Commun. Math. Phys. 149, 623–636 (1992)

Communications in Mathematical Physics

Calculation of General p-Adic Feynman Amplitude*

V. A. Smirnov

Nuclear Physics Institute of Moscow State University, Moscow 119899, Russia

Received November 4, 1991

Abstract. The general n-point massless p-adic Feynman amplitude with arbitrary parameters of analytic regularization for each line is calculated. This result is presented in the form of a sum over hierarchies of a given graph. The structure of ultraviolet and infrared divergences of p-adic Feynman amplitudes is characterized and the star-triangle uniqueness identity in the p-adic case is derived.

1. Introduction

In the past four years p-adic analysis [4, 15, 21] was applied in quantum theory [1, 7–10, 12, 13, 16–20, 22, 27, 28]. Several approaches were used. Within each of these approaches something was considered to be p-adic rather than real. For example, this can be the world sheet of a string (see, e.g., [7–10, 17]), or, our space-time itself [28]. In a paper by Lerner and Missarov [16] a generalized Koba-Nielsen amplitude was explicitly calculated and it was explained how an arbitrary one-dimensional p-adic massless Feynman amplitude can be computed.

The purpose of this work is to calculate the general p-adic massless Feynman amplitude with arbitrary parameters of analytic regularization in arbitrary space-time dimensions. The result will be written as a sum over hierarchies of the set of vertices of the given graph.

The paper is organized as follows. In the next section the main definitions of p-adic analysis are given and basic integrals are listed. In Sect. 3 auxiliary vacuum-type p-adic Feynman integrals are computed, and in Sect. 4 calculation of the general massless Feynman amplitude is presented. In Sect. 5 results of Sect. 4 are applied to simplest Feynman amplitudes. Furthermore, ultraviolet and infrared divergences of Feynman amplitudes are characterized and the star-triangle uniqueness identity is derived. Finally, in the Conclusion, the possibility of adelic formulae for massless Feynman amplitudes is discussed.

* Supported by Alexander von Humboldt-Stiftung

2. p-Adic Analysis and Basic Integrals

Let Q be the field of rational numbers, and p a fixed prime number. Any $x \in Q$ can be represented as $x = p^\nu m/n$ with integer ν, and integers m and n which are not divisible by p. By definition [4, 15, 21] the p-adic norm is $|x|_p = p^{-\nu}$. (A remarkable theorem due to Ostrowski states that on Q there are only two nontrivial versions of the norm: $|x|_p$ and the absolute value $|x| \equiv |x|_\infty$.) The completion of Q in respect to the p-adic norm gives the field of p-adic numbers Q_p. Any $x \in Q_p$ can be represented as $p^\nu \sum\limits_{i=0}^{\infty} a_i p^i$, with a_i integers, $0 \le a_i < p$, $a_0 \ne 0$.

To define the Fourier transformation the p-adic exponential [4, 11, 26] $\chi(x) = \exp(2\pi\{x\})$ is used. Here $\{x\} = p^\nu \sum\limits_{i=0}^{-\nu-1} a_i p^i$ is the non-integer part of the p-adic number. The integration of functions on Q_p with values in the field of complex numbers is presented in [11, 26]. Furthermore, in [26] the space of distributions in the p-adic case is described. For example, the distributions $|x|_p^\lambda$ and $\delta(x)$ are defined. The corresponding Fourier transforms are

$$\int\limits_{Q_p} |x|_p^\lambda \chi(qx)\,\mathrm{d}x = \Gamma(\lambda+1)\,|q|_p^{-\lambda-1}, \qquad \int\limits_{Q_p} \delta(x)\chi(qx)\,\mathrm{d}x = 1\,,$$

where $\Gamma(\lambda) = (1-p^{\lambda-1})/(1-p^{-\lambda})$ is a version the p-adic Γ-function [4, 11, 26]. The d-dimensional p-adic space Q_p^d is defined with the norm $|x|_p = \max\limits_{1\le i\le d} |x_i|_p$, $x = (x_1, \ldots, x_d) \in Q_p^d$. The corresponding Fourier transform is defined with the exponent $\chi(qx) = \chi(q_1x_1 + \ldots + q_dx_d)$ [26].

Feynman rules of a perturbative diagram technique are determined by a suitable choice of a propagator which is inserted into the functional integral. *A priori* there is no unique choice in the p-adic case. In [22] a p-adically natural propagator $|q,m|_p^{-2}$, with $|q,m|_p = \max\{|q|_p, |m|_q\}$, was proposed. In the d-dimensional case the Fourier transform of the analytically regularized propagator looks like [22]

$$D(x,m) = \int\limits_{Q_p^d} |q,m|_p^\lambda \chi(qx)\,\mathrm{d}x = \Gamma_d(\lambda+d)\,\omega(mx)\,(|x|_p^{-\lambda-d} - |pm|_p^{\lambda+d})\,. \tag{1}$$

Here $\omega(x) = 1$ for $|x|_p \le 1$ and $\omega(x) = 0$ otherwise. Furthermore,

$$\Gamma_d(\lambda) = \frac{1-p^{\lambda-d}}{1-p^{-\lambda}} \tag{2}$$

is a version of multidimensional p-adic Γ-function. In the massless case without analytic regularization (i.e. for $\lambda = -2$) we have

$$D(x) = \Gamma_d(d-2)\,|x|_p^{2-d}. \tag{3}$$

For example, in four-dimensional space $D(x) = |x|_p^{-2}$.

