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$N = 2$ Supergravity, Type IIB Superstrings, and Algebraic Geometry

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Abstract. The geometry of $N = 2$ supergravity is related to the variations of Hodge structure for "formal" Calabi–Yau spaces. All known results in this branch of algebraic geometry are easily recovered from supersymmetry arguments. This identification has a physical meaning for a type IIB superstring compactified on a Calabi–Yau 3-fold. We give exact (non-perturbative) results for the string effective Lagrangian. Our geometrical framework suggests a re-formulation of the Gepner conjecture about (2,2) superconformal theories as the solution to the *Schottky problem* for algebraic complex manifolds having trivial canonical bundle.

I. Introduction

It is well known that some classical problems in mathematics can be studied using ideas arising from supersymmetry [1]. A partial list of results contains the Witten formulation [2] of Morse theory, the simple proof of the index theorems [3] given by Alvarez–Gaume [4], and the theory of elliptic genera [5].

On the physical side, the understanding of the mathematical implications of susy is also crucial, since it may give non-perturbative insight on the theory [6].

These results arise from *rigid* supersymmetry. It is an easy guess that *local* supersymmetry (supergravity) should be even more powerful as a "mathematical trick." What is not easy to figure out, is what are the mathematical problems related to supergravity.

In this paper we give a partial answer to this question for $N = 2$ supergravity. We show that this theory is deeply connected with the period maps [7] for algebraic complex threefolds having trivial canonical bundle (Calabi–Yau spaces [8]). Everything is known [9, 10] about these maps, turns out to be a well known fact of supergravity. $N = 2$ SUGRA is a "physicist's" approach to the theory of Hodge structures and Hodge bundles over moduli spaces.

To give a rough idea of the period map and its relevance for physics, let us consider the situation where its meaning is obvious. Consider the compactification [11] of the type IIB superstring [12] on a Calabi–Yau (CY) 3-fold X. The 4D

effective theory has $N=2$ (space-time) supersymmetry. The scalar fields of the $N=2$ vector multiplets, are associated to harmonic $(2,1)$ forms on X. So they parametrize the inequivalent complex structures of X (the Kuranishi [13] deformation space S). By definition of CY space, for each complex structure (specified by a point $s \in S$) there is a holomorphic $(3,0)$ form $\varepsilon(s)$. The period map associates to a point of S the line in $PH^3(X, \mathbb{C})$ generated by $\varepsilon(s)$. From this map we easily reconstruct the low-energy interactions in a purely geometric way.

The period map should satisfy a number of conditions. First of all, there are the usual Riemann–Hodge bilinear relations. In addition, there are *universal* differential constraints, known as *infinitesimal period relations*. A variation of Hodge structure [9] is a map satisfying these conditions, but which needs not to arise as the period map for some manifold. The *Schottky problem* is to characterize the variations corresponding to *real* manifolds.

Below we show that *any* $N=2$ supergravity can be written in this "geometrical" way for some variation of Hodge structure. Conversely, any such variation defines a sugra model. The formal period map is interpreted as the equation of motion for the auxiliary field $T_{\mu\nu}$. The scalars' Kähler metric is the Weil–Peterson (WP) metric on the formal deformation space. So, the theorem [14] giving the WP metric in terms of the period maps is re-interpreted as the usual susy relation between the kinetic terms for the vector fields and for their scalar partners. This physical interpretation can be extended to more general results about the geometry of Hodge bundles.

The case of a two dimensional CY space ($K3$ surface), is completely understood from the mathematical side [15], as well as from the physical one [16]. Compactifying on $K3$ a type II superstring, we get $N=4$ space-time SUSY. For $K3$ there are no infinitesimal period relations. This corresponds to the *uniqueness* of the $N=4$ Lagrangian [17].

There are many applications for our geometrical point of view.

Our first motivation was to get the low-energy theory for a superstring compactified on a CY space. In first approximation, this is just a Kaluza–Klein problem. In Kaluza–Klein theory we need the internal metric. Unfortunately, no example of a CY metric is known. Since such a metric is uniquely fixed by the complex structure and the Kähler class [8], it is natural to look for a formula relating the effective Lagrangian directly to these data rather than to the metric itself. This is similar to what Strominger [18] and Candelas [19] did by relating the effective couplings to topological invariants of X.

We give exact formulae for the various couplings in the low-energy Lagrangian. Unfortunately, these formulae are quite involved, even in the simplest cases.

A related application is the moduli problem for two-dimensional $N=2$ superconformal theories, and their Zamolodchikov metric [20]. The starting point is Seiberg's remark [16] that this moduli space is equal to the scalars' sigma-model of the low-energy theory for the superstring defined by the given superconformal theory. The Zamolodchikov metric can be read directly from the effective kinetic terms [16]. The metric is constrained by (space-time) $N=2$ supersymmetry.

In this way, we get a relation between the (Zamolodchikov) geometry of superconformal moduli and the geometry of $N=2$ supergravity. Then, our interpretation of this geometry allows to identify the superconformal moduli space with a Hodge variation (of the Calabi–Yau type).

The correspondence between the superconformal and "formal CY" moduli recalls a conjecture by Gepner [21]; every (2,2) $c = 9$ superconformal model is equivalent to a "sigma-model" on a CY threefold. Indeed, our argument associates to each such conformal theory a "CY space" (in the sense of *formal* Hodge theory).

Assuming (a weak version of) the conjecture, our results strongly suggest a "physical" solution to the Schottky problem for Calabi–Yau threefolds. The variations of Hodge structure arising from geometry are precisely those associated to the $N = 2$ supergravities which can be realized as low-energy limits of type IIB superstrings (neglecting the hypermultiplet sector).

Recently, there was some progress towards a proof of this conjecture [22, 23]. These works also use ideas from algebraic geometry. However, the relation between the two approaches is unclear.

The paper is organized as follows. In Sect. II we recall some mathematical facts about Hodge variations and all that. In Sect. III we review the geometry of $N = 2$ sugra and relate it to Hodge variations. In Sect. IV we consider the string. The abstract ideas have a direct meaning here. This leads to a re-formulation of the Gepner conjecture as the solution to Schottky. We discuss the Yukawa couplings for the heterotic string. Finally, in Sect. V we discuss the application to low-energy effective Lagrangians.

II. Some Mathematical Preliminaries

Variations of Hodge Structure. Let X be a compact complex threefold with trivial canonical bundle

$$K_X \cong \mathcal{O}_X, \quad q(X) = 0. \tag{II.1}$$

We assume X to be Kähler, and fix a reference Kähler class $\omega_0 \in H^2(X, \mathbb{R})$. Without loss of generality, we take $b_1(X) = 0$. Then the cohomology in $H^3(X, \mathbb{C})$ is primitive.

