriented closed plane wave fronts. For a fixed orientation of Γ, one of the curves γ and γ^* is exponentially stable and the other exponentially unstable. The unstable curve γ is the closed path of the bike ridden backward.*

Proof: Since γ is closed, the monodromy M has a fixed point, and since γ is generic, M is hyperbolic. Then M has another fixed point, corresponding to the closed
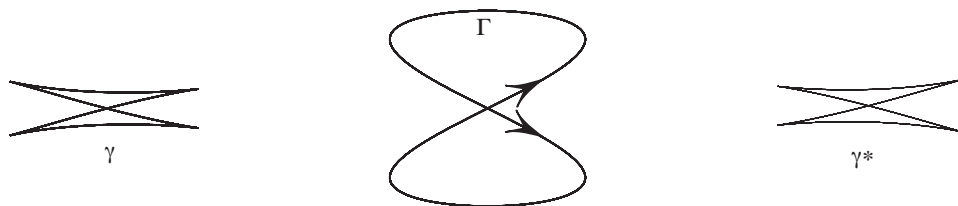
FIGURE 8. The unstable curve is on the right. If the direction of traversal of the figure eight is reversed, the two curves exchange stability.
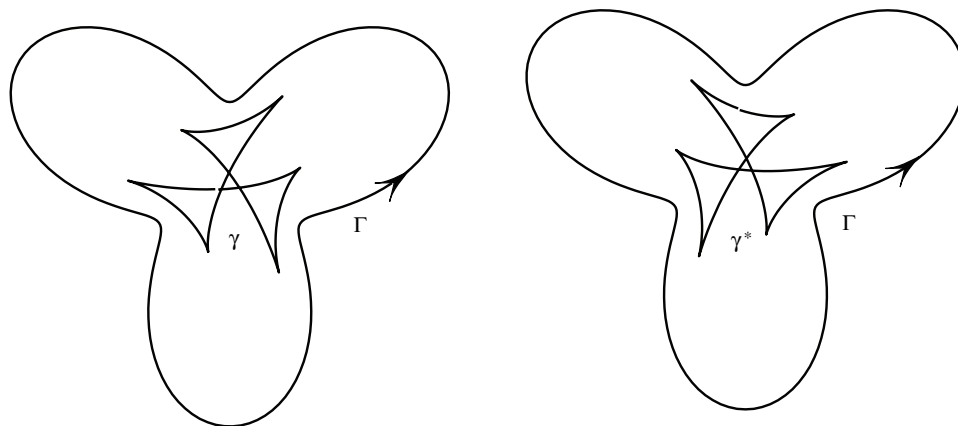

FIGURE 9. The stable and the unstable rear trajectories for the shamrock, just before the bifurcation when γ and γ^* coalesce. The shamrock is given by $x = r(t) \cos t$, $y = r(t) \cos t$ with $r = 0.94(1 - 0.5 \sin 3t)$.

trajectory γ^*. One of these fixed points is exponentially stable and the other unstable.   □

Corollary 3.5 is illustrated by Figures 8 and 9.

We precede the next observation with a remark: for any Möbius map with two fixed points, the derivatives at the two fixed points are reciprocal to each other. This, according to the next theorem, implies that γ and γ^* have the same length (up to sign).

Let γ be a closed cooriented wave front (the rear wheel track) and let $\Gamma = T(\gamma)$ be the front wheel track. Let M be the monodromy of the curve Γ and let L be the perimeter of Γ.

Theorem 3.6. *Let M be hyperbolic or parabolic, and let γ be the closed path of the rear wheel corresponding to a fixed point θ_0 of the Möbius circle map $\theta \mapsto M(\theta)$. Then*

$$M'(\theta_0) = e^{-\text{length}(\gamma)}. \tag{3–1}$$

Corollary 3.7. *If M is hyperbolic and γ and γ^* are the rear tracks corresponding to the two fixed points, then the curves γ and γ^* have equal lengths.*

Proof of the corollary.: For the fixed points θ_0, θ_0^* of any Möbius map one has $M'(\theta_0)M'(\theta_0^*) = 1$, and the statement follows from Theorem 3.6.   □

Remark 3.8. The case $\gamma = \gamma^*$ is quite interesting: in this case, one cannot tell which way the bicycle went from the closed tire tracks of the front and rear wheels; see Section 1.