To calculate an arbitrary massless p-adic Feynman amplitude we will compute a number of auxiliary integrals. First, reading the formula (1) from right to left, expressing the Fourier transform of the right-hand side as the integrand of the left-hand side, and putting then $q = 0$ and $m = 1$ we obtain

$$\int\limits_{|x|\leq 1} |x|^{\lambda}\,\mathrm{d}x = p^{\lambda} + \Gamma_d(\lambda+d) \equiv \frac{1-p^{-d}}{1-p^{-\lambda-d}}\,. \tag{4}$$

From now on we omit for brevity the index p in the p-adic norm. (As usual, the same symbol denotes also the number of elements in the corresponding finite set, but this should not lead to misunderstanding.) Using a suitable change of variables and linear relations between involved integrals we have the following list of basic d-dimensional p-adic integrals:

$$\int\limits_{|x|<p^n} |x|^{\lambda}\,\mathrm{d}x = p^{n(\lambda+d)}\frac{1-p^{-d}}{p^{\lambda+d}-1}\,, \tag{5}$$

$$\int\limits_{|x|=p^n} |x|^{\lambda}\,\mathrm{d}x = p^{n(\lambda+d)}\left(1-p^{-d}\right), \tag{6}$$

$$\int\limits_{|x|>p^n} |x|^{\lambda}\,\mathrm{d}x = p^{n(\lambda+d)}\frac{1-p^{-d}}{p^{-\lambda-d}-1}\,. \tag{7}$$

In particular, for $\lambda = 0$ and $n = 0$,

$$\int\limits_{|x|<1} \mathrm{d}x = p^{-d}\,, \qquad \int\limits_{|x|=1} \mathrm{d}x = 1 - p^{-d}\,. \tag{8}$$

Let us now take x_i, $i = 1, \ldots, k$ with $x_i \in Q_p^d$ and $|x_i - x_j| = 1$. Let us calculate the integral $\int\limits_{|x-x_i|=1\,\forall i} \mathrm{d}x$. It can be represented as

$$\int\limits_{|x|\leq 1} \mathrm{d}x - \sum_{i=1}^{k} \int\limits_{|x-x_i|<1} \mathrm{d}x\,.$$

Therefore, from (8),

$$\int\limits_{|x-x_i|=1\,\forall i} \mathrm{d}x = 1 - kp^{-d}\,. \tag{9}$$

Consider then the integral

$$I(V) = \int\limits_{\substack{|x_v - x_{v'}|=1 \\ \forall v,c'\in V}} \mathrm{d}\underline{x}\,, \tag{10}$$

where V is a finite set, $\underline{x} = \{x_v : v \in V\}$, $\mathrm{d}\underline{x} = \prod\limits_{v\in V, v\neq v_0} \mathrm{d}x_v$, and v_0 is a fixed element of V. Choosing some other $v_1 \in V$ we have

$$I(V) = \int\limits_{\substack{|x_v - x_{v'}|=1 \\ v,v'\in V\setminus v_1}} \prod_{v\in V, v\neq v_0, v_1} \mathrm{d}x_v \int\limits_{\substack{|x_v - x_{v_1}|=1 \\ v\in V\setminus v_1}} \mathrm{d}x_{v_1}\,.$$

Thus, by (9), we obtain

$$I(V) = (1 - (|V| - 1)p^{-d}) I(V \backslash \{v_1\}).$$

By repeated applications of (9) we come to the following result:

$$I(V) = c(V), \tag{11}$$

with

$$c(V) = (1 - p^{-d})(1 - 2p^{-d}) \ldots (1 - (|V| - 1)p^{-d}). \tag{12}$$

Let now $V_1 = V \cup \{v_1\}$, and

$$I(V, v_1) = \int\limits_{\substack{|x_v - x_{v'}|=1 \\ \forall v, v' \in V_1}} \mathrm{d}\underline{x}, \tag{13}$$

where $\mathrm{d}\underline{x}$ is the same as in (10) (i.e. written for the set V). Proceeding recursively in the same fashion as for $I(V)$ we obtain

$$I(V, v_1) = \frac{1 - |V| p^{-d}}{1 - p^{-d}} c(V). \tag{14}$$

Let us now define the domain

$$\mathscr{D}(V) = \{\underline{x} : |x_v - x_{v'}| \text{ is independent of } v, v' \in V\},$$

and let us for $\underline{x} \in \mathscr{D}(V)$ define $\mu_V(\underline{x}) = |x_v - x_{v'}|$. Using (11) we have

$$\int\limits_{\substack{|x_v - x_{v'}|=p^n \\ \forall v, v' \in V}} \mathrm{d}\underline{x} \mu_V(\underline{x})^\lambda = (p^n)^{\lambda + d(|V|-1)} c(V). \tag{15}$$

Furthermore, applying this formula, we come to

$$\int\limits_{\substack{\mu_1 < \mu_V(\underline{x}) < \mu_2 \\ \underline{x} \in \mathscr{D}(V)}} \mathrm{d}\underline{x} \mu_V(\underline{x})^\lambda$$

$$= c(V) \frac{1}{p^{\lambda + (|V|-1)d} - 1} \{\mu_2^{\lambda + d(|V|-1)} - (p\mu_1)^{\lambda + d(|V|-1)}\}, \tag{16}$$

where $\mu_{1,2} = p^{n_{1,2}}$ for some integers n_1 and n_2. In particular, for $\mu_1 = -\infty$ and $\mu_2 = \infty$ this, respectively, gives

$$\int\limits_{\substack{\mu_V(\underline{x}) < \mu \\ \underline{x} \in \mathscr{D}(V)}} \mathrm{d}\underline{x} \mu_V(\underline{x})^\lambda = c(V) \frac{1}{p^{\lambda + (|V|-1)d} - 1} \mu^{\lambda + d(|V|-1)}, \tag{17}$$

$$\int\limits_{\substack{\mu_V(\underline{x}) > \mu \\ \underline{x} \in \mathscr{D}(V)}} \mathrm{d}\underline{x} \mu_V(\underline{x})^\lambda = c(V) \frac{1}{p^{-\lambda - (|V|-1)d} - 1} \mu^{\lambda + d(|V|-1)}. \tag{18}$$

