Commun. Math. Phys. 176, 383–399 (1996)

Communications in
Mathematical
Physics

P-adic Theta Functions and Solutions of the KP Hierarchy

Takashi Ichikawa

Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840, Japan.
E-mail address: ichikawa@math.ms.saga-u.ac.jp

Received: 2 August 1994 / Accepted: 7 July 1995

Abstract: Based on Schottky uniformization theory of Riemann surfaces, we construct a universal power series for (Riemann) theta function solutions of the KP hierarchy. Specializing this power series to the coordinates associated with Schottky groups over p-adic fields, we show that the p-adic theta functions of Mumford curves give solutions of the KP hierarchy.

Introduction

The KP (Kadomtsev–Petvyashvili) hierarchy is a system of infinitely many Lax type partial differential equations,

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L]; \qquad L = \partial_x + \sum_{i=1}^{\infty} u_i(x, t_1, t_2, \dots)\partial_x^{-i}$$

($\partial_x = \partial/\partial x, (L^n)_+$: the nonnegative part of L^n for ∂_x) which includes the KP equation

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial t_2^2} - \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial t_3} - \frac{1}{4}\frac{\partial^3 u_1}{\partial x^3} - 3u_1\frac{\partial u_1}{\partial x}\right) = 0 .$$

Our final goal in this paper is to show that the p-adic theta functions of p-adic (Mumford) curves give solutions of the KP hierarchy. These solutions are included in algebro-geometric solutions for p-adic curves constructed by Krichever [Kr]. However, one cannot express these Krichever solutions in terms of p-adic theta functions as is done in [Kr] for the complex case because there is no theory on curvilinear integrals in p-adic analysis.

Our construction of p-adic solutions of the KP hierarchy consists of 2 steps: the first step is to obtain a "universal" solution expressed by a formal theta function, and the second step is to specialize this universal solution to p-adic solutions. For example, in the genus 1 case, the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{u \in L - \{0\}} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2}\right) \quad (L := \mathbf{Z}(\pi\tau) + \mathbf{Z}\pi)$$

for complex numbers τ with positive imaginary part can be regarded as an element of $\mathbf{Q}[[z,q]][1/z](q := \exp(2\pi\sqrt{-1}\tau))$ which gives a universal elliptic solution of the KdV (Korteweg–de Vries) hierarchy

$$\frac{\partial L^2}{\partial t_{2n-1}} = [(L^{2n-1})_+, L^2]; \qquad L^2 = \partial_x^2 + 2u_1(x,t_1,t_3,\ldots)\,,$$

and hence for p-adic numbers a with $|a|_p < 1$, $\wp(z)|_{q=a}$ give p-adic solutions of KdV. In the genus $\geqq 2$ case, we need Schottky uniformization theory of algebraic curves over local fields for the construction of a universal solution and p-adic solutions. It is known in [H, Sc] that there exist universal expressions of meromorphic 1-forms and period integrals of Schottky uniformized Riemann surfaces with sufficiently small handles. Based on results in [I], we give power series expansions of these 1-forms and periods with respect to the Koebe coordinates (the fixed points and the eigenvalues of generators of the associated Schottky groups) which, combined with the description given in [Kr] of quasi-periodic solutions of KP in terms of Riemann theta functions, induce a *universal power series* for solutions of KP from Riemann surfaces with square roots of canonical bundles. In this expression, the KP hierarchy is reduced to identities between certain formal power series including the universal periods given in [I]. Since pinching handles of Riemann surfaces induces degenerations of the solutions, one can obtain the solitonic degeneration from the universal solution which has been described in [Mum2] for KdV and in [Go] for KP.

As an application of the universal solution, we will construct formal solutions of the KP hierarchy with coefficients in p-adic fields. We show that over p-adic fields, the universal 1-forms and periods converge for the Koebe coordinates of any Schottky group, which implies the convergence of the universal solution. Hence this specialization of the universal solution gives solutions of KP which are seen to be expressed explicitly by the p-adic theta functions of Schottky uniformized algebraic curves over p-adic fields. It is shown in [Mum1] that an algebraic curve over p-adic fields can be Schottky uniformized if and only if this special fiber consists of projective lines, in which case, such a curve is called a *Mumford curve*. Therefore, we can see that the p-adic theta functions of Mumford curves give solutions of the KP hierarchy.

Based on the above result, we propose two problems. The first is to characterize these p-adic solutions of the KP hierarchy and the KP equation, which is concerned with a p-adic version of the Novikov conjecture mainly studied in [A-D, Mul and Sh]. The second is to construct a theory on Mumford curves "of infinite genus" uniformized by infinitely generated Schottky groups over p-adic fields, which will give solutions of the KP hierarchy conjectured to have infinite dimensional orbits.

1. Riemann Surfaces and the KP Hierarchy

Let g be a positive integer. We fix a Riemann surface C of genus g, a canonical basis $\{a_i,b_i\}_{1\leqq i\leqq g}$ of $H_1(C,\mathbf{Z})$ (i.e., $(a_i,b_j)=\delta_{ij}$, $(a_i,a_j)=(b_i,b_j)=0$), a point $P \in C$, and a local coordinate u at P such that $u(P)=0$. We denote the whole data by $X=(C,\{a_i,b_i\}_{1\leqq i\leqq g},P,u)$. Then there exists a unique basis $\{\omega_1,\ldots,\omega_g\}$

of $H^0(C,\Omega_C^1)$ satisfying $\int_{a_i}\omega_j = 2\pi\sqrt{-1}\delta_{ij}$ $(i,j=1,\ldots,g)$, and

$$Z = (Z_{ij})_{1\leqq i,j\leqq g} = \left(\frac{1}{2\pi\sqrt{-1}}\int_{b_i}\omega_j\right)_{1\leqq i,j\leqq g}$$

is the period matrix of $(C,\{a_i,b_i\}_{1\leqq i\leqq g})$. Take $r_{jm}\in\mathbf{C}$ $(j=1,\ldots,g,\ m\in\mathbf{N})$ such that

$$\omega_j = \sum_{m=1}^{\infty} r_{jm}\,u^{m-1}du \quad \text{at } P\,.$$

For each $n\in\mathbf{N}$, there exists a unique meromorphic 1-form $\omega^{(n)}$ on C satisfying

$$\omega^{(n)} \text{ is holomorphic outside } P\,, \tag{1.1}$$

$$\omega^{(n)} = \left(\frac{1}{u^{n+1}} + \sum_{m=1}^{\infty}\frac{q_{nm}}{n}u^{m-1}\right)du \quad \text{at } P \text{ for some } q_{nm}\in\mathbf{C}\,, \tag{1.2}$$

$$\int_{a_i}\omega^{(n)} = 0 \quad \text{for any } i=1,\ldots,g\,. \tag{1.3}$$

For $\mathbf{c}=(c_i)_{1\leqq i\leqq g}\in(\mathbf{C}^{\times})^g$ and a vector $\vec{z}=(z_i)_{1\leqq i\leqq g}$ of g indeterminates, we denote the Riemann theta function of $(C,\{a_i,b_i\}_{1\leqq i\leqq g})$ by

$$\Theta(\mathbf{c}\cdot\exp(\vec{z})) = \sum_{\vec{v}\in\mathbf{Z}^g}\left\{\prod_{i,j=1}^{g}\exp(\pi\sqrt{-1}Z_{ij})^{v_iv_j}\prod_{i=1}^{g}c_i^{v_i}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{i=1}^{g}v_iz_i\right)^n\right\}\,,$$

where $\vec{v}=(v_i)_{1\leqq i\leqq g}$. Let $t_m (m\in\mathbf{N})$ be indeterminates, and put $\mathbf{t}=(t_1,t_2,t_3,\ldots)$. Then we define the τ-function $\tau(\mathbf{t},X_{\mathbf{c}})$ as an element of $\mathbf{C}[[\mathbf{t}]]=\mathbf{C}[[t_1,t_2,\ldots]]$ by

$$\tau(\mathbf{t},X_{\mathbf{c}}) = \exp\left(\frac{1}{2}\sum_{n,m=1}^{\infty}q_{nm}t_nt_m\right)\cdot\Theta\left(\mathbf{c}\cdot\exp\left(\sum_{m=1}^{\infty}t_m\vec{r}_m\right)\right)\,,$$

where $\vec{r}_m=(r_{jm})_{1\leqq j\leqq g}$. For $\mathbf{c}\in(\mathbf{C}^{\times})^g$ with $\Theta(\mathbf{c})\neq 0$, we put