Following ref. [9], we define $H_{\mathbb{Z}} = H^3(X, \mathbb{Z})/\text{torsion}$, $H = H_{\mathbb{Z}} \otimes \mathbb{C}$ and $H^{p,q} = H^{p,q}(X, \mathbb{C})$, $(p + q = 3)$. In this notation, the Hodge decomposition reads

$$H = \bigoplus_{p+q=3} H^{p,q}, \quad H^{p,q} = \bar{H}^{q,p},$$
$$\dim_{\mathbb{C}} H^{3,0} = 1, \qquad \dim_{\mathbb{C}} H^{2,1} \equiv h^{2,1} = m. \tag{II.2}$$

The data $\{H_{\mathbb{Z}}, H^{p,q}\}$ is called [9] a *Hodge structure* (of weight 3). In this paper, we are interested in the supersymmetry interpretation of a *variation of Hodge structure* [9]. It is more convenient to work with the *Hodge filtration* of H, $\{F^p\}_{p=0,\ldots,3}$, where

$$H \supset F^p = \bigoplus_{k \geqq 0} H^{p+k, 3-p-k}. \tag{II.3}$$

On the lattice $H_{\mathbb{Z}}$ there is a natural skew-symmetric form

$$Q: H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}, \quad Q(\phi, \psi) = -\int_X \phi \wedge \psi \tag{II.4}$$

which satisfy the *Hodge–Riemann bilinear relations*:

$$\begin{aligned} &1.\ Q(H^{p,q},H^{p',q'})=0 \quad \text{unless } p'=3-p,\quad q'=3-q,\\ &2.\ (i)^{p-q}Q(\psi,\bar\psi)>0 \quad \text{for all (non-zero)} \quad \psi\in H^{p,q}. \end{aligned} \tag{II.5}$$

The data $\{H_{\mathbb{Z}},\{F^p\},Q\}$ is called a *polarized* Hodge structure [9] (of weight 3). In terms of the filtration the first bilinear relation reads

$$\begin{cases} F^1=(F^3)^{\perp} \\ F^2=(F^2)^{\perp}, \end{cases} \tag{II.6}$$

where $\perp$ denotes the orthogonal complement with respect to Q. Equation (II.6) implies that the filtration $\{H\equiv F^0\supset F^1\supset F^2\supset F^3\}$ is uniquely determined by the "short" filtration $\{H\supset F^2\supset F^3\}$.

Since $Q(\cdot,\cdot)$ is non-degenerate, it induces a *symplectic structure* on H. To make manifest the connection with $N=2$ sugra, we choose canonical coordinates on H

$$\begin{gathered} X^0,X^1,\dots,X^m;P_0,P_1,\dots,P_m\in H^* \\ \text{such that } \Lambda^2H^*\ni Q=\sum_{I=0}^{m}X^I\wedge P_I. \end{gathered} \tag{II.7}$$

In practice, we can start with a canonical homology basis[1] $a_{0,-},a_m;b^0,b^1,_-,b^m\in H_3(X,\mathbb{Z})$. For $\psi\in H$ we put

$$X^I(\psi)=\int_{a_I}\psi \quad P_I(\psi)=-\int_{b_I}\psi \quad \psi\in H, I=0,\dots,m. \tag{II.8}$$

There are three basic classifying spaces (period domains) we shall need below. First we have the *classical period domain* $\tilde D$,

$$\tilde D=\{\text{complex line } l\in H\,|\,\forall\varphi\in l,\varphi\neq 0,-iQ(\varphi,\bar\varphi)>0\}. \tag{II.9}$$

$\tilde D$ depends only on the topological type of X. It is an open domain in $\mathbb{P}H$.

The *Griffiths period domain* D is the space of all filtrations

$$\{F^3\subset F^2\subset F^1\subset F^0\equiv H\} \tag{II.10}$$

with $\dim(F^p)=\sum_{k\geqq 0}h^{p+k,3-p-k}$ and satisfying the bilinear relations. D is an open subset of a homogeneous complex manifold. The *compact dual* $\check D$ is defined as D, except that the second Riemann–Hodge relation is not required. Clearly, D is open in $\check D$.

Let $H_{\mathbb{R}}=H_{\mathbb{Z}}\otimes\mathbb{R}$, and $G_{\mathbb{R}}=\mathrm{Aut}(H_{\mathbb{R}},Q)\approx \mathrm{Sp}(2m+2,\mathbb{R})$. $G_{\mathbb{R}}$ is identified in supergravity with the Gaillard–Zumino duality group [24].

Then we have [9]

$$D\cong G_{\mathbb{R}}/V, \tag{II.11}$$

where $V\subset G_{\mathbb{R}}$ is the stabilizer of a point in D.

There is a natural projection

$$\begin{gathered} \pi:D\to\tilde D, \\ \pi(\{F^p\}_{p=0,\dots,3})=[F^3]\subset\tilde D. \end{gathered} \tag{II.12}$$

[1] I.e. the intersection numbers are $\#(a_I,a_J)=\#(b_I,b_J)=0$, $\#(a_I,b_J)=\delta_{IJ}$

The last definition we want to recall is that of *period map*. Let $(\mathfrak{X}, \varpi, S)$ be the local universal deformation family of X (Kuranishi [13] family). For a CY space the local universal deformation space S is smooth [14]: it is isomorphic to an open set in

$$H^1(X, \Theta) \cong H^1(X, \Omega^2) \cong H_{\bar{\partial}}^{2,1}(X). \tag{II.13}$$

If $\theta \in H^1(X, \Theta)$, $\theta \rfloor \omega_0 = 0$. Indeed, $\theta \rfloor \omega_0 \in H_{\bar{\partial}}^{0,2}(X, \mathbb{C})$ and $h^{0,2} = h^{1,0} = b_1/2 = 0$. Thus all the deformations preserve the Kähler class. This fact is crucial for the physical applications. It implies that the moduli space for the *Einstein metrics* on a CY 3-fold (with $b_1 = 0$) is the product of the moduli of complex structures with those of Kähler classes. Thus, the scalars' manifold for the effective IIB theory is a product space. This corresponds to the existence of two types of matter multiplets in $N = 2$ SUGRA: vector-multiplets and hypermultiplets. The scalars of each kind of multiplets parameterize one factor space. The Zamolodchikov metric also is a product metric, one factor being Kählerian and the other one quaternionic. The assumption $b_1 = 0$ is crucial. This is *physically* obvious: if $b_1 \neq 0, X$ should be (locally) reducible and by ref. [25] we have $N \geqq 4$ space-time SUSY. For $N = 4$ we have just one type of matter multiplets. Since the $N = 4$ case is trivial, we assume $b_1 = 0$. Then $h^{2,0} = 0$. By a theorem of Kodaira [26] X is *projective algebraic*. So, our physical problem is equivalent to computing the period integrals for a family of *algebraic* manifolds and can (in principle) be solved by algebraic-geometric techniques.