Similarly,

$$\int\limits_{\substack{\mu_V(\underline{x})=\mu \\ \underline{x}_{V_1}\in\mathscr{D}(V)}} \mathrm{d}\underline{x}_V \mu_{V_1}(\underline{x}_{V_1})^\lambda = \frac{1-|V|p^{-d}}{1-p^{-d}}\, c(V)\, \mu^{\lambda+d(|V|-1)}\,, \tag{19}$$

and

$$\int\limits_{\substack{\mu_V(\underline{x})<\mu \\ \underline{x}_{V_1}\in\mathscr{D}(V)}} \mathrm{d}\underline{x}_V \mu_{V_1}(\underline{x}_{V_1})^\lambda = \frac{1-|V|p^{-d}}{1-p^{-d}}\, c(V)\, \frac{1}{p^{\lambda+(|V|-1)d}-1}\, \mu^{\lambda+d(|V|-1)}\,, \tag{20}$$

where

$$\underline{x}_V = \{x_v : v\in V\}\,, \quad V_1 = V\cup\{v_1\}\,, \quad \mathrm{d}\underline{x}_V = \prod_{v\in V, v\neq v_0} \mathrm{d}\underline{x}_v\,.$$

By the same technique it is easy to calculate the following integral with v_1 a fixed vertex:

$$\int\limits_{\substack{|x_{v_1}+\eta|<\mu_V(\underline{x})=|\eta| \\ \underline{x}\in\mathscr{D}(V)}} \mathrm{d}\underline{x}\mu_V(\underline{x})^\lambda\, |x_{v_1}+\eta|^{\lambda_1} = c(V)\frac{1}{p^{\lambda_1+d}-1}\, |\eta|^{\lambda+\lambda_1+d(|V|-1)}\,. \tag{21}$$

3. Partial Vacuum Feynman Amplitudes

Let us now consider some "partial" vacuum Feynman amplitudes which will be later used in the next section when calculating the general Feynman amplitude. The first of them is a d-dimensional version of the integral which is a generalization of the Koba-Nielsen string amplitude

$$F_{\mathscr{V}} = \int\limits_{\substack{|x_v|<1 \\ \forall v\in\mathscr{V}}} \mathrm{d}\underline{x} \prod_{v,v'\in\mathscr{V}} |x_v - x_{v'}|^{a(v,v')}\,. \tag{22}$$

Here $\mathscr{V}$ is a given vertex set. As before $\mathrm{d}\underline{x}$ is the product of $\mathrm{d}x_v$ except for some fixed vertex v_0. Due to translation invariance we may put $x_{v_0}=0$. For $d=1$ the integral (22) was calculated in [16]. In the case of arbitrary dimension the following result holds.

Lemma 1.

$$F_{\mathscr{V}} = \sum_A \prod_{V\in A'} c(V_-)\, r(b(V))\,, \tag{23}$$

where the sum goes over hierarchies on $\mathscr{V}$, $A' = \{V\in A : |V|>1\}$, $V_- = \{V'\in A : V'_+ = V\}$, V_+ is the minimal element of A incuding given $V\in A$,

$$r(b) = 1/(p^b-1)\,, \quad b(V) = a(V) + (|V|-1)d\,, \tag{24}$$

and [16]

$$a(V) = a(V,V)\,, \quad a(V,V') = \sum_{v\in V, v'\in V'} a(v,v')\,. \tag{25}$$

We recall the definition of the hierarchy (see, e.g., [16]). A set A of subsets V of a given finite set $\mathscr{V}$ is an hierarchy on $\mathscr{V}$ if i) $\mathscr{V} \in A$ and $\{v\} \in A$ for each $v \in \mathscr{V}$; ii) for any $V, V' \in A$ either $V \cap V' = \emptyset$, or $V \subset V'$, or $V' \subset V$.

Proof of Lemma 1. We use a multidimensional generalization of the technique applied in [16]. Let us first decompose the integration domain in (22) into subdomains (sectors) $\mathscr{D}_A$ associated to hierarchies on $\mathscr{V}$:

$$\mathscr{D}_A = \{\underline{x} : |x_v - x_{v'}| < |x_v - x_{v''}| \ \forall c, v' \in V, v'' \overline{\in} V; V \in A\}. \tag{26}$$

If we use the notation [16]

$$M(V) = \max_{v,v' \in V} \{|x_v - x_{v'}|\}, \tag{27}$$

then $|x_v - x_{v'}| = M(V'')$, where V'' is the minimal element of A including both v and v' [16]. The sets $\mathscr{D}_A$ form a partition of Q_q^{nd} [16], i.e.

$$\mathscr{D}_A \cap \mathscr{D}_{A'} = \emptyset, \qquad \bigcup_A \mathscr{D}_A = \{\underline{x} \in Q_p^{nd}\}.$$

Thus,

$$F_{\mathscr{V}} = \sum_A F^A, \qquad F^A = \int\limits_{\substack{\underline{x} \in \mathscr{D}_A \\ |x_v| < 1}} \mathrm{d}\underline{x} \prod_{v,v' \in \mathscr{V}} |x_v - x_{v'}|^{a(v,v')}. \tag{28}$$

Let us in $\mathscr{D}_A$ introduce the new (sector) variables

$$\underline{\xi} = \{\underline{\xi}^V\}_{V \in A'}; \qquad \underline{\xi}^V = \{\xi^V_{V'}\}_{V' \in V_-, V' \neq \sigma(V)}.$$

Here σ is an arbitrary function $\sigma : V \to \sigma(V) \in V_-$. Thus, there will no variable $\xi^V_{\sigma(V)}$. However it is convenient, formally, to put $\xi^V_{\sigma(V)} = 0$. Now, we define the new variables by

$$x_v = \sum_{V \ni v} \xi_V = \xi_{\{v\}} + \xi_{\{v\}_+} + \dots. \tag{29}$$

Let us define

$$\underbrace{\sigma^n = \sigma(\sigma(\dots(\sigma}_{n}(V))\dots)).$$

Clearly, $\forall V \exists \varrho(V)$ such that $|\sigma^{\varrho(V)}(V)| = 1$. Let us introduce the function $\sigma^{\varrho(V)}(V) = \tau(V)$. For example, $\tau(\mathscr{V}) = \{v_0\}$. With this notation it is easy to derive that

$$\xi_V = x_{\tau(V)} - x_{\tau(V_+)}. \tag{30}$$

Note that the above partition of the integration domain and introduction of the new integration variables resembles the well-known procedure used for real-space Feynman amplitudes. For example, in the α-representation technique one uses sectors and auxiliary sector variables to resolve the complicated structure of ultraviolet and infrared divergences – see, e.g., [5, 23, 24, 29].