$$\frac{\tau(\mathbf{t}-[z],X_{\mathbf{c}})}{\tau(\mathbf{t},X_{\mathbf{c}})} = 1+\sum_{k=1}^{\infty}w_k(\mathbf{t},X_{\mathbf{c}})z^k\,, \tag{1.4}$$

where $[z]=(z,z^2/2,z^3/3,\ldots)$, and define two micro-differential operators

$$W(\mathbf{t},X_{\mathbf{c}}) = 1+\sum_{k=1}^{\infty}w_k(\mathbf{t},X_{\mathbf{c}})\partial_x^{-k}$$

and

$$L(\mathbf{t},X_{\mathbf{c}}) = W(\mathbf{t}+x,X_{\mathbf{c}})\cdot\partial_x\cdot W(\mathbf{t}+x,X_{\mathbf{c}})^{-1}$$

with coefficients in $\mathbf{C}[[x,\mathbf{t}]]$, where $\mathbf{t}+x=(t_1+x,t_2,t_3,\ldots)$. Then it is known (cf. [Kr, S-S]) that $L(\mathbf{t},X_{\mathbf{c}})$ satisfies the KP hierarchy

$$\frac{\partial L}{\partial t_n} = [(L^n)_+,L] \quad (n\in\mathbf{N})\,.$$

In particular,

$$u_1(x,t_2,t_3)=\frac{\partial^2}{\partial x^2}\log\Theta(\mathbf{c}\cdot\exp(x\vec{r}_1+t_2\vec{r}_2+t_3\vec{r}_3))+q_{11}$$

satisfies the KP equation

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial t_2^2}-\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial t_3}-\frac{1}{4}\frac{\partial^3 u_1}{\partial x^3}-3u_1\frac{\partial u_1}{\partial x}\right)=0\,.$$

2. Schottky Uniformization

2.1. Let K be $\mathbf{C}$ or a complete nonarchimedean valuation field with multiplicative valuation $|\ |$. Let $PGL_2(K)$ act on $\mathbf{P}^1(K)$ by the Möbius transformation. A subgroup Γ of $PGL_2(K)$ is called a *Schottky group* of rank g over K if there exist its free generators $\gamma_1,\dots,\gamma_g$ and $2g$ open domains bounded by Jordan curves if $K=\mathbf{C}$ (resp. $2g$ open disks if K is a nonarchimedean valuation field) $D_{\pm 1},\dots,D_{\pm g}\subset\mathbf{P}^1(K)$ such that

$$\overline{D}_i\cap\overline{D}_j=\emptyset\ (i\neq j),\qquad \gamma_k(\mathbf{P}^1(K)-D_{-k})=\overline{D}_k\ (k=1,\dots,g)\,,$$

and then Γ is called to be *marked* if selecting such a sequence $\gamma_1,\dots,\gamma_g$. Then for each $k=1,\dots,g$, we can take uniquely $\alpha_{\pm k}\in D_{\pm k}$ and $\beta_k\in K^\times$ such that

$$\gamma_k=\begin{pmatrix}\alpha_k & \alpha_{-k}\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \beta_k\end{pmatrix}\begin{pmatrix}\alpha_k & \alpha_{-k}\\ 1 & 1\end{pmatrix}^{-1}\operatorname{mod}(K^\times)\,, \tag{2.1}$$

i.e.,

$$\frac{\gamma_k(z)-\alpha_k}{\gamma_k(z)-\alpha_{-k}}=\beta_k\frac{z-\alpha_k}{z-\alpha_{-k}}\quad(z\in\mathbf{P}^1(K))\,.$$

We call $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$ the *Koebe coordinates* of $(\Gamma;\gamma_1,\dots,\gamma_g)$. A Schottky group Γ is called *classical* if we can take $D_{\pm k}$ $(k=1,\dots,g)$ as open disks (hence all Schottky groups over nonarchimedean valuation fields are automatically classical (cf. [G-v.d.P])). Then there exist $\mu_{\pm 1},\dots,\mu_{\pm g}<1$ such that

$$D_{\pm k}=\{z\in\mathbf{P}^1(K);\,|z-\alpha_{\pm k}|<\mu_{\pm k}|z-\alpha_{\mp k}|\}\quad(k=1,\dots,g)\,.$$

In particular,

$$|\beta_k|<\min\{|[\alpha_k,\alpha_{-k};\alpha_i,\alpha_j]|;\,i,j\neq\pm k\}\quad(k=1,\dots,g)\,,$$

where

$$[a,b;c,d]=\frac{(a-c)(b-d)}{(a-d)(b-c)}$$

denotes the cross ratio of four points. Put

$$F_\Gamma=\mathbf{P}^1(K)-\bigcup_{k=1}^{g}(D_k\cup\overline{D_{-k}}),\qquad H_\Gamma=\bigcup_{\gamma\in\Gamma}\gamma(F_\Gamma)\,.$$

Then Γ acts on H_Γ freely and properly discontinuously, and the quotient K-analytic space $C_\Gamma=H_\Gamma/\Gamma$ is obtained from $\mathbf{P}^1(K)-\cup_{k=1}^{g}D_{\pm k}$ by identifying ∂D_k and ∂D_{-k} via γ_k $(k=1,\dots,g)$. It is known that the K-analytic space C_Γ has naturally the structure of a proper and smooth algebraic curve of genus g over K. Moreover, it is known that any Riemann surface can be Schottky uniformized, and that an algebraic curve over a nonarchimedean valuation field can be Schottky uniformized if and only if this curve is a Mumford curve, i.e., the special fiber consists of projective lines (cf. [Ko, Mum1]). Let S_K be the subset of $(\mathbf{P}^1(K) \times \mathbf{P}^1(K) \times K^\times)^g$ consisting of the Koebe coordinates of all marked Schottky groups over K. For $k = -1, \dots, -g$, put $\beta_k = \beta_{-k}$ and $\gamma_k = (\gamma_{-k})^{-1}$. Then γ_k satisfies (2.1) also.

2.2. Proposition. *Let $(\Gamma; \gamma_1, \dots, \gamma_g)$ be a marked Schottky group over K with Koebe coordinates $(\alpha_{\pm k}, \beta_k)_{1 \leqq k \leqq g}$, and let Γ' be the Schottky group generated by $\gamma_1, \dots, \gamma_{g-1}$. Then under $\beta_g \to 0$, C_Γ becomes the degenerate algebraic curve C' obtained by identifying α_g and α_{-g} in $C_{\Gamma'}$.*

Proof. For any $\varepsilon \in K^\times$ with $|\varepsilon| \leqq 1$, let $D_{\pm g}(\varepsilon)$ be the domains containing $\alpha_{\pm g}$ given by

$$D_{\pm g}(\varepsilon) = \{z \in \mathbf{P}^1(K); |z - \alpha_{\pm g}| < \sqrt{|\varepsilon|}\, \mu_{\pm g}\, |z - \alpha_{\mp g}|\},$$

and put

$$\gamma_g(\varepsilon) = \begin{pmatrix} \alpha_g & \alpha_{-g} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\beta_g \end{pmatrix} \begin{pmatrix} \alpha_g & \alpha_{-g} \\ 1 & 1 \end{pmatrix}^{-1} \bmod (K^\times).$$

Then $\gamma_g(\varepsilon)(\partial D_{-g}(\varepsilon)) = \partial D_g(\varepsilon)$. Let $\Gamma(\varepsilon)$ be the subgroup of $PGL_2(K)$ generated by $\gamma_1, \dots, \gamma_{g-1}$ and $\gamma_g(\varepsilon)$. Then $\Gamma(\varepsilon)$ is a Schottky group of rank g and $C_{\Gamma(\varepsilon)}$ is isomorphic to the K-analytic space obtained by identifying ∂D_k and ∂D_{-k} via γ_k $(k = 1, \dots, g-1)$, $\partial D_g(\varepsilon)$ and $\partial D_{-g}(\varepsilon)$ via $\gamma_g(\varepsilon)$. Therefore, if $|\varepsilon| \to 0$, then $D_{\pm g}(\varepsilon)$ tend to $\alpha_{\pm g}$, and hence $C_{\Gamma(\varepsilon)}$ becomes C'.