Proof of Theorem 3.6.: Using the notation of Section 2, consider equation (2–1) (with $\ell = 1$). This equation has an L-periodic solution $\alpha(x)$. Consider an infinitesimal perturbation $\alpha(x) + \varepsilon\beta(x)$; the derivative of the monodromy map is given by $M'(\theta_0) = \beta(L)/\beta(0)$. But β satisfies the linearized equation $\beta' + \beta \cos \alpha = 0$, from which we obtain

$$M'(\theta_0) = \frac{\beta(L)}{\beta(0)} = e^{-\int_0^L \cos \alpha(x)dx}.$$

Recall that $\cos \alpha(x)$ is the speed of the rear wheel, and thus $\int_0^L \cos \alpha(x)dx = \text{length}(\gamma)$.   □

Remark 3.9. It is interesting that the monodromy may be the identity; that is, there exist closed trajectories of the front wheel for which every trajectory of the rear
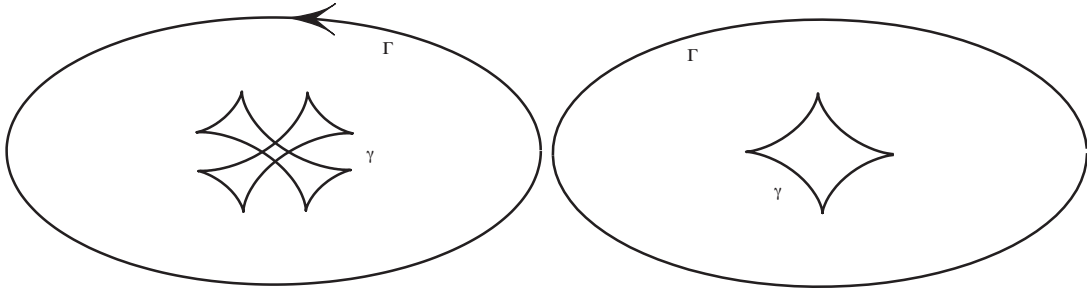
FIGURE 10. A saddle-node bifurcation: the ellipse on the left is (slightly) larger than the one on the right. The length of the coalesced curve on the right is zero, in accordance with Theorem 3.10. In this particular case this is seen directly: the four arcs are congruent by symmetry, and their signs alternate.

wheel is closed. To construct such an example, let Γ be a small simple closed curve. Then the monodromy M is elliptic; see the analysis in [Foote 98]. (This also follows from equation (2–1): in the limit $\ell \to \infty$, the equation becomes $\alpha'(x) = \kappa(x)$, and since $\int \kappa(x)\, dx = 2\pi$, the function $\alpha(x)$ cannot be periodic.) Slightly deforming Γ if necessary one may assume that M has a rational rotation number. Since an elliptic isometry is a rotation of the hyperbolic plane, M is actually a periodic map. Then, traversing Γ an appropriate number of times, the monodromy becomes the identity.

In contrast, if Γ is a closed immersed curve (not necessarily simple) and ℓ is sufficiently small, one has a hyperbolic monodromy. Indeed, in the limit $\ell \to 0$, equation (2–1) becomes $\sin \alpha = 0$ and has two solutions $\alpha(x) = 0$ and $\alpha(x) = \pi$, corresponding to the forward and backward tangent vectors to Γ. The two exponentially stable and unstable solutions survive for ℓ small enough.

As a limiting case of Theorem 3.6 for the parabolic monodromy, we have the following.

Theorem 3.10. *The monodromy M is parabolic if and only if the total algebraic length of γ is zero.*

FIGURE 11. Curve with turning number π.