The sector $\mathscr{D}_V$ in the new variables looks like

$$\mathscr{D}_V = \{\underline{\xi} \in \mathscr{D}(V^*) : \mu_{V^*}(\underline{\xi}^V) < \mu_{V'^*}(\underline{\xi}^{V'}) \ \forall V \subset V'; V, V' \in A\}. \tag{31}$$

Here $V^* = V/V_-$. However, for the sake of brevity, let us omit this asterisk. Therefore, $\underline{\xi} \in \mathscr{D}(V)$ will mean that $|\xi_{V'} - \xi_{V''}|$ is independent of V', $V'' \in V_-$.

By expressing the integrand in (28) in the new variables we thus obtain

$$F^A = \int\limits_{\substack{\underline{\xi}\in\mathscr{D}_A \\ |\xi_V|<1\,\forall V}} \mathrm{d}\underline{\xi} \prod_{V\in A'} \mu_V(\underline{\xi}^V)^{s(V)}, \tag{32}$$

where

$$s(V) = \sum_{\substack{V',V''\in V_- \\ V'\neq V''}} a(V',V'').$$

The integration is performed with the help of the formulae of the previous section. We begin with minimal elements of A'. If V is such an element we apply (17) to obtain

$$\int\limits_{\mu_V(\underline{\xi}^V)<\mu_{V_+}(\underline{\xi}^{V_+})} \mathrm{d}\underline{\xi}^V \mu_V(\underline{\xi}^V)^{s(V)} = c(V)\frac{1}{p^{a(V)+(|V|-1)d}-1}\,\mu_{V_+}(\underline{\xi}^{V_+})^{a(V)+(|V|-1)d}.$$

Then the integration is performed "at the next level," i.e. for those V which contain in V_- only minimal elements of A' and elements from $A\backslash A'$. Here it is necessary to take into consideration contributions to the power of μ_V from the result of previous integrations. Using the formula

$$s(V) + \sum_{V'\in V_-}(a(V') + (|V'|-1)d) = a(V) + (|V|-1)d - (|V_-|-1)d$$

we finally come to (23).

Suppose now that V_0 is a minimal element of A'. Another partial Feynman amplitude which will be later necessary looks like

$$F^{A;V_0,\mu_0,\mu_{k+1}} = \int\limits_{\substack{\underline{x}\in\mathscr{D}_A \\ \mu_0<M(V_0)<\ldots<M(V_k)<\mu_{k+1}}} \mathrm{d}\underline{x} \prod_{v,v'\in\mathscr{V}} |x_v - x_{v'}|^{a(v,v')}, \tag{33}$$

where $\mathscr{N} = \{V_i\}_{i=1,\ldots,k} = \{V_1 \subset \ldots \subset V_k \equiv \mathscr{V}\}$ is the nest of elements of A including V_0. The following assertion is valid.

Lemma 2.

$$F^{A;V_0,\mu_0,\mu_{k+1}} = \prod_{V\in A'} c(V_-) \prod_{V\in A''} r(b(V))\, f(\mu_0,\mu_{k+1}), \tag{34}$$

$$f = \mu_{k+1}^{b_k}\left\{\prod_{l=0}^{k} r(b_l) + \sum_{l=0}^{k}(\mu_0/\mu_{k+1})^{b_l} r(-b_i)\prod_{l'\neq l} r(b_{l'}-b_l)\right\} \tag{35}$$

with $A'' = A'\backslash\mathscr{N}$, $b_i = b(V_i)$.

Proof. Let us use the same change of variables as in the above proof of Lemma 1. We thus obtain

$$F^{A;V_0,\mu_0,\mu_{k+1}} = \int\limits_{\substack{\underline{\xi}\in\mathscr{D}_A \\ \mu_0<\mu_{V_0}(\underline{\xi}^{V_0})<\ldots<\mu_{k+1}}} \mathrm{d}\underline{\xi} \prod_{V\in A'} \mu_V(\underline{\xi}^V)^{s(V)}. \tag{36}$$

The integration over the variables $\{\underline{\xi}^V, V \in A''\}$ is performed as in the case of the integral (32). The "first" integration connected with the nest $\mathscr{N}$ is over $\underline{\xi}^{V_0}$. Using (16) with $V = V_0$ we have (for $\mu = \mu_0$)

$$\int\limits_{\mu<\mu_V(\underline{\xi}^V)<\mu_{V_+}(\underline{\xi}^{V_+})} \mathrm{d}\underline{\xi}^V \mu_V(\underline{\xi}^V)^{a(V)} = c(V)\left\{\mu_{V_+}(\underline{\xi}^{V_+})^{b(V)} - (p\mu)^{b(V)}\right\} r(b(V)) .$$

The rest of the integrations over $\underline{\xi}^{V_i}$ for $i = 1, \ldots, k-1$ is also performed by use of (16), and the last integration over $\underline{\xi}^{V_k}$ by (18). To prove the explicit result (35) it suffices to use induction on the number k.