2.3. Let $K = \mathbf{C}$, and let the notation be as in 2.1. Taking the conjugation $\gamma_i \mapsto \rho\gamma_i\rho^{-1} (i = 1, \dots, g)$ by a certain $\rho \in PGL_2(\mathbf{C})$, we may assume that $\infty \in F_\Gamma$ which implies that $\alpha_{\pm k} \neq \infty$ $(k = 1, \dots, g)$. For each $i = 1, \dots, g$, let a_i be the closed path ∂D_i counterclockwise oriented, and let b_i be an oriented path in F_Γ from a certain point x_i of ∂D_{-i} to $\gamma_i(x_i)$ such that $b_i \cap b_j = \emptyset$ $(i \neq j)$. Then $\{a_i, b_i\}_{1 \leqq i \leqq g}$ becomes a canonical basis of $H_1(C_\Gamma, \mathbf{Z})$. Let p be a point of $F_\Gamma - \{\infty\}$, let P be the point of C_Γ corresponding to p, and put $u = z - p$. Assume that $\beta_1, \dots, \beta_g$ are sufficiently small. Then we express $\omega^{(n)}$, ω_j, and Z_{ij} explicitly for $(C_\Gamma, \{a_i, b_i\}_{1 \leqq i \leqq g}, P, u)$ according to [H]. It is shown in [Sc] (see also [A]) that $\Sigma_{\gamma \in \Gamma} |\gamma'(z)|$ is uniformly convergent on $H_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\infty)$ under the above assumption. Hence for each integer $n \geqq 0$, one can define a meromorphic 1-form

$$\omega^{(n)} = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{(\gamma(z) - p)^{n+1}} dz$$

on H_Γ. Since $\omega^{(n)}$ is Γ-invariant, this induces a meromorphic 1-form on C_Γ which we denote by the same symbol, and a term-by-term integration shows that $\int_{a_i} \omega^{(n)} = 0$ $(i = 1, \dots, g)$. It is easy to see that $\omega^{(0)}$ is holomorphic except for simple poles of residue 1 (resp. -1) at P (resp. ∞), and that $\omega^{(n)} (n \geqq 1)$ satisfy (1.1) $-$(1.3). For each $j = 1, \dots, g$, let ω_j be the holomorphic 1-form on C_Γ such that $\int_{a_i} \omega_j = 2\pi\sqrt{-1}\delta_{ij}$ $(i = 1, \dots, g)$. Then by the classical period relation,

$$\int_\infty^P \omega_j = \int_{b_j} \omega^{(0)} = \int_{x_j}^{\gamma_j(x_j)} \omega^{(0)}.$$

Hence

$$\omega_j = \sum_{\gamma\in\Gamma}\left(\frac{1}{z-(\gamma\gamma_j)(x_j)} - \frac{1}{z-\gamma(x_j)}\right)dz$$
$$= \sum_{\gamma\in\Gamma_j}\left(\frac{1}{z-\gamma(\alpha_j)} - \frac{1}{z-\gamma(\alpha_{-j})}\right)dz$$
$$= \sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{\gamma\in\Gamma_j}\left(\frac{\gamma(\alpha_j)-\gamma(\alpha_{-j})}{(\gamma(\alpha_j)-p)^k(\gamma(\alpha_{-j})-p)^{m+1-k}}\right)u^{m-1}du\,,$$

where Γ_j $(j=1,\dots,g)$ denotes a complete set of representatives of the cosets $\Gamma/\langle\gamma_j\rangle$. Since

$$Z_{ij} = \frac{1}{2\pi\sqrt{-1}}\int_{b_i}\omega_j = \frac{1}{2\pi\sqrt{-1}}\int_{x_i}^{\gamma_i(x_i)}\omega_j \ (i,j=1,\dots,g)\,,$$

taking appropriate branches of the logarithm, we have

$$Z_{ij} = \frac{1}{2\pi\sqrt{-1}}\sum_{\gamma\in\Gamma_j}\log([\gamma_i(x),x;\gamma(\alpha_j),\gamma(\alpha_{-j})])$$
$$= \frac{1}{2\pi\sqrt{-1}}\sum_{\gamma\in\Gamma_{ij}}\log(\psi_{ij}(\gamma))\,,$$

where Γ_{ij} $(i,j=1,\dots,g)$ denotes a complete set of representatives of the cosets $\langle\gamma_i\rangle\backslash\Gamma/\langle\gamma_j\rangle$, and ψ_{ij} is the map $\Gamma_{ij}\to\mathbf{C}^\times$ given by

$$\psi_{ij}(\gamma) = \begin{cases}\beta_i & (\text{if } i=j \text{ and } \gamma\in\langle\gamma_i\rangle)\\ [\alpha_i,\alpha_{-i};\gamma(\alpha_j),\gamma(\alpha_{-j})] & (\text{otherwise})\,.\end{cases}$$

For each $i,j=1,\dots,g$, put $p_{ij}=\exp(2\pi\sqrt{-1}Z_{ij})$. Then

$$p_{ij} = \prod_{\gamma\in\Gamma_{ij}}\psi_{ij}(\gamma)\,.$$

We note that in [M-D], the above ω_j and p_{ij} are obtained as holomorphic 1-forms and multiplicative periods of Schottky uniformized Mumford curves (cf. Corollary 4.4).

3. Universal Solution of KP

3.1. Let $x_{\pm k}, y_k$ $(k=1,\dots,g)$, p, and z be variables, and put $u=z-p$. Let A be the ring of formal power series over $\mathbf{Z}[x_k,\Pi_{i<j}1/(x_i-x_j)]$ $(i,j,k\in\{\pm1,\dots,\pm g\})$ with variables $y_1,\dots,y_g$, i.e.,

$$A = \mathbf{Z}\left[x_k,\prod_{i<j}\frac{1}{x_i-x_j}\right][[y_1,\dots,y_g]]\,,$$

and put

$$A_p = A\left[\prod_{k=1}^{g}\frac{1}{(x_k-p)(x_{-k}-p)}\right]\,.$$

Let I be the ideal of A generated by $y_1, \dots, y_g$. For each $k = \pm 1, \dots, \pm g$, put $y_k = y_{|k|}$, and let f_k be the element of $GL_2(\Omega)$ (Ω : the quotient field of A) given by

$$f_k = \begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_k \end{pmatrix} \begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix}^{-1} .$$

3.2. Proposition. *For any* $f = f_{k(1)} f_{k(2)} \cdots f_{k(l)}$ *satisfying* $k(s) \neq -k(s+1)$ $(s = 1, \dots, l-1)$, *put* $y = y_{k(1)} y_{k(2)} \cdots y_{k(l)}$. *Then we have*

(a) $f(x_j) \in A$, *and* $f(x_j) \in x_{k(1)} + I$ *for* $f \notin \langle f_j \rangle$,

(b) $f(x_j) - f(x_{-j}) \in yA$ *for* $j \neq \pm k(l)$,

(c) $[x_i, x_{-i}; f(x_j), f(x_{-j})] \in 1 + yA$ *for* $i \neq \pm k(1), j \neq \pm k(l)$,

(d) $\frac{f'(z)}{(f(z) - p)^{n+1}} \in y(A_p[[u]])$.

Proof. (a) We prove this by induction on l. Assume that $f \notin \langle f_j \rangle$ and $f(x_j) = x_{k(1)} + a$ for some $a \in I$. Then for any $i \neq -k(1)$,

$$(f(x_j) - x_{-i})^{-1} = \left\{ (x_{k(1)} - x_{-i}) \left(1 + \frac{a}{x_{k(1)} - x_{-i}} \right) \right\}^{-1} \in A .$$

Hence

$$(f_i f)(x_j) = \left\{ x_i - \frac{(f(x_j) - x_i) x_{-i} y_i}{f(x_j) - x_{-i}} \right\} \left\{ 1 - \frac{(f(x_j) - x_i) y_i}{f(x_j) - x_{-i}} \right\}^{-1}$$

belongs to $x_i + I$. Assume that $f \in \langle f_j \rangle$. Then $f(x_j) = x_j$, and hence for any $i \neq \pm j$, $(f_i f)(x_j) \in x_i + I$.

(b) We prove this by induction on l. Assume that $j \neq \pm k(l)$ and $f(x_j) - f(x_{-j}) \in yA$. Then by (a), $f(x_j) = x_{k(1)} + b$ for some $b \in I$, and hence for any $i \neq -k(1)$,

$$\{f(x_j) - x_{-i} - y_i(f(x_j) - x_i)\}^{-1} = \{(x_{k(1)} - x_{-i}) + (b - y_i(f(x_j) - x_i))\}^{-1}$$

belongs to A. Similarly, $\{f(x_{-j}) - x_{-i} - y_i(f(x_{-j}) - x_i)\}^{-1}$ belongs to A. Hence

$$\begin{aligned} &(f_i f)(x_j) - (f_i f)(x_{-j}) \\ &= \frac{(x_i - x_{-i})^2 (f(x_j) - f(x_{-j})) y_i}{\{f(x_j) - x_{-i} - y_i(f(x_j) - x_i)\}\{f(x_{-j}) - x_{-i} - y_i(f(x_{-j}) - x_i)\}} \end{aligned}$$

belongs to $y y_i A$.