Proof: At the fixed point θ_0 we have $M'(\theta_0) = 1$; comparison with (3–1) shows that length$(\gamma) = 0$. $\square$

Corollary 3.11. *In the parabolic case, the curve γ has cusps.*

An example of a wave front γ yielding parabolic monodromy is depicted in Figure 11. The curve γ has total turning number π, so for Γ to close up, one traverses γ twice. This "doubled" front γ obviously has zero total length.

An example of the saddle-node bifurcation from the hyperbolic to the elliptic case, as the size of Γ decreases, is shown in Figure 10.

Remark 3.12. Computation of the monodromy amounts to multiplying infinitely many 2×2 matrices corresponding to infinitesimal arcs of the curve Γ (if Γ is a polygon, one has a finite product of hyperbolic elements in $\mathrm{SL}(2,\mathbb{R})$). A similar problem concerning the group of isometries of the sphere $\mathrm{SO}(3)$ is treated in [Levi 96, Levi 93]; we plan to extend this work to the group of isometries of the hyperbolic plane.

4.   PROOF OF THE MENZIN CONJECTURE

Theorem 4.1. *If Γ is a closed convex curve bounding a region with area greater than π, then the respective monodromy is hyperbolic.*

Proof: By approximation, we may assume that Γ is an oval, that is, a smooth closed strictly convex curve. We need to prove that if the monodromy M is elliptic or parabolic, then area$(\Gamma) \leq \pi$. As we already mentioned, if Γ is large enough, the monodromy M is hyperbolic. Hence, if M is elliptic, we can make Γ larger (say, by
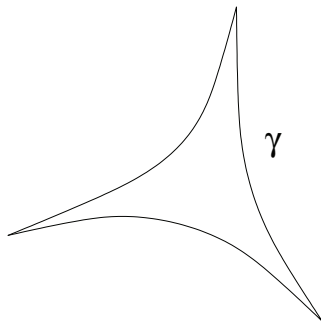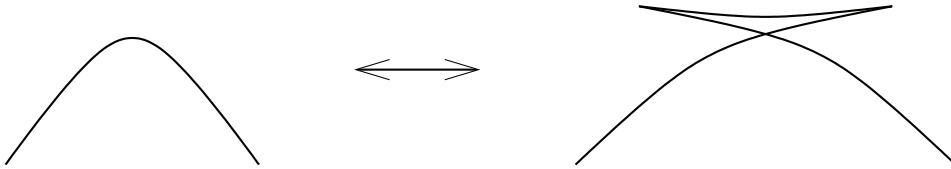
FIGURE 12. Birth and death of a pair of cusps.

homothety) and render M parabolic. Therefore it suffices to prove that if M is parabolic, then area$(\Gamma) \leq \pi$.

The proof is based on two observations:

- area$(\Gamma) = $ area$(\gamma) + \pi$, so that area$(\Gamma) \leq \pi$ is equivalent to area$(\gamma) \leq 0$.

- If length$(\gamma) = 0$ then area$(\gamma) \leq 0$.

We proceed with the detailed proof. Include Γ in a one-parameter family of homothetic nested ovals Γ_s, starting with a very large oval Γ_0 and ending with the given oval Γ. Let $s = 1$ be the first value of the parameter for which the monodromy is parabolic. Since the monodromy for Γ is elliptic or parabolic, Γ lies inside Γ_1 and bounds a smaller area than Γ_1. We want to show that the latter does not exceed π.

Since the monodromies M_s for $s \in [0, 1)$ are hyperbolic, one has a family of wave fronts γ_s (the closed trajectories of the rear wheel). Since Γ_0 is large enough, γ_0 is also an oval. The Legendrian liftings of the fronts γ_s form a continuous family of immersed Legendrian curves in the space of contact elements. Therefore, the Maslov index of γ_1 equals that of γ_0, that is, zero. Likewise, the rotation number $\rho(\gamma_1)$ equals one. The number of cusps may change in the family γ_s; see Figure 12.

The following holds due to the convexity of Γ.