A point $s \in S$ stands for a compact Kähler manifold $X_s = \varpi^{-1}(s)$, diffeomorphic to $X \equiv X_0$, $c_1(X_s) = 0$. For each $s \in S$ we have a Hodge filtration $\{F^p(s)\}_{p=0,\ldots,3}$, according to Eq. (II.3).

Take a diffeomorphism

$$\phi_s : X \to X_s. \tag{II.14}$$

Obviously, ϕ induces the isomorphism

$$\phi_s^* \{H^3(X_s, \mathbb{C})\} \to H^3(X, \mathbb{C}) \equiv H \tag{II.15}$$

(locally in S; in the large ϕ_s is defined only up to "modular transformations" $=$ $\mathrm{Aut}\,[H_{\mathbb{Z}}]$).

The *Griffiths period map* is given by

$$\Omega : S \to D \quad \Omega(s) = \{\phi_s^*(F^p(s))\}_{p=0,\ldots,3} \tag{II.16}$$

and the *classical period map* by

$$\tilde{\Omega}(s) \equiv \pi[\Omega(s)] \in \tilde{D}. \tag{II.17}$$

These maps are holomorphic [7]. This also follows from supersymmetry.

The rest of this section is devoted to the description in this mathematical framework of the basic physical constructions: the vector and scalars' kinetic terms and the superpotential.

The Weil–Peterson Metric on the Universal Deformation Space. The WP metric on S is defined in the usual way. For $\psi, \phi \in H_{\bar{\partial}}^{2,1}(X_s) \approx T_s(S)$ the WP metric is [14]

$$G_{\mathrm{WP}}(s)(\psi, \phi) = \int_X \psi \wedge * \bar{\phi}. \tag{II.18}$$

The physical interpretation of the WP metric is clear from its definition. Consider the compactification of the type IIB superstring on X. In the resulting $4D$ theory [16] the vector-multiplets' scalar fields z^A take value in S

$$z\colon(\text{Space–Time})\to S.$$

In the low-energy effective Lagrangian their kinetic terms read

$$G_{\mathrm{WP}}(z,\bar{z})_{A\bar{B}}\partial_\mu z^A\partial^\mu\bar{z}^B, \tag{II.19}$$

as it is easy to see from Kaluza–Klein arguments. See e.g. refs. [18, 19].

The resulting effective theory has *two* local supersymmetries. Then the WP metric should be of the special kind allowed in $N=2$ SUGRA. One point of the present paper is to show that virtually *everything* known about the deformations of CY 3-folds [10] is contained in the statement that the WP metric is consistent with $N=2$ supergravity.

The first condition that follows from $N=2$ susy is that G_{WP} is Kähler. In fact, it should be Hodge [27]. That this is true for a Calabi–Yau manifold was shown by G. Tian [14]. This author also gives the explicit form of the Kähler potential in terms of the classical period map.

Since the metric is Hodge, the Kähler form ω_{WP} is the Ricci form for some metric h on a line bundle $\mathfrak{l}$. The most efficient way of describing the metric G_{WP} is to specify $\mathfrak{l}$ and h. Using the notations as above, let L_0 be the tautological bundle over $\mathbb{P}H$ and $\tilde{L}_0$ its restriction to the open domain $\tilde{D}\subset\mathbb{P}H$. The quadratic form $-iQ(\cdot,\cdot)$ induces a metric h along the fibers of $\tilde{L}_0$. This metric is positive-definite by definition of $\tilde{D}$.

The Kähler form ω_{WP} is given by [14]

$$\begin{aligned}&\omega_{\mathrm{WP}}=i\tilde{\Omega}^*\,\mathrm{Ric}(\tilde{h}),\\ &\text{where}\quad \mathrm{Ric}(\tilde{h})=-\partial\bar{\partial}\ln\tilde{h}.\end{aligned} \tag{II.20}$$

The Hodge property is not the only condition on the scalars' metric coming from $N=2$ supergravity. The most interesting properties of $N=2$ metrics stem from the physical fact that the vector multiplet contains a propagating vector. The scalars' kinetic terms are related by SUSY to the vectors' kinetic terms. We shall see that they are specified by the period map. The expression of the WP metric in terms of the period map, Eq. (II.20), is nothing else than the $N=2$ relation between the scalars' and vectors' kinetic terms. Conversely, *any* $N=2$ SUGRA has the above form for some formal period map.

Infinitesimal Period Relations for a CY 3-*fold*: *Contact Systems and Legendre Manifolds* [10]. A *contact manifold* is a complex manifold M of dimension $2m+1$ together with a holomorphic line sub-bundle $L\subset T^*(M)$ such that if ω is a local generator of $\vartheta(L)\subset\Omega^1_M$, then

$$\omega\wedge(d\omega)^m\neq 0.$$

Given a contact manifold (M,L) a *Legendre manifold* is given by an m-dimensional complex manifold S together with an immersion

$$f\colon S\to M$$

such that

$$f^*(\omega) = 0.$$

We can choose local holomorphic coordinates $(q^1, \ldots, q^m, S, p_1, \ldots, p_m)$ for M such that

$$\omega = dS - \sum_{i=1}^{m} p_i dq^i. \tag{II.21}$$

Then a general Legendre manifold is given parametrically by

$$q \to (q, S(q), p(q)), \quad p_i = \frac{\partial S(q)}{\partial q^i} \ [S(q) \text{ holomorphic}]. \tag{II.22}$$

From a $\mathbb{C}$-space $H(\dim H = 2m + 2)$ equipped with a skew-symmetric form Q,

$$Q: H \otimes H \to \mathbb{C}, \tag{II.23}$$

we construct a canonical contact system. Let $P \equiv \mathbb{P}H \approx P\mathbb{C}^{2m+1}$. Consider the projection

$$\pi: (H \backslash \{0\}) \to P. \tag{II.24}$$

Let s be a local holomorphic section of this bundle. We set [10]

$$\omega = s^* Q(dz, z). \tag{II.25}$$

Since $Q(z, z) = 0, \omega$ is well-defined (up to a multiple).

By infinitesimal Torelli [28], the classical period map

$$\tilde{\Omega}: S \to P \cong \mathbb{P}H^3(X, \mathbb{C}) \tag{II.26}$$

is an *immersion*. The universal deformation space S, together with the period map $\tilde{\Omega}$, is a *Legendre manifold* for the canonical contact structure induced by the form Q in Eq. (II.4). In fact, the infinitesimal period relations read

$$\frac{\partial v(s)}{\partial s^i} \in F^{p-1}(s), \tag{II.27}$$

where $v(s) \in F^p(s)$ and $s^1, \_, s^m$ are local holomorphic coordinates on S.