(c) By (a), $(f(x_j) - x_{-i})^{-1}$ and $(f(x_{-j}) - x_i)^{-1}$ belong to A. Therefore, by (b),

$$[x_i, x_{-i}; f(x_j), f(x_{-j})] = 1 + \frac{(x_i - x_{-i})(f(x_j) - f(x_{-j}))}{(f(x_j) - x_{-i})(f(x_{-j}) - x_i)}$$

belongs to $1 + yA$.

(d) Put

$$f = \begin{pmatrix} a_f & b_f \\ c_f & d_f \end{pmatrix} .$$

Then

$$\frac{f'(z)}{(f(z)-p)^{n+1}}=\frac{(a_f d_f-b_f c_f)\{c_f p+d_f+c_f u\}^{n-1}}{\{(a_f p+b_f-c_f p^2-d_f p)+(a_f-c_f p)u\}^{n+1}}\,.$$

Since

$$f_k=\frac{1}{x_k-x_{-k}}\left\{\begin{pmatrix} x_k & -x_k x_{-k}\\ 1 & -x_{-k}\end{pmatrix}-\begin{pmatrix} x_{-k} & -x_{-k}x_k\\ 1 & -x_k\end{pmatrix}y_k\right\}$$

and

$$\begin{pmatrix} \alpha & -\alpha\beta\\ 1 & -\beta\end{pmatrix}\begin{pmatrix} \gamma & -\gamma\delta\\ 1 & -\delta\end{pmatrix}=(\gamma-\beta)\begin{pmatrix} \alpha & -\alpha\delta\\ 1 & -\delta\end{pmatrix},$$

a_f, b_f, c_f and d_f belong to A, and these constant terms are $x_{k(1)}t, -x_{k(1)}x_{-k(l)}t, t$ and $-x_{-k(l)}t$ respectively, where

$$t=\frac{\prod_{s=2}^{l}(x_{k(s)}-x_{-k(s-1)})}{\prod_{s=1}^{l}(x_{k(s)}-x_{-k(s)})}\in A^\times\,.$$

Hence the constant terms of $c_f p+d_f$ and $a_f p+b_f-c_f p^2-d_f p$ are $-(x_{-k(l)}-p)t\in A_p^\times$ and $-(x_{k(1)}-p)(x_{-k(l)}-p)t\in A_p^\times$ respectively. This implies (d).

3.3. Put $\varphi_k=f_k \mathrm{mod}(\Omega^\times)\in PGL_2(\Omega)$ for $k=\pm1,\dots,\pm g$, and let Φ be the free subgroup of $PGL_2(\Omega)$ with generators $\varphi_1,\dots,\varphi_g$. Then $\varphi_{-k}=\varphi_k^{-1}$ and $\varphi_k(z)=f_k(z)$. Let Φ_j and Φ_{ij} be the subsets of Φ given by

$$\Phi_j=\{\varphi_{k(1)}\cdots\varphi_{k(l)};\ k(s)\neq-k(s+1),k(l)\neq\pm j\}\,,$$
$$\Phi_{ij}=\{\varphi_{k(1)}\cdots\varphi_{k(l)};\ k(s)\neq-k(s+1),k(1)\neq\pm i,k(l)\neq\pm j\}\,.$$

Then Φ_j (resp. Φ_{ij}) is a complete set of representatives of the cosets $\Phi/\langle\varphi_j\rangle$ (resp. $\langle\varphi_i\rangle\backslash\Phi/\langle\varphi_j\rangle$). Hence by Proposition 3.2, one can define two 1-forms with coefficients in A_p and an element of A as follows:

$$\Omega^{(n)}=\sum_{\varphi\in\Phi}\frac{\varphi'(z)}{(\varphi(z)-p)^{n+1}}du\ (n\geqq 0)\,,$$

$$\Omega_j=\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{\varphi\in\Phi_j}\frac{(\varphi(x_j)-\varphi(x_{-j}))u^{m-1}}{(\varphi(x_j)-p)^k(\varphi(x_{-j})-p)^{m+1-k}}du\ (j=1,\dots,g)\,,$$

$$P_{ij}=\prod_{\varphi\in\Phi_{ij}}\psi_{ij}(\varphi)\ (i,j=1,\dots,g)\,,$$

where

$$\psi_{ij}(\varphi)=\begin{cases} y_i & (\text{if } i=j \text{ and } \varphi\in\langle\varphi_i\rangle)\\ [x_i,x_{-i};\varphi(x_j),\varphi(x_{-j})] & (\text{otherwise})\,.\end{cases}$$

We introduce g variables $y_i^{1/2}$ $(i=1,\dots,g)$ which are square roots of y_i, and define square roots $P_{ii}^{1/2}\in A[y_1^{1/2},\dots,y_g^{1/2}]\hat{\otimes}_{\mathbf{Z}}\mathbf{Q}$ of P_{ii} $(i=1,\dots,g)$ by

$$P_{ii}^{1/2}=y_i^{1/2}\sum_{n=0}^{\infty}\binom{1/2}{n}\left\{\prod_{\varphi\in\Phi_{ii}-\{1\}}\psi_{ii}(\varphi)-1\right\}^n\,.$$

Then for two sequences $\mathbf{w} = (w_i)_{1 \leqq i \leqq g}$ and $\vec{z} = (z_i)_{1 \leqq i \leqq g}$ of g indeterminates, the universal theta function is defined by

$$\Theta(\mathbf{w} \cdot \exp(\vec{z})) = \sum_{\vec{v} \in \mathbf{Z}^g} \left\{ \prod_{i=1}^{g} P_{ii}^{v_i^2/2} \prod_{i<j} P_{ij}^{v_i v_j} \prod_{i=1}^{g} w_i^{v_i} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^{g} v_i z_i \right)^n \right\} .$$

Let Q_{nm} and R_{jm} be the elements of A_p such that

$$\Omega^{(n)} = \left(\frac{1}{u^{n+1}} + \sum_{m=1}^{\infty} \frac{Q_{nm}}{n} u^{m-1} \right) du \ (n \geqq 0) ,$$

$$\Omega_j = \sum_{m=1}^{\infty} R_{jm} u^{m-1} du \ (j = 1, \dots, g) ,$$

and put $\vec{R}_m = (R_{jm})_{1 \leqq j \leqq g}$. Then we define the *universal τ-function* by

$$\tau(\mathbf{t}) = \exp\left(\frac{1}{2} \sum_{n,m=1}^{\infty} Q_{nm} t_n t_m \right) \cdot \Theta\left(\mathbf{w} \cdot \exp\left(\sum_{m=1}^{\infty} t_m \vec{R}_m \right) \right)$$

which is an element of

$$B = A_p[y_1^{1/2}, \dots, y_g^{1/2}, w_1^{\pm 1}, \dots, w_g^{\pm 1}] \hat{\otimes}_{\mathbf{Z}} \mathbf{Q}[[\mathbf{t}]] .$$

Since $\tau(\mathbf{t}) - 1$ belongs to the ideal of B generated by $y_1^{1/2}, \dots, y_g^{1/2}$, as in (1.4)–(1.6), $\tau(\mathbf{t})$ defines a micro-differential operator $L(\mathbf{t})$ with coefficients in $B[[x]]$.