Lemma 4.2. *The wave front γ_1 has no inflections.*

Proof: Assume that γ_1 has an inflection point. Note that γ_0 is convex. Let τ be the first value of the parameter s for which the curvature of γ_s vanishes. Then, for s slightly greater than τ, the curve γ_s has a "dimple," and Γ_s is not convex; see Figure 13. $\square$
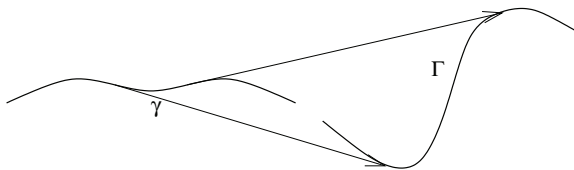

FIGURE 13. Inflections of γ.

Thus γ_1 is a wave front made of an even number of convex smooth arcs; the adjacent arcs form cusps. The total turning of the tangent direction to γ_1 is 2π. The arcs are marked by $\pm$; the sign changes at each cusp. By Theorem 3.10, the algebraic length of γ_1 vanishes: length$(\gamma_1) = 0$.

Consider a smooth arc of γ_1 in the arc-length parameterization; abusing notation, call this arc $\gamma_1(t)$. The respective arc of Γ_1 is $\Gamma_1(t) = \gamma_1(t) + \sigma\gamma_1'(t)$, where $\sigma = \pm$ is the sign of the arc γ_1. Therefore $\Gamma_1' = \gamma_1' + \sigma\gamma_1''$, and hence

$$\Gamma_1 \times \Gamma_1' = \gamma_1 \times \gamma_1' + \sigma\gamma_1 \times \gamma_1'' + \sigma^2\gamma_1' \times \gamma_1''.$$

Note that $\gamma_1 \times \gamma_1'' = (\gamma_1 \times \gamma_1')'$ and that $\gamma_1' \times \gamma_1'' = k$, the curvature of γ_1.

For a closed parametric curve $\Gamma(t)$, twice the area bounded by Γ is given by the integral $\int(\Gamma \times \Gamma')\,dt$. Applying this to Γ_1, we get

$$2\,\text{area}(\Gamma_1) = \sum_i \Big(\int \gamma_{1,i}(x) \times \gamma_{1,i}'(x)\,dx \qquad (4\text{--}1)$$
$$+\, \sigma_i\Delta_i(\gamma_{1,i} \times \gamma_{1,i}') + \theta_i\Big),$$

where the sum is taken over the smooth arcs of $\gamma_{1,i}$, where σ_i is the sign of the ith arc, Δ_i is the difference of the momenta $\gamma_1 \times \gamma_1'$ at the endpoints of the ith arc, and θ_i is the turning angle of the ith arc.

Note that the sum of integrals in (4–1) is $2\,\text{area}(\gamma_1)$. Note also that $\sum\theta_i = 2\pi$. Finally note that the terms Δ cancel out: $\sum\sigma_i\Delta_i(\gamma_1 \times \gamma_1') = 0$. Therefore the inequality area$(\Gamma_1) \leq \pi$ is equivalent to area$(\gamma_1) \leq 0$.

To prove the latter inequality, let $p(\varphi)$ be the support function of the front γ_1 (the signed distance from the origin to the tangent line to γ_1 as a function of the direction of this line; see, e.g., [Santalo 04] for the theory of support functions). The support function exists because γ_1 is free from inflections and makes one full turn. One has the following formulas:

$$\text{length}(\gamma_1) = \int_0^{2\pi} p(\varphi)\,d\varphi,$$

$$\text{area}(\gamma_1) = \frac{1}{2}\int_0^{2\pi} (p^2(\varphi) - p'^2(\varphi)\,d\varphi.$$

Thus we need to show that if

$$\int_0^{2\pi} p(\varphi)\, d\varphi = 0,$$

then

$$\int_0^{2\pi} p^2(\varphi)\, d\varphi \le \int_0^{2\pi} p'^2(\varphi)\, d\varphi.$$