Let $0 \neq z(s) \in F^3(s)$. Then $\partial z(s)/\partial s^i \in F^2(s)$, which by Eq. (II.6) implies

$$0 = \sum_i Q\left(\frac{\partial z(s)}{\partial s^i}, z(s)\right) ds^i = \tilde{\Omega}^* s^* Q(dz, z). \tag{II.28}$$

Since $\dim S = m, S$ is a Legendre manifold. Given that $\tilde{\Omega}$ is an immersion,

$$F^2(s) = \operatorname{span}\left\{z(s), \frac{\partial z(s)}{\partial s^1}, \ldots, \frac{\partial z(s)}{\partial s^m}\right\}. \tag{II.29}$$

Then (see the remark after Eq. (II.6)), the full period map Ω is determined by the classical one $\tilde{\Omega}(s) = \{z(s)\} \in \mathbb{P}H$. I.e. the period map Ω is the first prolongation [10] of the Legendre manifold $\tilde{\Omega}: S \to \mathbb{P}H$. Conversely, any m-dimensional submanifold of D, satisfying the differential system (II.27), arises as the first prolongation of such a Legendre manifold [10].

Let us summarize what we have learned. One starts with a vector space H equipped with a symplectic form Q. From these data we construct the canonical contact structure $(\mathbb{P}H, L)$, where L is generated (locally) by $s^*Q(dz, z)$. Let (S, Ω) be one of its Legendre submanifolds. The WP Kähler form on S has the explicit form

$$\omega_{\mathrm{WP}} = -i\partial\bar{\partial}\tilde{\Omega}^* s^* \ln[-iQ(z, \bar{z})] \tag{II.30}$$

and is independent of the particular local section s. [In physical terms, s is a choice of superconformal gauge.]

We shall see below that Eq. (II.30) describes *all* $N=2$ supergravity theories coupled to m vector-multiplets.

Given this identification for the Kähler metric of $N=2$ sugra, we can re-state the known properties of the WP metric as supergravity theorems.

For instance, from Eq. (II.30) we see that the WP metric is the pullback via $\tilde{\Omega}$ of a $G_{\mathbb{R}}$-invariant metric on D. This implies [29] that the holomorphic sectional curvatures of the WP metric (and hence of all $N=2$ SUGRA metrics) are *negative and bounded away from zero.* Then by Ahlfors lemma [30], if $f: \Delta \to S$ is a holomorphic map (Δ is the unit disk)

$$f^*(ds^2_{\mathrm{WP}}) \leqq C(ds^2_{\mathrm{P}}),$$

where (ds^2_{P}) is the Poincaré metric and C some positive constant. This distance-decreasing property holds for *any* $N=2$ supergravity. One can also show that all these metrics are complete [14b]. These properties are crucial for the physical consistency of $N=2$ supergravity.

Infinitesimal Deformations of Hodge Structures and the Cubic Form (= *Yukawa Couplings*). There is another aspect of deformation theory which is relevant to physics: the cubic form Ξ [10]. Its physical interpretation is the following. Compactify on X the heterotic string [11]. Then the cubic form is just the effective superpotential for the antifamilies of E_6. Hodge deformation theory implies that there is a simple relation between this superpotential and the Kähler potential for the IIB string.

Combining the Kodaira–Spencer map [31], $\rho_s: T_s(S) \to H^1(X_s, \Theta)$, and the cup product map

$$k_s: H^1(X_s, \Theta_s) \to \bigoplus_{p+q=3} \mathrm{Hom}\{H^{p,q}(X_s), H^{p-1,q+1}(X_s)\}, \tag{II.31}$$

we get a map

$$\delta_s = k_s \cdot \rho_s: T_s(S) \to \bigoplus_{p+q=3} \mathrm{Hom}\{H^{p,q}(X_s), H^{p-1,q+1}(X_s)\} \subset T_{\Omega(s)}(D). \tag{II.32}$$

δ_s is the differential of the period map, $\delta_s = \Omega_*(s)$. [Compare with Eq. (II.27).]

Let $T = T_0(S)$, and $\delta = \delta_0$. δ has two fundamental properties: (i) $\delta(\xi)$ and $\delta(\eta)$ commute; (ii) δ preserves the form Q.

The data $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ (satisfying i, ii) are called an *infinitesimal variation of Hodge structure*. A variation of Hodge structure has no *algebraic invariants*, since $G_{\mathbb{R}}$ is transitive in D. Instead, an infinitesimal variation of Hodge structure has many invariants. Physically, they are the Yukawa couplings.

The infinitesimal variation δ induces three maps [10]

$$T \otimes H^{3,0} \to H^{2,1}, \quad T \otimes H^{2,1} \to H^{2,1*}, \quad T \otimes H^{2,1*} \to H^{3,0*},$$

the first being an isomorphism, the last its dual map and the middle one symmetric. Combining the three maps we get

$$\delta^{(3)} : T \otimes T \otimes T \to \otimes^2 H^{0,3} \equiv \mathbb{C}, \tag{II.33}$$

$\delta^{(3)}$ is symmetric by (i). The cubic form Ξ associated to the given infinitesimal variation δ is just $\delta^{(3)}$ restricted to the symmetric product $\mathrm{Sym}^3\, T$. Ξ contains all the algebraic invariants of δ.

For each $s \in S$ we have a cubic form $\Xi(s)$. For $\xi = \Sigma \xi^i \partial/\partial s^i \in T_s(S)$ we have [10]

$$\Xi(s)[\xi] = Q\left(z(s), \xi^i \xi^j \xi^k \frac{\partial^3 z(s)}{\partial s^i \partial s^j \partial s^k} \right). \tag{II.34}$$

$\Xi(s)$ is well defined. In the geometrical case, $T \approx H^1(X, \Theta)$ and the map Ξ is given explicitly by

$$\det : H^1(X, \Theta) \to H^3(X, K_X^*) \cong \otimes^2 H^{0,3}(X),$$

$$\det\left(\theta^i_{\bar{j}} \frac{\partial}{\partial z^i} \otimes d\bar{z}^j \right) = \det \| \theta^i_{\bar{j}} \| \wedge^i \frac{\partial}{\partial z^i} \otimes \wedge^j d\bar{z}^j. \tag{II.35}$$

The isomorphism $\lambda : H^3(X, K^*) \to \mathbb{C}$ is given by $\int_X [(\cdot) \rfloor \varepsilon] \wedge \varepsilon$, where $\varepsilon = z(0) \in H^{3,0}(X)$. Then $g = \lambda[\det(\cdot)]$ is the superpotential for the antifamilies. See refs. [18, 19].