3.4. Theorem.

(a) *Let $(\Gamma; \gamma_1, \dots, \gamma_g)$ be a marked Schottky group over* $\mathbf{C}$ *with Koebe coordinates $(\alpha_{\pm k}, \beta_k)_{1 \leqq k \leqq g}$, and let $X = (C_\Gamma, \{a_i, b_i\}_{1 \leqq i \leqq g}, P, u)$ be as in* 2.3 *with period matrix $(Z_{ij})_{1 \leqq i,j \leqq g}$. Assume that $\beta_1, \dots, \beta_g$ are sufficiently small, and take $\beta_1^{1/2}, \dots, \beta_g^{1/2}$ such that*

$$\beta_i^{1/2} \sum_{n=0}^{\infty} \binom{1/2}{n} \left\{ \prod_{\gamma \in \Gamma_{ii} - \{1\}} \psi_{ii}(\gamma) - 1 \right\}^n = \exp(\pi \sqrt{-1} Z_{ii}) .$$

Then for any $\mathbf{c} = (c_i)_{1 \leqq i \leqq g} \in (\mathbf{C}^\times)^g$ with $\Theta(\mathbf{c}) \neq 0$,

$$L(\mathbf{t}) \Big|_{x_k = \alpha_k, y_k^{1/2} = \beta_k^{1/2}, w_k = c_k} = L(\mathbf{t}, X_{\mathbf{c}}) .$$

(b) *$L(\mathbf{t})$ satisfies the KP hierarchy* (1.7). *In particular,*

$$u_1(x, t_2, t_3) = \frac{\partial^2}{\partial x^2} \log \Theta(\mathbf{w} \cdot \exp(x\vec{R}_1 + t_2 \vec{R}_2 + t_3 \vec{R}_3)) + Q_{11}$$

satisfies the KP equation (1.8).

Proof. Assertion (a) follows from the definition of $L(\mathbf{t})$. Hence as seen in Sect. 1, $\partial L(\mathbf{t})/\partial t_n$ and $[(L(\mathbf{t})^n)_+, L(\mathbf{t})]$ coincide for $x_k = \alpha_k, y_k^{1/2} = \beta_k^{1/2}, w_k = c_k,\ p \in F_\Gamma$ if $(\alpha_{\pm k}, \beta_k)_{1 \leqq k \leqq g} \in S_K, \beta_1, \dots, \beta_g$ are sufficiently small and $\mathbf{c}$ is generic (i.e., $\Theta(\mathbf{c}) \neq 0$). Therefore, $\partial L(\mathbf{t})/\partial t_n = [(L(\mathbf{t})^n)_+, L(\mathbf{t})]$.

3.5. By Proposition 2.2, for a marked Schottky group $(\Gamma;\gamma_1,\dots,\gamma_g)$ with Koebe coordinates $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$, if $\beta_1,\dots,\beta_g\to 0$, then C_Γ becomes a degenerate curve obtained from $\mathbf{P}^1(K)$ by identifying α_k and $\alpha_{-k}(k=1,\dots,g)$. By Proposition 3.2,

$$\tau(\mathbf{t})|_{y_1=\cdots=y_g=0}=1\ ,$$

and hence $L(\mathbf{t})|_{y_1=\cdots=y_g=0}=\partial_x$ which we call the *trivial degeneration.* On the other hand, generalizing a modified theta function in [Go, Mum2], we define

$$\Theta_\delta(\mathbf{w}\cdot\exp(\vec{z}))=\sum_{\vec{v}\in\mathbf{Z}^g}\left\{\prod_{i=1}^g P_{ii}^{(v_i^2-v_i)/2}\prod_{i<j}P_{ij}^{v_iv_j}\prod_{i=1}^g w_i^{v_i}\sum_{n=0}^\infty\frac{1}{n!}\left(\sum_{i=1}^g v_iz_i\right)^n\right\}$$

and

$$\tau(\mathbf{t})=\exp\left(\frac{1}{2}\sum_{n,m=1}^\infty Q_{nm}t_nt_m\right)\cdot\Theta_\delta\left(\mathbf{w}\cdot\exp\left(\sum_{m=1}^\infty t_m\vec{R}_m\right)\right)\ .$$

Then by Proposition 3.2,

$$\tau_\delta(\mathbf{t})|_{y_1=\cdots=y_g=0}=\sum_{\vec{v}\in\{0,1\}^g}\left[\prod_{i<j}[x_i,x_{-i},x_j,x_{-j}]^{v_iv_j}\prod_{i=1}^g w_i^{v_i}\right.$$
$$\left.\times\sum_{n=0}^\infty\frac{1}{n!}\left\{\sum_{i=1}^g v_i\sum_{m=1}^\infty\left(\frac{1}{(x_{-i}-p)^m}-\frac{1}{(x_i-p)^m}\right)t_m\right\}^n\right]$$

which induces the soliton solution (cf. [Go, Mum2]).

4. *P*-adic Solutions of KP

4.1. Let K be a complete nonarchimedean valuation field, and let Γ be a Schottky group of rank g over K. Then as seen in 2.1 (cf. [Ge]), there exist free generators $\gamma_1,\dots,\gamma_g$ of Γ whose Koebe coordinates $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$ satisfy $\alpha_i\neq\alpha_j(i\neq j)$ and

$$|\beta_k|<\min\{|[\alpha_k,\alpha_{-k};\alpha_i,\alpha_j]|;i,j\neq\pm k\}\quad(k=1,\dots,g)\,.$$

Taking the conjugation $\gamma_i\mapsto\rho\gamma_i\rho^{-1}(i=1,\dots,g)$ by a certain $\rho\in PGL_2(K)$, we may assume that $\alpha_{\pm1},\dots,\alpha_{\pm g}\neq\infty$. Then we can take $D_{\pm1},\dots,D_{\pm g}$ as in 2.1 such that $F_\Gamma\ni\infty$. Hence for any $k=\pm1,\dots,\pm g$ and $x,y\notin D_k\cup\overline{D_{-k}}$,

$$\left|\frac{\alpha_k-x}{\alpha_{-k}-x}\beta_k\right|,\ |[\alpha_k,\alpha_{-k};x,y]\beta_k|<1\ .$$

Put

$$r=\max\left\{\left|\frac{\alpha_k-\alpha_i}{\alpha_{-k}-\alpha_i}\beta_k\right|,|[\alpha_k,\alpha_{-k};\alpha_i,\alpha_j]\beta_k|;\ i,j\neq\pm k\right\}\ .$$

Then $r<1$.

4.2. Lemma. *Let* $k(1),\dots,k(l)\in\{\pm1,\dots,\pm g\}$ *such that* $k(s)\neq-k(s+1)$ $(s=1,\dots,l-1)$, *and put*

$$\tau=\frac{\prod_{s=2}^l(\alpha_{k(s)}-\alpha_{-k(s-1)})}{\prod_{s=1}^l(\alpha_{k(s)}-\alpha_{-k(s)})}\ .$$

Then

(a) *For any* $x, y \in F_\Gamma$,

$$\left| \frac{\prod_{s=1}^{l} \beta_{k(s)}}{\tau^2(\alpha_{k(1)} - x)(\alpha_{-k(l)} - y)} \right| < r^{l-2} .$$

(b) *For any* $j \neq \pm k(l)$ *and* $z \in F_\Gamma$,

$$\left| \frac{(\alpha_j - \alpha_{-j}) \prod_{s=1}^{l} \beta_{k(s)}}{\tau^2(\alpha_j - \alpha_{-k(l)})(\alpha_{-j} - \alpha_{-k(l)})(\alpha_{k(1)} - z)} \right| < r^{l-1} .$$

(c) *For any* $i \neq \pm k(1)$ *and* $j \neq \pm k(l)$,

$$\left| \frac{(\alpha_i - \alpha_{-i})(\alpha_j - \alpha_{-j}) \prod_{s=1}^{l} \beta_{k(s)}}{\tau^2(\alpha_{k(1)} - \alpha_i)(\alpha_{k(1)} - \alpha_{-i})(\alpha_j - \alpha_{-k(l)})(\alpha_{-j} - \alpha_{-k(l)})} \right| \leqq r^l .$$