But this is the well-known Wirtinger inequality, which concludes the proof. $\square$

Remark 4.3. The Wirtinger inequality is intimately related to the isoperimetric inequality. Consider an oval γ with area A and perimeter L. Consider the one-parameter family of equidistant fronts γ_t inside the oval (that is, consider γ as a source of light propagating inward). The support function of γ_t is that of γ minus t. One has

$$\text{length}(\gamma_t) = L - 2\pi t, \quad \text{area}(\gamma_t) = A - Lt + \pi t^2.$$

By the Wirtinger inequality, when $\text{length}(\gamma_t) = 0$, one has $\text{area}(\gamma_t) \le 0$. Therefore, if $t = L/2\pi$, then $A - Lt + \pi t^2 \le 0$, that is, $A \le L^2/4\pi$, which is the isoperimetric inequality.

Remark 4.4. One has two involutions on the space of cooriented wave fronts: one $\gamma \leftrightarrow \gamma^*$ described in Corollary 3.5, and a second that is coorientation-reversing. The composition of these involutions is an interesting mapping of the space of cooriented wave fronts. This mapping has (at least) two integrals: signed area and length. Are there more?

5.   OSCILLATION OF UNICYCLE TRACKS

Recall Finn's construction described in Section 1. Let $\gamma(t)$, $t \in [0, L]$, be an arc-length parameterized smooth curve in $\mathbb{R}^2$ such that the ∞-jets of $\gamma(t)$ coincide, for $t = 0$ and $t = L$, with the ∞-jets of the x-axis at points $(0, 0)$ and $(1, 0)$, respectively. We use γ as a "seed" trajectory of the rear wheel of a bicycle. Then $\Gamma = T(\gamma) = \gamma + \gamma'$ is also tangent to the horizontal axis with all derivatives at its endpoints $(1, 0)$ and $(2, 0)$. Iterating this procedure yields a smooth infinite forward bicycle trajectory $\mathcal{T}$ such that the tracks of the rear and the front wheels coincide. We shall study oscillation properties of $\mathcal{T}$. For starters, we note that the length of each new arc of $\mathcal{T}$ increases compared to the previous one.

Lemma 5.1. *The length of Γ equals*

$$\int_0^L \sqrt{1 + k^2(t)}\, dt > L,$$

where $k(t) = |\gamma''(t)|$ is the curvature of γ.

Proof: One has

$$\Gamma'(t) = \gamma'(t) + \gamma''(t), \quad |\Gamma'(t)|^2 = 1 + |\gamma''(t)|^2;$$

therefore the length of Γ is

$$\int_0^L |\Gamma'(t)|\, dt = \int_0^L \sqrt{1 + k^2(t)}\, dt.$$

$\square$

Denote by $Z(\gamma)$ the number of intersection points of the curve $\gamma(t)$, $t \in (0, L)$, with the x-axis (we exclude the endpoints); assume that $Z(\gamma)$ is finite.

Proposition 5.2. *One has $Z(\Gamma) > Z(\gamma)$.*

Proof: Note that

$$e^{-t} \left(e^t \gamma(t)\right)' = \Gamma(t). \tag{5–1}$$

Let $Z(\gamma) = n$ and let $t_0 = 0 < t_1 < \cdots < t_n < t_{n+1} = L$ be the consecutive moments of intersection of $\gamma(t)$ with the x-axis. Then t_i are also the consecutive moments of intersection of the curve $\Delta(t) := e^t \gamma(t)$ with the x-axis. By a version of Rolle's theorem, see Figure 14, for each $i = 0, 1, \ldots, n$, there is $t \in (t_i, t_{i+1})$ for which the curve $\Delta(t)$ has a horizontal tangent, i.e., the vector $\Delta'(t)$ is horizontal. It follows from (5–1) that $\Gamma(t)$ lies on the x-axis, and we are done. $\square$

Consider the problem of extending the curve $\mathcal{T}$ backward, that is, inverting the operator T. It turns out that usually T can be inverted only finitely many times. Namely, one has the following corollary of Proposition 5.2.