III. Geometry of $N = 2$ Supergravity and Legendre Manifolds

In this section we show that the $N = 2$ Kähler σ-models correspond to Legendre manifolds for the canonical contact structure and that their Kähler metric is given by Eq. (II.30).

For didactical reasons, we begin with the simpler case of $N = 2$ rigid SUSY [32]. In this case, the vector field-strength $F_{\mu\nu}$ is contained in an $N = 2$ chiral superfield X,

$$X = x + \cdots + \theta \sigma_{\mu\nu} \theta F^{\mu\nu} + \cdots,$$
$$\bar{D}^i_\alpha X = 0. \tag{III.1}$$

However, a generic $N = 2$ chiral multiplet does not describe a vector-multiplet. Indeed, the tensor appearing in its θ-expansion needs not to be the curl of a *real* vector field A_μ: one has to impose the Abelian Bianchi identity $\partial^\mu {}^*F_{\mu\nu} = 0$. At the superfield level, this identity reads

$$D_{ij} X = \varepsilon_{ik} \varepsilon_{jl} D^{-kl} \bar{X}, \tag{III.2}$$

where $D_{ij} = D^\alpha_i D_{j\alpha}$.

Just as in the $N = 1$ case, the first components of the chiral multiplets X^A ($A = 1, \_, m$) can be seen as holomorphic coordinates on a complex manifold M. Although a holomorphic function $f(X^A)$ is still a chiral multiplet, in general it does not satisfy the Bianchi identity, Eq. (III.2). Thus on M there exist *preferred* complex coordinates, those satisfying the Bianchi identity. The preferred coordinates induce canonically a *structure* on M which is consistent only with *special* Kähler metrics.

Naively, it appears that the $N = 2$ formalism is covariant only under real linear transformations of the X^A. However, it is not so. The point is that in X^A we have the field-strengths $F^A_{\mu\nu}$ rather than the gauge-potentials. At the level of field-strengths, there is still the possibility of duality transformations [24]. To appreciate the intrinsic geometry of $N = 2$ SUSY one should work in a manifestly duality-invariant first-order formalism.

The $N = 2$ Lagrangian for a system of m Abelian vector-multiplets is

$$L = \tfrac{1}{2}\int F(X^1, \ldots, X^m) d^4\theta + \text{h.c.}, \tag{III.3}$$

where F is a *holomorphic* function of its arguments and $\int d^4\theta$ is the integration over chiral superspace.

We introduce *dual* chiral multiplets P_A by

$$P_A = -\frac{i}{2}\frac{\partial F}{\partial X^A}, \quad A = 1, \ldots, m. \tag{III.4}$$

We use $\omega^\Lambda (\Lambda = 1, \_, 2m)$ as a shorthand for (X^A, P_A). All the ω^Λ are chiral superfields, since F is holomorphic. This notation unifies the equations of motion with the Bianchi identities into a single equation [32]

$$D_{ij}\omega^\Lambda = \varepsilon_{ik}\varepsilon_{jl}\bar{D}^{kl}\bar{\omega}^\Lambda. \tag{III.5}$$

These equations are covariant under arbitrary *real* linear transformations of the ω^Λ. On the space of the ω^Λ we have a natural symplectic form $dX^I \wedge dP_I$. In fact, this symplectic structure is induced by the physical energy-momentum tensor. This is more easily seen at the component level. The dual multiplets P_I contain as 2-form components the dual field-strengths $\tilde{G}_{A\mu\nu} \equiv \partial L/\partial F^A_{\mu\nu}$. Defining

$$(\mathscr{F}_{\mu\nu})^T = (F^A_{\mu\nu}, G_{B\mu\nu}),$$

the energy momentum reads

$$\begin{aligned} T^\mu_\nu &= \tfrac{1}{2}(\mathscr{F}^T)^{\mu\sigma}\Omega\tilde{\mathscr{F}}_{\sigma\nu} + \text{terms at most linear in } \mathscr{F} \\ &\equiv \tfrac{1}{2}Q(\mathscr{F}, \tilde{\mathscr{F}})^\mu_\nu + \cdots \\ \Omega &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{III.6}$$

This is the physical origin of the symplectic structure. $\mathrm{Sp}(2m, \mathbb{R})$ is the maximal symmetry group of the energy-momentum tensor. Only the linear transformations which leave invariant this form are true invariances of the formalism [24].

Now we turn to the much more interesting case of local $N = 2$ susy. The idea is to use the superconformal approach [33]. One constructs a theory which is invariant under a larger gauge group – the superconformal group $SU(2,2|2)$ – and then eliminates the "spurious" gauge invariances $SU(2,2|2)$/[superPoincaré] by a suitable gauge fixing. At the superconformal level, dilatation and R-symmetry are gauge invariances. The action of these two symmetries on the vector-multiplets

X^I is fixed by the gauge algebra itself. On the first components it reads

$$X^I(x) \to e^{\alpha(x) + i\beta(x)} X^I(x) \tag{III.7}$$

with $\alpha(x)$ and $\beta(x)$ arbitrary. Then, the two scalar field configurations $\{X^I(x)\}$ and $\{\lambda(x)X^I(x)\}$ are physically equivalent, being related by a gauge symmetry. In particular, this implies that to have m physical vector-multiplets we need $m+1$ superfields X^I, since one of their scalars components can be gauged away. Instead all $m+1$ vectors are physical (the additional one is the "graviphoton"). We work in a first order formalism. Let $P_I (I = 0, \_, m)$ be the dual superfields and $\omega^\Pi = (X^I, P_J)$. Consistency of gauge with duality transformations requires the scalar field configurations $\{\omega^\Pi(x)\}$ and $\{\lambda(x)\omega^\Pi(x)\}$ to be identified.

ω^Π can be seen as the coordinates in a complex vector space H ($\dim H = 2m+2$). The above gauge-invariances imply that the physically relevant space is not H but $P = \mathbb{P}H$.

Again, on the vector space H we have a symplectic form

$$Q = X^I \wedge P_I \tag{III.8}$$

defined by the energy-momentum tensor, i.e. by the spin-1 kinetic terms.

From the data (H, Q) we construct the canonical complex contact manifold (P, L) as in Sect. II. Recall that $\vartheta(L)$ is locally generated by $s^* Q(d\omega, \omega)$, with s *any* (local) holomorphic section. Note that s is just a gauge slice for dilatation and R-symmetry.