4. General Feynman Amplitude

Consider now an arbitrary n-point massless Feynman amplitude

$$F_{\mathscr{V}}(\underline{x}) = \int \mathrm{d}\underline{x}' \prod_{v,v' \in \mathscr{V}} |x_v - x_{v'}|^{a(v,v')} , \tag{37}$$

where $\underline{x} = \{x_v : v \in \mathscr{V}^{\text{ext}}\}$, $\mathrm{d}\underline{x}' = \prod_{v \in \mathscr{V}^{\text{int}}} \mathrm{d}x_v$, $\mathscr{V}^{\text{ext}}(\mathscr{V}^{\text{int}})$ is the set of external (internal) vertices of the given graph with $\mathscr{V} = \mathscr{V}^{\text{ext}} \cup \mathscr{V}^{\text{int}}$. In the p-adic case it is explicitly calculable for arbitrary values of dimension and regularization parameters $a(v, v')$.

Theorem . *Let A_0 be a fixed hierarchy on $\mathscr{V}^{\text{ext}}$. If $\underline{x} \in \mathscr{D}_{A_0}$, then*

$$\begin{aligned} F_{\mathscr{V}}(\underline{x}) = \prod_{V \in A_0'} c(V_-)^{-1} & \sum_{A \in A\{\mathscr{V};A_0\}} \prod_{V \in A'} c(V_-) \prod_{V \in A''} r(b(V)) \\ & \times \prod_{V \in A_0} f^A_{\bar{V}_+/\bar{V}}(M(V), M(V_+)) , \end{aligned} \tag{38}$$

where the first sum is over the set $A\{\mathscr{V}; A_0\}$ of hierarchies A on $\mathscr{V}$ such that the family $\{V : V = V' \backslash \mathscr{V}^{\text{int}}, V' \in A\}$ coincides with A_0, the function $f^A_{\bar{V}_+/\bar{V}}$ is given by (35) *with the nest $V_0 = \bar{V}/\bar{V}_-$, $V_k = \bar{V}_+/\bar{V}_-$, $\bar{V}$ is the minimal element of the hierarchy A including the set $V \in A_0$, $A'' = A' \backslash \bigcup_{V \in A_0} \mathscr{N}_{\bar{V}}$, and $\mathscr{N}_{\bar{V}}$ are the nests generated by elements $\bar{V}$, $V \in A_0$.*

Proof. We shall restrict ourselves to the cases $n = 2$ and $n = 3$. Then it will be clear how the arguments are generalized for arbitrary n. In the case of the general two-point Feynman amplitude ($n = 2$), (38) is rewritten as

$$\begin{aligned} F_{\mathscr{V}}(x_1 - x_0) &\equiv \int \mathrm{d}\underline{x} \prod_{v,v' \in \mathscr{V}} |x_v - x_{v'}|^{a(v,v')} \\ &= (1 - p^{-d})^{-1} |x_1 - x_0|^{a(\mathscr{V})+d(|\mathscr{V}|-2)} \sum_A \prod_{V \in A'} \frac{c(V)}{p^{-\Omega(V)} - 1} , \end{aligned} \tag{39}$$

where $\mathrm{d}\underline{x} = \prod_{v \in V, v \neq v_{0,1}} \mathrm{d}x_v$, $\mathscr{V}' = \mathscr{V} \backslash \{v_1\}$, $\underline{x}' = \{x_v : v \in \mathscr{V}'\}$, $\Omega(V) = -b(V)$ if $V \subset V_{01}$ or $V \cap V_{01} = \emptyset$, $\Omega(V) = b(\mathscr{V}) - b(V)$ if $V_{01} \subseteq V \subset \mathscr{V}$, and V_{01} is the minimal element of the hierarchy A including both external vertices.

We first decompose the integral in (39) into subintegrals associated with hierarchies on the set $\mathscr{V}'$:

$$F_{\mathscr{V}} = \sum_A \int_{\underline{x}' \in \mathscr{D}_A} \mathrm{d}\underline{x} \prod_{v, v' \in \mathscr{V}} |x_v - x_{v'}|^{a(v,v')} . \tag{40}$$

Second, let us apply the following partition (which was introduced in [16] for another purpose):

$$\{x_1 \in Q_p^d\} = \bigcup_{V \in A} \mathscr{D}^V , \qquad \mathscr{D}^V = \mathscr{D}_1^V \cup \mathscr{D}_2^V \tag{41}$$

with

$$\mathscr{D}_1^V = \{x_1 \in Q_p^d : |x_1 - x_v| = M(V) \ \forall v \in V\} , \tag{42}$$

$$\mathscr{D}_2^V = \{x_1 \in Q_p^d : M(V) < |x_1 - x_v| < M(V_+) \ \forall v \in V\} \tag{43}$$

and $M(V)$ given by (27).

Consequently, the Feynman amplitude is represented as

$$F_{\mathscr{V}} = \sum_A \sum_{V \in A'} \sum_{l=1,2} F_l^{A,V} . \tag{44}$$

Observe that the fixed element V of the given hierarchy A on $\mathscr{V}'$ generates the nest $\mathscr{N}_V$ of elements of A including V. Let us enumerate them in the natural order: $V \equiv V_1 \subset V_2 \subset \ldots \subset V_N = \mathscr{V}$, and let V_n be the minimal element including both V and v_0. It is easy to show that $M(V_n) = |x_1 - x_0|$. Then, as in the previous section, we turn to the variables $\{\underline{\xi}^{V'}\}_{V' \in A'}$, by use of (29, 30), and obtain

$$F_1^{A,V} = \int_{\substack{\underline{\xi} \in \mathscr{D}_A \\ x_1 \in \mathscr{D}_1^V}} \mathrm{d}\underline{\xi} \prod_{V' \in A'} \mu_{V'}(\underline{\xi}^{V'})^{s(V')} \prod_{i=1}^{N} \mu_{V_i}(\underline{\xi}^{V_i})^{a(V_i,1) - a(V_{i-1},1)} . \tag{45}$$

For $F_2^{A,V}$, in the corresponding representation it is necessary to substitute the factor for $i = 1$ in the integrand by $|\xi_{V_1} + \eta|^{a(V_1,1)}$ with $\eta = \xi_{V_2} + \ldots + \xi_{V_{n-1}} - (x_1 - x_0)$.