Proof. We prove only (a) because one can show (b) and (c) in the same way. Since

$$\left| \frac{\alpha_{k(1)} - \alpha_{-k(1)}}{\alpha_{k(1)} - x} \right| \leqq \max \left\{ 1, \left| \frac{\alpha_{-k(1)} - x}{\alpha_{k(1)} - x} \right| \right\} ,$$

$$\left| \frac{\alpha_{k(s)} - \alpha_{-k(s)}}{\alpha_{k(s+1)} - \alpha_{-k(s)}} \right| \leqq \max \left\{ 1, \left| \frac{\alpha_{k(s)} - \alpha_{k(s+1)}}{\alpha_{-k(s)} - \alpha_{k(s+1)}} \right| \right\} \quad (s = 1, \ldots, l-1) ,$$

$$\left| \frac{\alpha_{k(s)} - \alpha_{-k(s)}}{\alpha_{k(s)} - \alpha_{-k(s-1)}} \right| \leqq \max \left\{ 1, \left| \frac{\alpha_{-k(s)} - \alpha_{-k(s-1)}}{\alpha_{k(s)} - \alpha_{-k(s-1)}} \right| \right\} \quad (s = 2, \ldots, l) ,$$

$$\left| \frac{\alpha_{k(l)} - \alpha_{-k(l)}}{y - \alpha_{-k(l)}} \right| \leqq \max \left\{ 1, \left| \frac{\alpha_{k(l)} - y}{\alpha_{-k(l)} - y} \right| \right\} ,$$

we have

$$\left| \frac{\prod_{s=1}^{l} \beta_{k(s)}}{\tau^2(\alpha_{k(1)} - x)(y - \alpha_{-k(l)})} \right| \leqq \prod_{s=1}^{l} v_{k(s)} ,$$

where

$$v_{k(1)} = |\beta_{k(1)}| \max \left\{ 1, \left| \frac{\alpha_{-k(1)} - x}{\alpha_{k(1)} - x} \right|, \left| \frac{\alpha_{k(1)} - \alpha_{k(2)}}{\alpha_{-k(1)} - \alpha_{k(2)}} \right| , \right.$$
$$\left. |[\alpha_{k(1)}, \alpha_{-k(1)}; \alpha_{k(2)}, x]| \right\} ,$$

$$v_{k(s)} = |\beta_{k(s)}| \max \left\{ 1, \left| \frac{\alpha_{k(s)} - \alpha_{k(s+1)}}{\alpha_{-k(s)} - \alpha_{k(s+1)}} \right|, \left| \frac{\alpha_{-k(s)} - \alpha_{-k(s-1)}}{\alpha_{k(s)} - \alpha_{-k(s-1)}} \right| , \right.$$
$$\left. |[\alpha_{k(s)}, \alpha_{-k(s)}; \alpha_{k(s+1)}, \alpha_{-k(s-1)}]| \right\} \quad (s = 2, \ldots, l-1) ,$$

$$v_{k(l)} = |\beta_{k(l)}| \max \left\{ 1, \left| \frac{\alpha_{-k(l)} - \alpha_{-k(l-1)}}{\alpha_{k(l)} - \alpha_{-k(l-1)}} \right|, \left| \frac{\alpha_{k(l)} - y}{\alpha_{-k(l)} - y} \right| , \right.$$
$$\left. |[\alpha_{k(l)}, \alpha_{-k(l)}; y, \alpha_{-k(l-1)}]| \right\} ,$$

which implies (a).

4.3. Theorem. *Let* $(\Gamma;\gamma_1,\dots,\gamma_g)$ *and* $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$ *be as in* 4.1, *let* p *be a point of* $F_\Gamma-\{\infty\}$, *and put* $u=z-p$. *Then*

(a) *For any* $n\geqq 0$,

$$\omega^{(n)}=\sum_{\gamma\in\Gamma}\frac{\gamma'(z)}{(\gamma(z)-p)^{n+1}}du$$

is uniformly convergent for $z\in H_\Gamma-\cup_{\gamma\in\Gamma}\gamma(\{p,\infty\})$ ($H_\Gamma-\cup_{\gamma\in\Gamma}\gamma(p)$ *if* $n\geqq 1$) *in the wider sense, and any coefficient of* $\omega^{(n)}\in K[[u]]du$ *is convergent in* K.

(b) *For any* $j=1,\dots,g$,

$$\omega_j=\sum_{\gamma\in\Gamma}\left(\frac{1}{z-\gamma(\alpha_j)}-\frac{1}{z-\gamma(\alpha_{-j})}\right)du$$

is uniformly convergent for $z\in H_\Gamma$ *in the wider sense, and any coefficient of*

$$\omega_j=\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{\gamma\in\Gamma_j}\left(\frac{\gamma(\alpha_j)-\gamma(\alpha_{-j})}{(\gamma(\alpha_j)-p)^k(\gamma(\alpha_{-j})-p)^{m+1-k}}\right)u^{m-1}du$$

in $K[[u]]du$ *is convergent in* K.

(c) *For any* $i,j=1,\dots,g$, $p_{ij}=\Pi_{\gamma\in\Gamma_{ij}}\psi_{ij}(\gamma)$ *is convergent.*

Proof. (a) Let $f=f_{k(1)}f_{k(2)}\cdots f_{k(l)}$ such that $k(s)\neq -k(s+1)(s=1,\dots,l-1)$, and put

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}=f|_{x_{\pm k}=\alpha_{\pm k},y_k=\beta_k}\,.$$

Then in the same way as the proof of Proposition 3.2(d), one can show that

$$\begin{aligned} f|_{x_{\pm k}=\alpha_{\pm k},y_k=\beta_k} &= \tau\left\{\begin{pmatrix}\alpha_{k(1)} & -\alpha_{k(1)}\alpha_{-k(l)}\\ 1 & -\alpha_{-k(l)}\end{pmatrix}\right.\\ &-A\frac{\alpha_{k(2)}-\alpha_{k(1)}}{\alpha_{k(2)}-\alpha_{-k(1)}}\beta_{k(1)}\begin{pmatrix}\alpha_{-k(1)} & -\alpha_{-k(1)}\alpha_{-k(l)}\\ 1 & -\alpha_{-k(l)}\end{pmatrix}\\ &-B\frac{\alpha_{-k(l)}-\alpha_{-k(l-1)}}{\alpha_{k(l)}-\alpha_{-k(l-1)}}\beta_{k(l)}\begin{pmatrix}\alpha_{k(1)} & -\alpha_{k(1)}\alpha_{k(l)}\\ 1 & -\alpha_{k(l)}\end{pmatrix}\\ &\left.+C\frac{\alpha_{k(2)}-\alpha_{k(1)}}{\alpha_{k(2)}-\alpha_{-k(1)}}\frac{\alpha_{-k(l)}-\alpha_{-k(l-1)}}{\alpha_{k(l)}-\alpha_{-k(l-1)}}\beta_{k(1)}\beta_{k(l)}\begin{pmatrix}\alpha_{-k(1)} & -\alpha_{-k(1)}\alpha_{k(l)}\\ 1 & -\alpha_{k(l)}\end{pmatrix}\right\}. \end{aligned}$$

Here

$$\tau=\frac{\prod_{s=2}^{l}(\alpha_{k(s)}-\alpha_{-k(s-1)})}{\prod_{s=1}^{l}(\alpha_{k(s)}-\alpha_{-k(s)})}\,,$$

and A, B and C are the sums of products $\prod_{s=2}^{l-1}u_s$, where each u_s is either 1 or one of the following:

$$-[\alpha_{k(s)},\alpha_{-k(s)};\alpha_{\pm k(s+1)},\alpha_{\pm k(s-1)}]\beta_{k(s)},\qquad -[\alpha_{k(s)},\alpha_{-k(s)};\alpha_{\pm k(s+1)},\alpha_{\mp k(s-1)}]\beta_{k(s)}\,.$$

Hence $|A|, |B|, |C| \leqq 1$. Since

$$cz + d = \tau(z - \alpha_{-k(l)}) \left\{ 1 - A \frac{\alpha_{k(2)} - \alpha_{k(1)}}{\alpha_{k(2)} - \alpha_{-k(1)}} \beta_{k(1)} \right.$$
$$- B \frac{\alpha_{-k(l)} - \alpha_{-k(l-1)}}{\alpha_{k(l)} - \alpha_{-k(l-1)}} \frac{z - \alpha_{k(l)}}{z - \alpha_{-k(l)}} \beta_{k(l)}$$
$$\left. + C \frac{\alpha_{k(2)} - \alpha_{k(1)}}{\alpha_{k(2)} - \alpha_{-k(1)}} \frac{\alpha_{-k(l)} - \alpha_{-k(l-1)}}{\alpha_{k(l)} - \alpha_{-k(l-1)}} \frac{z - \alpha_{k(l)}}{z - \alpha_{-k(l)}} \beta_{k(1)} \beta_{k(l)} \right\}$$

and

$$az + b - cpz - dp$$
$$= \tau(\alpha_{k(1)} - p)(z - \alpha_{-k(l)}) \{ 1 - A[\alpha_{k(1)}, \alpha_{-k(1)}; \alpha_{k(2)}, p] \beta_{k(1)}$$
$$- B[\alpha_{k(l)}, \alpha_{-k(l)}; z, \alpha_{-k(l-1)}] \beta_{k(l)}$$
$$+ C[\alpha_{k(1)}, \alpha_{-k(1)}; \alpha_{k(2)}, p][\alpha_{k(l)}, \alpha_{-k(l)}; z, \alpha_{-k(l-1)}] \beta_{k(1)} \beta_{k(l)} \} \,,$$

for any $z \in F_\Gamma, |cz + d| = |\tau(z - \alpha_{-k(l)})|$ and $|az + b - cpz - dp| = |\tau(\alpha_{k(1)} - p)(z - \alpha_{-k(l)})|$. Therefore, by Lemma 4.2 (a),

$$\left| \left(\frac{f'(z)}{(f(z) - p)^{n+1}} \right) \Bigg|_{x_{\pm k} = \alpha_{\pm k}, y_k = \beta_k} \right| = \left| \frac{(ad - bc)(cz + d)^{n-1}}{(az + b - cpz - dp)^{n+1}} \right|$$
$$= \left| \frac{\prod_{s=1}^{l} \beta_{k(s)}}{\tau^2 (z - \alpha_{-k(l)})^2 (\alpha_{k(1)} - p)^{n+1}} \right|$$
$$\leqq \frac{r^{l-2}}{|(z - \alpha_{-k(l)})(\alpha_{k(1)} - p)^n|} \to 0 \quad \text{if } l \to \infty \,.$$