P has complex dimension $2m+1$, whereas physically we have just m (complex) scalar fields. Thus, the (physical) scalars' sigma-model should be an m-dimensional complex submanifold of P. In fact it is a *Legendre submanifold* for the contact structure (P, L).

To see this, recall the definition of the dual vector-multiplets P_I, Eq. (III.4). Since P_I and X^I transform in the same way under R and scale invariances, F should be homogeneous of degree 2 in the X^I's [33].

Let Σ be the submanifold of H defined by the Eq. (III.4), and let

$$i_\Sigma : \Sigma \to H \tag{III.9}$$

be the corresponding immersion. We denote by S^0 the image of Σ under the projection $\pi: H \to P$. S^0 has dimension m. The physical scalar manifold S is an open subset of the manifold S^0. Which subset will be clear in a moment. Since

$$Q(d\omega, \omega) = P_I dX^I - X^I dP_I$$

we have

$$i_\Sigma^* Q(d\omega, \omega) = -\frac{i}{2}(F_I - X^J F_{JI}) dX^I \equiv 0, \tag{III.10}$$

since F is homogeneous of degree 2. Equation (III.10) shows that S^0 is a Legendre manifold for our contact structure. Thus, the physical scalars' manifold is (open in) a Legendre submanifold, as claimed.

Conversely, (using Pfaff–Darboux) one shows that a general Legendre manifold of (P, L) is associated to an $N = 2$ supergravity coupled to m vector-multiplets. The association is one-to-one.

In perfect analogy with Sect. II, we define the open set $\tilde{D} \subset P$ (the "classical period domain") to be the projection in P of the open set

$$\{\omega \in H \mid -iQ(\omega,\bar{\omega}) > 0\}. \tag{III.11}$$

Let $\tilde{S}$ be the restriction of the Legendre manifold S^0 to the open set $\tilde{D} \subset P$, and $\tilde{\Omega}_S$ the corresponding immersion. The map $\tilde{\Omega}_S$ has all the formal properties of the period map for a CY 3-fold (see Sect. II) *except one*: its first prolongation

$$\tilde{\Omega}_S^{(1)}: \tilde{S} \to \check{D},$$

whereas the Griffiths period map ($=1^{\text{st}}$ prolongation of the classical period map) has D as target space. In other words, $\tilde{\Omega}_S^{(1)}$ needs not to satisfy the 2^{nd} Riemann–Hodge relation. D is open in $\check{D}$. Let S be the largest open set in $\tilde{S}$ such that

$$\tilde{\Omega}_S^{(1)}(S) \subseteqq D, \tag{III.12}$$

and let $\tilde{\Omega} = \tilde{\Omega}_S|_S$ be the corresponding immersion. $\tilde{\Omega}$ has all the properties required by formal Hodge theory.

We claim that:

i) S is the target manifold for the (physical) scalars' σ-model.

ii) the Kähler metric on S predicted by local $N=2$ SUSY is the Weil–Peterson metric.

We show ii) before. We have

$$\begin{aligned}\omega_{\text{WP}} &= -i\tilde{\Omega}^* s^* \partial\bar{\partial}\ln[-iQ(\omega,\bar{\omega})]\\ &\equiv -ii_\Sigma^* \partial\bar{\partial}\ln[-iQ(\omega,\bar{\omega})]|_S\\ &\equiv -i\partial\bar{\partial}\ln[\tfrac{1}{2}(X^I\bar{F}_I + \bar{X}^I F_I)]|_S\\ &\equiv -i\partial\bar{\partial}\ln[2|X^0|^2 Y(Z^I,\bar{Z}^J)]|_S\\ &\equiv -i\partial\bar{\partial}\ln[Y(Z^I,\bar{Z}^J)]|_S,\end{aligned} \tag{III.13}$$

where $Z^I = X^I/X^0$, $(Z^0 = 1)$, and $Y(Z,\bar{Z})$ is defined by

$$|X^0|^2 Y(Z,\bar{Z}) \equiv \tfrac{1}{4}\{X^I\bar{F}_I + \bar{X}^I F_I\}. \tag{III.14}$$

The last line of Eq. (III.13) is exactly the usual expression for the Kähler metric in $N=2$ sugra, ref. [33]. To show i), we recall that in SUGRA the physical σ-model is defined to be the "positivity domain," i.e. domain in $\mathbb{C}^n$ where $Y > 0$ (positivity of spin-2 kinetic terms) and $G_{\text{WP}} > 0$ (positivity of spin-0 kinetic terms). Together, these conditions are equivalent to the 2^{nd} bilinear relation (compare with Eq. (II.18)). Then, (i) is the definition of S.

S may be interpreted as the formal "moduli space" associated to the variation of Hodge structures which defines the given $N=2$ SUGRA model.

Up to now we have studied only the formal aspects of supergravity and shown that the emerging geometrical structure is that of deformation theory for weight 3 Hodge structures with $h^{3,0} = 1$ (formal "Calabi–Yau spaces"). At this level, it may seem that the relationships between the two subjects are purely formal. But it is not so. Our next point is to explain *physically* the emergence of the period map as well as its close relationships with superstring theory.

The connection between supergravity and period maps is more general than $N=2$ sugra. The well known fact that for Abelian varieties and $K3$ surfaces there are no infinitesimal period relations, corresponds in physics to the well known fact that for $N = 4$, 8 only *one* scalar manifold is consistent with local SUSY. In fact, in the presence of differential constraints, the general solution contains arbitrary functions encoding the boundary conditions for the differential equations. The free function $F(X)$ of $N = 2$ supergravity arises just in this way.

IV. The Graviphoton as the Classical Period Map

The classical period map specifies the subspace $H^{3,0}(X_s)$ in $H^3(X, \mathbb{C})$ as a function of the moduli. Here we show that, in physical terms, this is equivalent to specify what vector is the "graviphoton."

The "graviphoton" field-strength $Z_{\mu\nu}$ is defined by the susy transformation of the gravitino

$$\delta\psi^i_\mu = 2D_\mu\varepsilon^i + \varepsilon^{ij}\sigma^{\rho\sigma}\gamma_\mu Z^-_{\rho\sigma}\varepsilon_j + \cdots \tag{IV.1}$$

($Z^\pm_{\mu\nu}$ are the (anti-)self-dual part of $Z_{\mu\nu}$). If susy is unbroken and $\Lambda = 0$, the flux at spatial infinity of the 2-form Z has the physical interpretation of a central charge. On-shell, $Z_{\mu\nu}$ is the same as the auxiliary field $T_{\mu\nu}$ of $N = 2$ supergravity [33]. So, the period map is just the equation of motion for $T_{\mu\nu}$.