The integration over $\underline{\xi}^{V'}$ with $V' \in A' \backslash \mathscr{N}_V$ is performed as in the proofs in the previous section. It results in the factors $c(V'_-) r(b(V'))$ and the corresponding contributions to the powers of μ_{V_i}. In the case $l = 1$ the integration over $\underline{\xi}^{V_1}$ is performed with the aid of (20), and the integrations over $\underline{\xi}^{V_i}$, $i = 2, \ldots, n$ with the aid of (19). For $l = 2$, all integrations over $\underline{\xi}^{V_i}$, $i = 1, \ldots, n$ are performed with the aid of (21). The integration over $\underline{\xi}^{V_i}$, $i = n+1, \ldots, N$ is the same in both cases and is performed by use of Lemma 2 with $\mu_0 = |x_1 - x_0|$.

Eventually, this gives

$$
\begin{aligned}
F_{\mathscr{V}}(x_1 - x_0) = |x_1 - x_0|^{a(V)+d(|\mathscr{V}|-2)} \sum_A \prod_{V' \in A'} c(V'_-) \\
\times \Bigg\{ \sum_{V \in A : |V|=1} \prod_{V' \in A''} r(b(V')) \prod_{i=1}^{n-1} r(b(V_i \cup \{1\})) h^A_{\mathscr{V}'/V_n} \\
+ \sum_{V \in A'} \prod_{V' \in A''} r(b(V')) \left[\frac{1 - |V_-| p^{-d}}{1 - p^{-d}} + r(b(V)) \right] \\
\times \prod_{i=1}^{n-1} r(b(V_i \cup \{1\})) h^A_{\mathscr{V}'/V_n} \Bigg\},
\end{aligned} \tag{46}
$$

where

$$
h^A_V = \prod_{V' \supseteq V} r(-b(V')) \,. \tag{47}
$$

Finally, the sum in (46) is rewritten as a sum over subsets V_n and over hierarchies including V_n on the whole set of vertices $\mathscr{V}$. (In fact, $\{x_v, v \in \mathscr{V}'\} \in \mathscr{D}_A$ and $x_1 \in \mathscr{D}^V_{1,2}$ for some $V \in A$ if, and only if $\{x_v, v \in \mathscr{V}\} \in \mathscr{D}_{A^*}$ with a certain uniquely defined hierarchy A^* on $\mathscr{V}$.) Consequently, this yields (39).

In the case $n = 3$ we have four hierarchies A_0 on the set of three external vertices. The corresponding domains of external coordinates are $|x_0 - x_1| < |x_0 - x_2| = |x_1 - x_2|$, the other two domains of this type, and the second-type domain $|x_0 - x_1| = |x_0 - x_2| = |x_1 - x_2| \equiv M$. For the first type the result (38) is written as

$$
\begin{aligned}
F_{\mathscr{V}}(x_0, x_1, x_2) = (1 - p^{-d})^{-2} |x_2 - x_0|^{a(\mathscr{V})+d(|\mathscr{V}|-3)} \\
\times \sum_{A \in A\{\mathscr{V}; A_0\}} \prod_{V \in A'} c(V_-) \prod_{V \in A''} r(b(V)) \\
\times f^A_{V_{012}/V_{01}}(|x_0 - x_1| / |x_0 - x_2|, 1) h^A_{\mathscr{V}/V_{012}} \,,
\end{aligned} \tag{48}
$$

where V_{01} is the minimal element of the hierarchy A including the external vertices v_0 and v_1, and V_{012} is the minimal element of the hierarchy A including all the external vertices. Furthermore, $f^A_{V_{012}/V_{01}}$ is given by (35) with the nest $V_0 = (V_{01})_+ / V_{01}, \ldots, V_{012}/V_{01}$.

For the second type the result looks like

$$
\begin{aligned}
F_{\mathscr{V}}(x_0, x_1, x_2) = (1 - p^{-d})^{-1} (1 - 2p^{-d})^{-1} M^{a(\mathscr{V})+d(|\mathscr{V}|-3)} \\
\times \sum_A \prod_{V \in A'} \frac{c(V_-)}{p^{-\Omega(V)} - 1} \,,
\end{aligned} \tag{49}
$$

where $\Omega(V) = -b(V)$ if $V \subset V_{012}$ or $V \cap V_{012} = \emptyset$ and $\Omega(V) = b(\mathscr{V}) - b(V)$ if $V_{012} \subseteq V \subset \mathscr{V}$.

A proof can be achieved as a generalization of the proof of the corresponding result for $n = 2$. The integral (37) is first decomposed into subintegrals associated with hierarchies on the set $\mathscr{V}' \equiv \mathscr{V}^{\text{int}} \cup v_0$. Then partition (41–43) is used both for $x = x_0$ and $x = x_1$. Consequently, the Feynman amplitude (37) is represented as a sum over V_1, V_2. If V_{01} is the minimal element containing v_0 and V_1 we have

$M(V_{01}) = |x_1 - x_0|$. Then one turns to the variables $\{\underset{\sim}{\xi}^{V'}\}_{V' \in A'}$ by use of Eqs. (28, 29).

In the case of the first-type domain A is represented as $A = A'' \cup \mathcal{N}$ with $\mathcal{N}$ the nest of elements $V_1, \dots, V_{01}, \dots, V_{012}, \dots, \mathscr{V}'$. The integration over $\underset{\sim}{\xi}^{V'}$ with $V' \in A' \cap A''$ is performed as in the proof of Lemma 1. For $V = V_1, \dots, V_{01}$ the integration procedure is the same as in the case $n = 2$. Then, for $V = (V_{01})_+, \dots, V_{012}$, it is necessary to apply Lemma 2. for $V = (V_{012})_+, \dots, \mathscr{V}'$, one uses the same lemma (the final integration is with $\mu_{k+1} = \infty$). In the case of the second-type hierarchy the integration is the same as for $n = 2$. Finally, the obtained results are naturally rewritten as sums over hierarchies on the whole set of vertices $\mathscr{V}$.