Corollary 5.3. *Let Γ be a curve whose endpoints are unit distance apart and that is tangent to the x-axis at the endpoints to all orders. Let $Z(\Gamma) = n$. Then for no curve γ whose endpoints are unit distance apart and that*
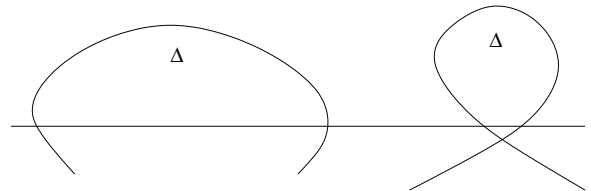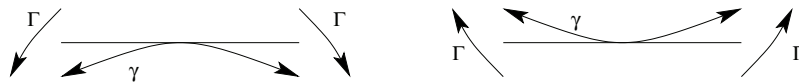

FIGURE 14. Rolle's theorem for curves.

FIGURE 15. Height extrema of the curve γ.

is tangent to the x-axis at the endpoints to all orders does one have $T^{n+1}(\gamma) = \Gamma$.

Here is another oscillation property of the curve $\mathcal{T}$. Let $E(\gamma)$ be the (finite) number of locally highest and lowest points of the curve γ. As before, $\Gamma = T(\gamma)$.

Proposition 5.4. *One has* $E(\Gamma) > E(\gamma)$.

Proof: At a locally highest point of γ, the curve Γ has the downward direction, and at a locally lowest point, it has the upward direction; see Figure 15. It follows that between consecutive locally highest and lowest points of γ, one has a locally lowest point of Γ, and between consecutive locally lowest and highest points of γ, one has a locally highest point of Γ. Considering the endpoints of γ as local extrema of the height function yields the result. $\square$

Conjecture 5.5. *It follows from Figure 15 that the maximum height of Γ is greater than that of γ, and likewise for the minimum height. We conjecture that the amplitude of the curve $\mathcal{T}$ is unbounded; in other words, unless γ is a segment, $\mathcal{T}$ is not contained in any horizontal strip. We also conjecture that unless γ is a segment, $\mathcal{T}$ is not the graph of a function (i.e., one of the curves $T^n(\gamma)$ has a vertical tangent line) and further, fails to be an embedded curve. One more conjecture: unless $\mathcal{T}$ is the horizontal axis, the curvature of $\mathcal{T}$ is unbounded.*

5.1 Configuration Space of Equilateral Forward Infinite Linkages

The construction of bicycle motion generating a single track can be interpreted as follows. Let $\mathcal{M}$ be the space of semi-infinite equilateral linkages $\{X = (x_0, x_1, x_2, \dots)\}$, where each x_i is a point in the plane and $|x_i - x_{i+1}| = 1$ for all i. Denote by v_i the unit vector $x_i x_{i+1}$ and by α_i the angle between v_{i-1} and v_i. Let $\mathcal{M}_0$ be an open subset of $\mathcal{M}$ given by the condition $\alpha_i \neq \pm\pi/2$ for all i.

Consider the constraint on $\mathcal{M}$ defined by the condition that the velocity of point x_i be proportional to v_i. If t_i is the speed of x_i, then the condition that all links remain of unit length is
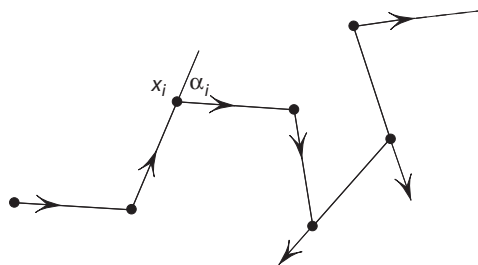
$$t_i = t_{i+1} \cos \alpha_{i+1} \qquad (5\text{--}2)$$

for all i. On $\mathcal{M}_0$, where $\cos \alpha_i \neq 0$, all the velocities are uniquely defined, up to a common factor, and one has a well-defined field of directions ξ, which can be normalized to a vector field by setting $t_0 = 1$. If $\alpha_i = \pi/2$ for some i, then the speeds of all x_j with $j < i$ must vanish; in particular, if $\alpha_i = \pi/2$ for infinitely many values of i, then such a configuration has no infinitesimal motions at all. See [Montgomery and Zhitomirskii 01, Montgomery and Zhitomirskii 09] for this nonholonomic system in relation to "monster tower" and Goursat flags.