In $N = 2$ supergravity, coupled to m vector-multiplets, we have $m + 1$ spin-1 particles. Let $F^I_{\mu\nu}$ ($I = 0, \dots, m$) be the corresponding (Abelian) field-strengths. The field-strengths

$$\mathscr{F}^\Lambda_{\mu\nu} = (F^I_{\mu\nu}, G_{J\mu\nu}) \quad [\Lambda = 0, 1, \dots, 2m + 1]$$

can be seen as a two-form taking value in a vector space H. Defining the multiplication by the imaginary unit i as the Hodge dual, H becomes a complex vector space. In fact, it is just the space H introduced in Sect. III, as it can be seen by expanding the chiral multiplets in components. It is also identified with the space H of the corresponding *formal* Hodge variation (Sect. II).

Here we want to show that this identification is *geometrical* when the $N = 2$ model arises as the low-energy limit of a IIB superstring compactified on a CY space X. This is already clear at the field-theoretic level. In this case, the 5-form field-strength of $10D$, IIB supergravity is given by

$$\mathscr{H} = \sum_{\Lambda=1}^{2m+2} \mathscr{F}^\Lambda \wedge \Psi_\Lambda, \tag{IV.2}$$

where the harmonic 3-forms Ψ_Λ form a basis for $H^3(X, \mathbb{C})$. The identification $H = H^3(X, \mathbb{C})$ then follows from the isomorphism $H^3(X, \mathbb{C})^* \approx H^3(X, \mathbb{C})$ given by the non-degenerate pairing Q. In particular, the holomorphic (3,0) form ε gets identified with the graviphoton field-strength $Z_{\mu\nu}$. More precisely, ε (respectively $\bar{\varepsilon}$) corresponds to the anti-self-dual (respectively self-dual) part of $Z_{\mu\nu}$. Then using the equations of motion for $T_{\mu\nu}$, we get the identifications

$$\varepsilon \leftrightarrow iZ^-_{\mu\nu} \equiv \frac{i}{2} F_I F^{I-}_{\mu\nu} - X^I G^-_{I\mu\nu},$$

$$\bar{\varepsilon} \leftrightarrow -iZ^+_{\mu\nu} \equiv -\frac{i}{2} \bar{F}_I F^{I+}_{\mu\nu} - \bar{X}^I G^+_{I\mu\nu} \tag{IV.3}$$

(cf. Eq. (3.35) of ref. [33a]). The other two $Sp(2m+2,\mathbb{R})$-invariant combinations vanish identically

$$\frac{i}{2}F_I F^{I+}_{\mu\nu} - X^I G^{+}_{I\mu\nu} \equiv 0,$$

$$-\frac{i}{2}\bar{F}_I F^{I-}_{\mu\nu} - \bar{X}^I G^{-}_{I\mu\nu} \equiv 0, \tag{IV.4}$$

as a consequence of the definition of $G_{I\mu\nu}$.

The $T_{\mu\nu}$ equations of motion associate to a scalar field configuration Z^A the field-strength $Z^{-}_{\mu\nu}$ (up to a multiple). Given the above correspondence between field-strengths and lines in $H^3(X,\mathbb{C})$, this map is nothing else but the classical period map for the family of CY treefolds X_s. It is manifest that the period map is holomorphic.

Moreover, from its explicit from—Eq. (IV.3)—we see that in this case the geometrical period map is equal to the "formal" period map associated to the corresponding 4D low-energy theory (Sect. III).

Thus, in the case of a superstring compactified on a Calabi–Yau manifold, the period map arises from geometry, i.e. it is a solution of the Schottky problem. In particular, the relationship between the period map and the Weil–Peterson metric is understood as the supersymmetry relation between the vectors' and the scalars' kinetic terms.

We emphasize that these results are exact (*non-perturbative*) even if Calabi–Yau compactifications are only *approximate* solutions to the string equations of motion [34].

In our language this can be seen as follows (for alternative arguments, see refs. [35]). CY compactifications become exact solutions in the weak coupling limit for the 2d sigma-model. This is the infinite volume limit for the internal manifold X [36]. But the volume of X is simply $V_X = \int \omega^3$, where ω is the reference Kähler form. Thus V_X depends only on the reference Kähler class. We can go to the infinite volume limit just by rescaling it, $[\omega] \to \lambda[\omega]$, $\lambda \to \infty$.

We saw in Sect. II that the moduli of the Kähler class are independent from the moduli of complex structures. The moduli of the Kähler class correspond to *hypermultiplets* not to vector-multiplets. In $N=2$ supergravity, the scalars' manifold is the direct product of the hypermultiplets' quaternionic manifold with a Kähler space parameterized by the vector scalars. The limit $\lambda \to \infty$ affects only the hypermultiplet sector and has no effect on the Kähler geometry. Thus—in type IIB strings—the field-theoretical results are *at best* approximate in the hypermultiplet sector, but they are exact, even non-perturbatively, in the vector-multiplet sector.

In the infinite volume regime, *all* Calabi–Yau spaces solve the string equations. Since the vector-multiplet kinetic terms are independent from the volume, this means that all the WP metrics arising from geometry can appear as Kähler metrics for the effective Lagrangian of some 4D type IIB superstring. For a detailed discussion see ref. [37]. We can rephrase this observation as follows. To each $N=2$ model we can associate, in a unique way, a "formal" period map. Consider the

$N=2$ sugras which arise as low-energy limits of type IIB superstrings. Then, the associated period maps *contain all the solutions to the Schottky problem.*

In this language, the Gepner conjecture [21] can be restated in a nice way:

Under the correspondence between $N=2$ supergravities and variations of Hodge structures, the solution to the Schottky problem for the algebraic 3-fold with trivial canonical bundle (and $b_1=0$) is given by the set of all $N=2$ supergravities which arise as low-energy limits of 4D, type IIB superstrings (compactified on (2,2) superconformal systems).

Two remarks are in order: i) the precise statement of the conjecture is not clear. However, the above statement requires only a rather weak form of it. It is enough that the model is continuously connected to weak coupling, or that the Yukawa couplings for the antifamilies are the "topological" ones (since we can reconstruct the period map from these couplings). This last condition was explicitly checked by Gepner in some model [38], ii) one datum, the lattice $H_{\mathbf{Z}}$, is difficult to reconstruct from supergravity arguments. However, it is likely that even this lattice can be found by a more stringy analysis. ($H_{\mathbf{Z}}$ is likely to require *quantum* arguments.)