As was demonstrated in [22] by simple examples one can introduce various renormalization schemes: analytic [24], dimensional [5, 25] and BPHZ-renormalizations [2, 3, 14, 30]. However, in the massless case, we have the explicit result (38). It is easy to observe that the divergences manifest themselves in the factors $r(b(V)) \equiv 1/(p^{b(V)} - 1)$, where V generally takes the form of a reduced graph V'/V''. If the subgraph generated by the set V does not involve isolated vertices, then $\omega(V) = -b(V) \equiv -a(V) - d(|V| - 1)$ is nothing but the (ultraviolet) degree of divergence ω_V. Therefore the amplitude does not contain ultraviolet and infrared divergences if the degree of divergence of each reduced graph of the form V'/V'' is not equal to zero. In other words, the p-adic Feynman amplitudes can possess only logarithmical ultraviolet and infrared divergences. (The infrared degree of divergence of a subgraph γ in Γ can be naturally defined as minus ultraviolet degree of some reduced graph – see [23].)

5. Examples

Let us apply the general formula (39) for the graph shown in Fig. 1. There are 24 hierarchies on the set of four vertices. For $a(0, 1) = 0$ and $a(i, i') = -2$ elsewhere, and arbitrary dimension $d = 4 + \delta$, we have

$$
\begin{aligned}
F(\underset{\sim}{x}) = |x_1 - x_0|^{a(\mathscr{V})+2d} \Bigg\{ & (1 - 2p^{-d})(1 - 3p^{-d})(1 - p^{-d})(1 - 2p^{-d}) \\
& \times \left[\frac{1}{p^{2-2\delta} - 1} + \frac{5}{p^{2+\delta} - 1} + \frac{2}{p^{2-\delta} - 1} + \frac{2}{p^{2+2\delta} - 1} \right] \\
& + (1 - p^{-d})^2 \left[\frac{4}{(p^{2-\delta} - 1)(p^{2+\delta} - 1)} - \frac{2}{(p^{2-2\delta} - 1)(p^{2-\delta} - 1)} \right. \\
& \left. + \frac{6}{(p^{2+2\delta} - 1)(p^{2+\delta} - 1)} + \frac{1}{(p^{2-2\delta} - 1)(p^{2+\delta} - 1)} + \frac{2}{(p^{2+\delta} - 1)^2} \right] \Bigg\} . \qquad (50)
\end{aligned}
$$

Consider now the star Feynman diagram with three external $\{1,2,3\}$ and one internal $\{4\}$ vertices with $a(i, 4) = \lambda_i$ and $a(i, i') = 0$ elsewhere. Using general results (48, 49) we have

$$
\begin{aligned}
F(\underset{\sim}{x}) = {} & B(\lambda_1, \lambda_2) |x_1 - x_2|^{\lambda_1+\lambda_2+d} \, |x_1 - x_3|^{\lambda_3} \\
& + B(\lambda_1 + \lambda_2, \lambda_3) |x_1 - x_3|^{\lambda_1+\lambda_2+\lambda_3+d} \qquad (51)
\end{aligned}
$$

for the domain $|x_1 - x_2| < |x_2 - x_3| = |x_1 - x_3|$, (here $B(\lambda_1, \lambda_2) = \Gamma(\lambda_1 + d)\Gamma(\lambda_2 + d)/\Gamma(\lambda_1 + \lambda_2 + 2d)$), and

$$F(\underline{x}) = |x_1 - x_2|^{\lambda_1+\lambda_2+\lambda_3+d} \times \left[1 - 3p^{-d} + (1 - p^{-d})\left(\frac{1}{p^{-\sum_i \lambda_i - d} - 1} + \sum_i \frac{1}{p^{\lambda_i + d} - 1}\right)\right] \tag{52}$$

for the domain $|x_1 - x_2| = |x_2 - x_3| = |x_1 - x_3|$. For $d = 4$ and $\lambda_i = -2$ in the first case we have

$$F(\underline{x}) = |x_1 - x_3|^{-2}\left\{(1 + 4p^{-2} + p^{-4}) + (1 - p^{-4})\log_p \frac{|x_1 - x_3|}{|x_1 - x_2|}\right\}, \tag{53}$$

and in the second case

$$F(\underline{x}) = |x_1 - x_3|^{-2}(1 + 4p^{-2} + p^{-4}). \tag{54}$$

Let now $\sum_i \lambda_i = -2d$. Using the above explicit results (51, 52) we obtain the following identity:

$$F(\underline{x}) = \prod_i \Gamma_d(\lambda_i + d)|x_1 - x_2|^{-\lambda_3 - d}|x_2 - x_3|^{-\lambda_1 - d}|x_3 - x_1|^{-\lambda_2 - d}, \tag{55}$$

which is generalization of the corresponding star-triangle identify for the real-space Feynman amplitudes [6].

6. Conclusion

In four-dimensional space we obtain a simple and symmetrical expression for the graph of Fig. 1:

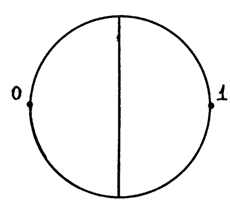

Fig. 1. The master two-loop diagram

$$F_p = \frac{1}{|x|_p^2}(1 + 10p^{-2} + 20p^{-4} + 10p^{-6} + p^{-8}). \tag{56}$$

(We now restore the index p as well as explicit dependence on p.) In the real space this diagram is not calculable for arbitrary values of the analytic regularization. In the absence of the regularization the corresponding result is

$$F_\infty = \frac{1}{|x|_\infty^2} 6\zeta(3) \tag{57}$$

with $\zeta(z)$ the Riemann zeta-function.