Since $\omega^{(n)}$ is Γ-invariant, it is uniformly convergent on $H_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\{p, \infty\})$ in the wider sense. Put

$$\sum_{i=0}^{\infty} s_i u^i = \frac{f'(z)}{(f(z) - p)^{n+1}} \Bigg|_{x_{\pm k} = \alpha_{\pm k}, y_k = \beta_k}$$
$$= \frac{(ad - bc)(cp + d)^{n-1}}{(ap + b - cp^2 - dp)^{n+1}} \left(1 + \frac{cu}{cp + d} \right)^{n-1} \left(1 + \frac{(a - cp)u}{ap + b - cp^2 - dp} \right)^{-n-1} .$$

Since $|c| = |\tau|, |cp + d| = |\tau(p - \alpha_{-k(l)})|, |a - cp| = |\tau(\alpha_{k(1)} - p)|$ and $|ap + d - cp^2 - dp| = |\tau(\alpha_{k(1)} - p)(p - \alpha_{-k(l)})|$,

$$|s_i| \leqq \left| \frac{(ad - bc)(cp + d)^{n-1}}{(ap + b - cp^2 - dp)^{n+1}} \right| \max \left\{ \left| \frac{c}{cp + d} \right|^i, \left| \frac{a - cp}{ap + b - cp^2 - dp} \right|^i \right\}$$
$$= \left| \frac{\prod_{s=1}^{l} \beta_{k(s)}}{\tau^2 (p - \alpha_{-k(l)})^{i+2} (\alpha_{k(1)} - p)^{n+1}} \right| ,$$

and hence by Lemma 4.2 (a),

$$|s_i| \leqq \frac{r^{l-2}}{|(p-\alpha_{-k(l)})^{i+1}(\alpha_{k(1)}-p)^n|} \to 0 \quad \text{if } l \to \infty .$$

Therefore, any coefficient of $\omega^{(n)} \in K[[u]]du$ is convergent.

(b) Let f be as in (a), and put $\gamma = f|_{x_{\pm k}=\alpha_{\pm k}, y_k=\beta_k} \bmod (K^\times)$. Then for $j \neq \pm k(l)$ and $i \neq -k(1)$,

$$\begin{aligned}|\gamma(\alpha_{\pm j})-\alpha_{-i}| &= \frac{|a\alpha_{\pm j}+b-c\alpha_{-i}\alpha_{\pm j}-d\alpha_{-i}|}{|c\alpha_{\pm j}+d|} \\ &= \frac{|\tau(\alpha_{k(1)}-\alpha_{-i})(\alpha_{\pm j}-\alpha_{-k(l)})|}{|\tau(\alpha_{\pm j}-\alpha_{-k(l)})|} = |\alpha_{k(1)}-\alpha_{-i}|\end{aligned}$$

and

$$\left|\frac{\gamma(\alpha_{\pm j})-\alpha_i}{\gamma(\alpha_{\pm j})-\alpha_{-i}}\beta_i\right| < 1 ,$$

because $\gamma(\alpha_{\pm j}) \notin D_{-i}$. Therefore,

$$\begin{aligned}&|(\gamma_i\gamma)(\alpha_j)-(\gamma_i\gamma)(\alpha_{-j})| \\ &= \left|\frac{(\gamma(\alpha_j)-\gamma(\alpha_{-j}))(\alpha_i-\alpha_{-i})^2\beta_i}{(\gamma(\alpha_j)-\alpha_{-i})(\gamma(\alpha_{-j})-\alpha_{-i})\left(1-\frac{\gamma(\alpha_j)-\alpha_i}{\gamma(\alpha_j)-\alpha_{-i}}\beta_i\right)\left(1-\frac{\gamma(\alpha_{-j})-\alpha_i}{\gamma(\alpha_{-j})-\alpha_{-i}}\beta_i\right)}\right| \\ &= \left|\frac{(\gamma(\alpha_j)-\gamma(\alpha_{-j}))(\alpha_i-\alpha_{-i})^2\beta_i}{(\alpha_{k(1)}-\alpha_{-i})^2}\right| ,\end{aligned}$$

and hence by induction on l, we have

$$|\gamma(\alpha_j)-\gamma(\alpha_{-j})| = \left|\frac{(\alpha_j-\alpha_{-j})\prod_{s=1}^{l}\beta_{k(s)}}{\tau^2(\alpha_j-\alpha_{-k(l)})(\alpha_{-j}-\alpha_{-k(l)})}\right| .$$

Since

$$\begin{aligned}|\gamma(\alpha_{\pm j})-z| &= \frac{|a\alpha_{\pm j}+b-cz\alpha_{\pm j}-dz|}{|c\alpha_{\pm j}+d|} \\ &= \frac{|\tau(\alpha_{k(1)}-z)(\alpha_{\pm j}-\alpha_{-k(l)})|}{|\tau(\alpha_{\pm j}-\alpha_{-k(l)})|} \\ &= |\alpha_{k(1)}-z|\end{aligned}$$

for any $z \in F_\Gamma$, by Lemma 4.2 (b),

$$\begin{aligned}\left|\frac{1}{z-\gamma(\alpha_j)}-\frac{1}{z-\gamma(\alpha_{-j})}\right| &= \left|\frac{\gamma(\alpha_j)-\gamma(\alpha_{-j})}{(z-\gamma(\alpha_j))(z-\gamma(\alpha_{-j}))}\right| \\ &= \left|\frac{(\alpha_j-\alpha_{-j})\prod_{s=1}^{l}\beta_{k(s)}}{\tau^2(\alpha_j-\alpha_{-k(l)})(\alpha_{-j}-\alpha_{-k(l)})(\alpha_{k(1)}-z)^2}\right| \\ &\leqq \frac{r^{l-1}}{|\alpha_{k(1)}-z|} \longrightarrow 0 \quad \text{if } l \to \infty .\end{aligned}$$

Since ω_j is Γ-invariant, it is uniformly convergent on H_Γ. Similarly, one can show that

$$\sum_{k=1}^{m}\sum_{\gamma\in\Gamma_j}\left(\frac{\gamma(\alpha_j)-\gamma(\alpha_{-j})}{(\gamma(\alpha_j)-p)^k(\gamma(\alpha_{-j})-p)^{m+1-k}}\right)$$

is convergent for any m.

(c) Let γ be as in (b), and assume that $j \neq \pm k(l)$ and $i \neq \pm k(1)$. Then by Lemma 4.2(c),

$$|[\alpha_i,\alpha_{-i};\gamma(\alpha_j),\gamma(\alpha_{-j})] \ -1| = \left|\frac{(\alpha_i-\alpha_{-i})(\gamma(\alpha_j)-\gamma(\alpha_{-j}))}{(\gamma(\alpha_j)-\alpha_{-i})(\gamma(\alpha_{-j})-\alpha_i)}\right|$$

$$= \left|\frac{(\alpha_i-\alpha_{-i})(\alpha_j-\alpha_{-j})\prod_{s=1}^{l}\beta_{k(s)}}{\tau^2(\alpha_{k(1)}-\alpha_i)(\alpha_{k(1)}-\alpha_{-i})(\alpha_j-\alpha_{-k(l)})(\alpha_{-j}-\alpha_{-k(l)})}\right|$$

$$\leqq r^l \longrightarrow 0 \quad \text{if } l\to\infty\,,$$

and hence p_{ij} is convergent.

4.4. Corollary. *$\omega_j (j=1,\dots,g)$ (resp. $\omega^{(n)}$ $(n \geqq 1), \omega^{(0)}$) are differential forms on C_Γ of the first (resp. second, third) kind.*

Proof. This follows from Theorem 4.3 and that $\omega_j, \omega^{(n)}$ are all Γ-invariant.