A forward bicycle motion generating a single track corresponds to a solution to our system. The above-described curve $\mathcal{T}$ yields an integral curve of the field ξ in $\mathcal{M}_0$. Indeed, $\alpha_i = \pi/2$ corresponds to a cusp of the trajectory of point x_{i-1}, whereas $\mathcal{T}$ is a smooth curve, as follows from Lemma 2.1.

The starting configuration X of the Finn construction consists of nonnegative integers on the horizontal axis, $x_i = (i, 0)$, and one has a variety of integral curves of ξ through $X \in \mathcal{M}_0$ (of which the simplest one is uniform motion along the horizontal axis). Thus one has nonuniqueness of solutions of the differential equation describing the field ξ.

Finn's construction can be easily generalized as follows. Let δ be an infinite jet of a curve at the point x_0. Consider the infinite jet $T(\delta)$ at the point $x_1 = T(x_0)$,

FIGURE 16. Note the change of direction when $\alpha > \pi/2$. Only the direction of motion, and not the speeds, is indicated; the latter becomes large for large values of i.

and let γ be a curve smoothly interpolating between δ and $T(\delta)$. Then the concatenation of the curves γ, $T(\gamma)$, $T^2(\gamma)$, etc., is a smooth unicycle track left by the bicycle motion with the seed curve γ.

The above construction provides a mapping $\Phi : J^\infty(x_0) \rightarrow \mathcal{M}_0$ from the space of infinite jets of curves at the point x_0 to unit forward infinite linkages $\{(x_0, x_1, \ldots)\}$.

Proposition 5.6. *The mapping Φ is a bijection.*

Proof: We construct the inverse map $\Psi : \mathcal{M}_0 \rightarrow J^\infty(x_0)$. Let $X = (x_0, x_1, \ldots) \in \mathcal{M}_0$ and set $C_i = \cos \alpha_i \neq 0$. Then, according to (5–2),

$$t_0 = 1, \quad t_k = \frac{1}{\Pi_{i=1}^k C_i};$$

hence the speeds of all points are determined.

We claim that for each $r \geq 1$, one has $x_j^{(r)} = F_{j,r}(x_i, C_i)$, where F is a polynomial in x_i and a Laurent polynomial in C_i for $i = 0, 1, \ldots$. This is proved by induction on r. For $r = 1$, one has $x_j' = t_j(x_{j+1} - x_j)$. If $x_j^{(r)} = F_{j,r}(x_i, C_i)$, then

$$x_j^{(r+1)} = \sum_i \frac{\partial F_{j,r}}{\partial x_i} x_i' + \frac{\partial F_{j,r}}{\partial C_i} C_i'.$$

The induction step will be completed if we show that C_i' is also a polynomial in x_i and C_i. Indeed, $C_i = (x_i - x_{i-1}) \cdot (x_{i+1} - x_i)$, and hence

$$C_i' = \big(t_i(x_{i+1} - x_i) - t_{i-1}(x_i - x_{i-1})\big) \cdot (x_{i+1} - x_i)$$
$$+ (x_i - x_{i-1}) \cdot \big(t_{i+1}(x_{i+2} - x_{i+1}) - t_i(x_{i+1} - x_i)\big),$$

as required.

In particular, X determines all the derivatives $x_0^{(r)}$, that is, the infinite jet of a curve at x_0. This is $\Psi(X)$. $\square$

We finish with another question: Is a straight line the only real analytic "unicycle" trajectory?

6.   ACKNOWLEDGMENTS

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