In [10] it was discovered that the product over all prime numbers of $N = 4$ p-adic string amplitudes A_p times the real-space string amplitude A_∞ is equal to one (see also [1, 8, 20]). Thus, the adelic string amplitude A_{adelic} seems to be very simple at the tree level. Attempts to generalize this adelic formula to other string amplitudes did not lead to such simple identities (see, e.g., [17]).

Let us now return to Feynman amplitudes. It is tempting to multiply the result (57) and the results (56) over all prime numbers p. The simplest product formula [4, 15] $\prod_p \frac{1}{1-p^{-z}} = \zeta(z)$ seems to be insufficient for this product to be calcualted. Nevertheless, this product *is convergent*. This fact allows one to suppose the existence of product formulae of adelic type. Perhaps, they are not so simple as the simplest adelic formula for strings. To guess what kind of adelic formulae can exist a calculational experience based on the derived explicit result for the general Feynman amplitude could play a decisive role. As a consequence of such adelic formulae one could obtain new calculation methods for real-space Feynman amplitudes.

Acknowledgements. The author is grateful to Professor W. Zimmermann for kind hospitality at the Max-Planck-Institut für Physik and to the Alexander von Humboldt-Stiftung for support.

References

1. Aref'eva, I.Ya., Dragovič, B., Volovich, I.V.: On the adelic string amplitudes. Phys. Lett. B **209**, 445–450 (1988)
2. Bogoliubov, N.N., Parasiuk, O.S.: Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. Acta Math. **97**, 227–266 (1957)
3. Bogoliubov, N.N., Shirkov, D.V.: Introduction to the theory of quantized fields. 4th ed. Moscow: Nauka 1984
4. Borevich, Z.I., Shafarevich. I.R.: Number theory. New York, London: Academic Press 1966
5. Breitenlohner, P., Maison, D.: Dimensional renormalization and the action principle. Commun. Math. Phys. **52**, 11 (1977)
6. D'Eramo, M., Peliti, L., Parisi, G.: Theoretical predictions for critical exponents at λ-point of Bose liquids. Lett. Nuovo Cim. **2**, 878–882 (1971)
7. Frampton, P.H., Okada, Y.: p-adic string N-point function. Phys. Rev. Lett. **60**, 484–486 (1988)
8. Frampton, P.H., Okada, Y., Ubriaco, M.R.: On adelic formulas for the p-adic string. Phys. Lett. B **213**, 260–262 (1988)
9. Freund, P.G.O., Olson, M.: Non-archimedean strings. Phys. Lett. B **199**, 186–190 (1987)
10. Freund, P.G.O., Witten, E.: Adelic string amplitudes. Phys. Lett. B **199**, 191–195 (1987)
11. Gel'fand, I.M., Graev, M.I., Pjatetski-Shapiro, I.I.: Theory of representations and automorphic functions. Moscow: Nauka 1966
12. Gracey, J.A., Verstegen, D.: The $O(N)$ Gross-Neveu and supersymmetric σ-models on p-adic fields. Mod. Phys. Lett. A **5**, 243–254 (1990)
13. Grossman, B.: p-adic strings, the Weyl conjectures and anomalies. Phys. Lett. B **197**, 101–106 (1987)
14. Hepp, K.: Proof of the Bogoliubov-Parasiuk theorem on renormalization. Commun. Math. Phys. **2**, 301–326 (1966)
15. Koblitz, N.: p-adic numbers, p-adic analysis and zeta-functions. 2nd ed. Berlin, Heidelberg, New York: Springer 1984
16. Lerner, E.Y., Missarov, M.D.: p-adic Feynman and string amplitudes. Commun. Math. Phys. **121**, 35–48 (1989)
17. Marinari, E., Parisi, G.: On the p-adic five-point function. Phys. Lett. B **203**, 52–56 (1988)
18. Meurice, Y.: Quantum mechanics with p-adic numbers. Int. J. Mod. Phys. A **4**, 5133–5147 (1989)
19. Parisi, G.: On p-adic functional integrals. Mod. Phys. Lett. A **3**, 639–643 (1988)
20. Ruelle, P., Thiran, E., Verstegen, D., Weyers, J.: Adelic string and superstring amplitudes. Mod. Phys. Lett. A **4**, 1745–1752 (1989)

21. Schikhof, W.H.: Ultrametric calculus. Cambridge: Cambridge University Press 1984
22. Smirnov, V.A.: Renormalization in p-adic quantum field theory. Mod. Phys. Lett. A **6**, 1421–1427 (1991)
23. Smirnov, V.A.: Renormalization and asymptotic expansions. Basel: Birkhäuser 1991
24. Speer, E.R.: Analytic renormalization. J. Math. Phys. **9**, 1404–1410 (1968)
25. 't Hooft, G., Veltman, M.: Regularization and renormalization of gauge fields. Nucl. Phys. B **44**, 189–213 (1972)
26. Vladimirov, V.S.: Distributions over the field of p-adic numbers. Usp. Mat. Nauk **43**, No. 5, 17–53 (1988)
27. Vladimirov, V.S.., Volovich, I.V.: p-adic quantum mechanics. Commun. Math. Phys. **123**, 659–676 (1989)
28. Volovich, I.V.: p-adic space-time and string theory. Teor. Mat. Fiz. **71**, 337–340 (1987); p-adic string. Class. Quant. Grav. **4**, L 83–L 85 (1987); Number theory as the ultimate physical theory. Preprint CERN-TH 4981/87
29. Zavialov, O.I.: Renormalized quantum theory. Kluwer Academic Publishers 1990
30. Zimmermann, W.: Composite operators in the perturbation theory of renormalizable interactions. Ann. Phys. **77**, 536–569 (1973)

Communicated by K. Gawedzki