4.5. We assume that K is of characteristic 0. Let $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$ be as in 4.1, and assume that

$$|\beta_k| < \min\{|4[\alpha_k,\alpha_{-k};\alpha_i,\alpha_j]| \ ; \ i,j \neq \pm k\} \quad (k=1,\dots,g)\,,$$

which is automatically satisfied if the residual characteristic of K is not 2. Then by the proof of Theorem 4.3 (c),

$$\left|\frac{1}{4}\left(\prod_{\gamma\in\Gamma_{ii}-\{1\}}\psi_{ii}(\gamma)-1\right)\right| < 1 \quad (i=1,\dots,g)\,.$$

For each $i=1,\dots,g$, fix square roots $\beta_i^{1/2}$ of β_i, and assume that $\beta_i^{1/2}\in K^\times$. Then

$$\beta_i^{1/2}\left\{\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\prod_{\gamma\in\Gamma_{ii}-\{1\}}\psi_{ii}(\gamma)-1\right)^n\right\}^{-1}$$

$$=\beta_i^{1/2}\left\{\sum_{n=0}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n\left(\prod_{\gamma\in\Gamma_{ii}-\{1\}}\psi_{ii}(\gamma)-1\right)^n\right\}^{-1}$$

is convergent and a square root of p_{ii} which we denote by $p_{ii}^{1/2}$. Let $(\Gamma;\gamma_1,\dots,\gamma_g)$ be the marked Schottky group over K with Koebe coordinates $(\alpha_{\pm k},\beta_k)_{1\leqq k\leqq g}$, let P be the point of C_Γ corresponding to a point p of $F_\Gamma-\{\infty\}$, and put $u=z-p$. Then from $X=(C_\Gamma,P,u)$, we construct a micro-differential operator as follows. By

Theorem 4.3 (b), there exist uniquely $q_{nm}, r_{jm} \in K$ such that

$$\omega^{(n)} = \left(\frac{1}{u^{n+1}} + \sum_{m=1}^{\infty} \frac{q_{nm}}{n} u^{m-1}\right) du \quad (n \geqq 0),$$

$$\omega_j = \sum_{m=1}^{\infty} r_{jm} u^{m-1} du \quad (j = 1, \ldots, g).$$

Since $\log |\Pi_{i=1}^{g} p_{ii}^{v_i^2/2} \Pi_{i<j} p_{ij}^{v_i v_j}|$ is a negative definite form for $\vec{v} = (v_i)_{1 \leqq i \leqq g} \in \mathbf{Z}^g$, (cf. [M-D], Sect. 4), for $\mathbf{c} = (c_i)_{1 \leqq i \leqq g} \in (K^\times)^g$ and a vector $\vec{z} = (z_i)_{1 \leqq i \leqq g}$ of g indeterminates,

$$\Theta(\mathbf{c} \cdot \exp(\vec{z})) = \sum_{\vec{v} \in \mathbf{Z}^g} \left\{ \prod_{i=1}^{g} p_{ii}^{v_i^2/2} \prod_{i<j} p_{ij}^{v_i v_j} \prod_{i=1}^{g} c_i^{v_i} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^{g} v_i z_i \right)^n \right\}$$

belongs to $K[[z_1, \ldots, z_g]]$. We define the *τ-function* for $(X, \mathbf{c})$ by

$$\tau(\mathbf{t}, X_\mathbf{c}) = \exp\left(\frac{1}{2} \sum_{n,m=1}^{\infty} q_{nm} t_n t_m\right) \Theta\left(\mathbf{c} \cdot \exp\left(\sum_{m=1}^{\infty} t_m \vec{r}_m\right)\right),$$

where $\vec{r}_m = (r_{jm})_{1 \leqq j \leqq g}$. Then as in (1.4)–(1.6), for any $\mathbf{c} \in (K^\times)^g$ with $\Theta(\mathbf{c}) \neq 0, \tau(\mathbf{t}, X_\mathbf{c})$ defines a micro-differential operator with coefficients in $K[[x, \mathbf{t}]]$ which we denote by $L(\mathbf{t}, X_\mathbf{c})$.

4.6. Theorem.

(a) $L(\mathbf{t}, X_\mathbf{c}) = L(\mathbf{t})|_{x_{\pm k} = \alpha_{\pm k}, y_k^{1/2} = \beta_k^{1/2}, w_k = c_k}$.

(b) $L(\mathbf{t}, X_\mathbf{c})$ *satisfies the KP hierarchy* (1.7). *In particular,*

$$u_1(x, t_2, t_3) = \frac{\partial^2}{\partial x^2} \log \Theta(\mathbf{c} \cdot \exp(x\vec{r}_1 + t_2\vec{r}_2 + t_3\vec{r}_3)) + q_{11}$$

satisfies the KP equation (1.8).

Proof. This follows from the definition of $L(\mathbf{t}, X_\mathbf{c})$ and Theorem 3.4 (b).

4.7. Remark. Since

$$\tau(\mathbf{t}, X_\mathbf{c}) = \tau(\mathbf{t})\Big|_{x_{\pm k} = \alpha_{\pm k}, y_k^{1/2} = \beta_k^{1/2}, w_k = c_k},$$

for any $\mathbf{c} \in (K^\times)^g$, it is easy to see that $\tau(\mathbf{t}, X_\mathbf{c})$ and

$$L(\mathbf{t}, X_\mathbf{c}) = L(\mathbf{t})\Big|_{x_{\pm k} = \alpha_{\pm k}, y_k^{1/2} = \beta_k^{1/2}, w_k = c_k}$$

satisfy the same relation as in (1.4)–(1.6) and that $L(\mathbf{t}, X_\mathbf{c})$ satisfies (1.7).

4.8. Remark. One can extend Theorem 4.6 for general local coordinates u without difficulty.

References

[A] Akaza, T.: Poincaré theta series and singular sets of Schottky groups. Nagoya Math. J. **24**, 43–65 (1964)

[A-D] Arbarello, E., De Concini, C.: On a set of equations characterizing Riemann matrices. Ann. Math. **120**, 119–140 (1984)

[Ge] Gerritzen, L.: Zur analytischen Beschreibung des Raumes der Schottky–Mumford–Kurven. Math. Ann. **255**, 259–271 (1981)

[Go] González-diéz, G.: Theta functions on the boundary of moduli space. In: Donagi, R. (ed.) Curves, Jacobians, and Abelian Varieties, Contemp. Math. **136**, Providence RI: Amer. Math. Soc., 1992, pp. 185–208

[G-v.d.P] Gerritzen, L., van der Put, M.: Schottky groups and Mumford curves, Lect. Notes Math. **817**, Berlin, Heidelberg, New York: Springer, 1980

[H] Hejhal, D.A.: On Schottky and Teichmüller spaces. Adv. Math. **15**, 133–156 (1975)

[I] Ichikawa, T.: The universal periods of curves and the Schottky problem. Compos. Math. **85**, 1–8 (1993)

[Ko] Koebe, P.: Über die Uniformisierung der algebraischen Kurven IV. Math. Ann. **75**, 42–129 (1914)

[Kr] Krichever, I.M.: Methods of algebraic geometry in the theory of nonlinear equations. Russ. Math. Surv. **32**, 185–214 (1977)

[M-D] Manin, Y., Drinfeld, V.: Periods of p-adic Schottky groups. J. Reine Angew. Math. **262/263**, 239–247 (1973)

[Mul] Mulase, M.: Cohomological structure in soliton equations and jacobian varieties. J. Differential Geom. **19**, 403–430 (1984)

[Mum1] Mumford, D.: An analytic construction of degenerating curves over complete local fields. Compos. Math. **24**, 129–174 (1972)

[Mum2] Mumford, D.: Tata lectures on theta II, Progress in Math. **43**, Basel: Birkhäuser, 1983

[S-S] Sato, M., Sato, Y.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifold. Lax, P.D. (ed.) Nonlinear Partial Differential Equations in Applied Sciences, Lect. Notes Num. Appl Anal. **5**, Amsterdam: North-Holland and Tokyo: Kinokuniya, 1982, pp. 259–271

[Sc] Schottky, F.: Über eine spezielle Funktion, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt. J. Reine Angew. Math. **101**, 227–272 (1887)

[Sh] Shiota, T.: Characterization of jacobian varieties in terms of soliton equations. Invent. Math. **83**, 333–382 (1986)

Communicated by M. Jimbo