The above considerations lead us to an explicit formula for the 4D effective Lagrangian for the IIB superstring compactified on a CY space X,

$$L=\tfrac{1}{2}[F(X^I)]_{\text{last component density}}+\text{h.c.}, \tag{IV.5}$$

where the function $F(X^I)$ is given as follows. We work with the canonical coordinates on $H^3(X,\mathbb{C})$ introduced in Sect. II. Comparing Eqs. (IV.2) with the results of Sect. II, we get the following identifications for the superconformal fields X^I and P_J:

$$X^I=\int_{a_I}\varepsilon(s),\quad P_I\equiv-\frac{i}{2}F_I(X)=\int_{b_I}\varepsilon(s), \tag{IV.6}$$

where $\varepsilon(s)$ is *any* generator of $H^{3,0}(X_s)$, $s\in S$ and (a_I,b_J) is a canonical homology basis. By homogeneity $F(X)=X^I F_I/2$. Then

$$X^I=\int_{a_I}\varepsilon(s),\quad F(X)=i\sum_{I=0}^{m}\int_{a_I}\varepsilon(s)\int_{b_I}\varepsilon(s). \tag{IV.7}$$

The conformal gauge acts in Eq. (IV.6) as a change of the generator $\varepsilon(s)$ of $H^{3,0}(X_s)$. Then from Eqs. (IV.7) it is obvious that $F(X)$ is holomorphic and homogeneous of degree 2.

We give a simple formula for the scalars' Kähler form

$$\omega_{\mathrm{WP}}(s,\bar{s})=-i\partial\bar{\partial}\ln\left[i\int_X\varepsilon(s)\wedge\bar{\varepsilon}(\bar{s})\right]. \tag{IV.8}$$

This is just Eq. (II.30). In fact in the geometrical case the Q form is just the intersection matrix and (X^I,P_J) are the coordinates of the (3,0) from $\varepsilon(s)$ in $H^{3,0}(X_s)$. In Eq. (IV.8) the map $s\to\varepsilon(s)$ is chosen holomorphic. The advantage of Eq. (IV.8) is that it is valid independently of the local complex coordinates we use on S. This should be contrasted with most of the formulae in $N=2$ SUGRA which are manageable enough only in complex coordinates adapted to the contact structure. In practical computations it may be difficult to construct these preferred coordinates. So it is convenient to have a formula which is valid for arbitrary complex coordinates.

The relation between the graviphotons and the period map is more general than $N=2$ 4D supergravity. In the 4D, $N=4$ case this can be understood also in terms of the equations of motion for the auxiliary fields $T^{ij}_{\mu\nu}$. Let us see how this connection arises in the cases relevant for type II strings. The simplest way to make contact with complex geometry is by compactifying the 10D type IIB superstring on the relevant complex manifold. The argument is analogous to that for the $N=2$ case. Since for $N>2$ there are no infinitesimal period relations (e.g. for $K3$ we have A. Todorov's surjectivity theorem for the period map [15]), we recover two well-known results:

i) In the low-energy theory the scalars' manifold is a coset space.
ii) The 2d σ-model is conformal (i.e. hyper Kähler supersymmetric σ-models are finite).

Superpotential in the Heterotic string. We have already seen that the superpotential for the E_6 antifamilies of the heterotic string (compactified on X) is given by the cubic form $\Xi(s)$,

$$\Xi(s)[\xi] = Q\left(\varepsilon(s), \xi^i\xi^j\xi^k \frac{\partial^3\varepsilon(s)}{\partial s^i\partial s^j\partial s^k}\right). \tag{IV.9}$$

In particular, if X is a complete intersection of polynomials in $P\mathbb{C}^n$ these couplings can be computed using the Grothendieck residue symbol [39] (for a physicist's review of this method see ref. [19]).

What we want to show is that the knowledge of the Yukawa couplings allows (at least in principle) to reconstruct the associated period map, i.e. the function $F(X)$. Using our canonical coordinates X^I and P_J and putting $X^0=1$ as a choice of section (conformal gauge), we have

$$\varepsilon(Z) = \left(1, Z^1, \ldots, Z^m, -\frac{i}{2}F_0, -\frac{i}{2}F_1, \ldots, -\frac{i}{2}F_m\right). \tag{IV.10}$$

Then, $(i,j,k=1,\ldots,m)$,

$$\Xi(Z)[\delta X] = -\frac{i}{2}F_{ijk}(Z)\delta X^i\delta X^j\delta X^k. \tag{IV.11}$$

Thus, once given $\Xi(s)$ we reconstruct $F(X)$ modulo a term of the form $\eta_{IJ}X^IX^J$, η_{IJ} real.

Notice that although $\otimes^2 H^{0,3}(X_s) \approx \mathbb{C}$, there is no *natural* isomorphism. In other words, the Yukawa coupling are not normalized in a natural way. However, picking out a particular isomorphism is a choice of superconformal gauge. Since the observables are gauge-independent, the particular choice of normalization is physically irrelevant. This is the good news. The bad news is that the superpotential $\Xi(s)(\delta X)$ is simply related to the function F only in the above canonical coordinates. In generic complex coordinates this relation is messy. Usually we have the Yukawa couplings in coordinates whose relation with the canonical ones is rather involved.

Thus, in many practical situations it is not easy to reconstruct the $N = 2$ effective Lagrangian from $\Xi(s)$.

V. Simple Examples of String Effective Lagrangians

Finally we apply the above discussion to the computation of the low-energy effective Lagrangian for type IIB superstrings compactified on Calabi–Yau manifolds.

First of all, we check the results of ref. [40] for the untwisted sectors of the $\mathbb{Z}_N$-orbifolds. In these cases the moduli spaces and period maps are just truncations of the well known ones for the complex tori. However, it is convenient to analyze them using the general method above. In the $\mathbb{Z}_N$-orbifold we have at most one untwisted (2,1) form $dz^1 \wedge dz^2 \wedge d\bar{z}^3$ surviving the $\mathbb{Z}_N$-projection. Calling s the moduli, the solution to the Kodaira–Spencer structure equation [31]

$$\bar{\partial}\varphi(s) = \tfrac{1}{2}[\varphi(s), \varphi(s)] \tag{V.1}$$

corresponding to the Kuranishi family is

$$\varphi(s) = s\, d\bar{z}^3 \otimes \frac{\partial}{\partial z^3}. \tag{V.2}$$

The germs of holomorphic functions $f_s(z)$ on X_s satisfy [31]

$$[\bar{\partial} - \varphi(s)] f_s = 0. \tag{V.3}$$

From Eq. (V.2) we get the following expression for the (3,0) holomorphic form:

$$\varepsilon(s) = dz^1 \wedge dz^2 \wedge dz^3 + s\, dz^1 \wedge dz^2 \wedge d\bar{z}^3. \tag{V.4}$$

We choose as the canonical homology basis the dual basis to

$$\begin{aligned} \alpha_0 &= \mathrm{Re}(dz^1 \wedge dz^2 \wedge dz^3), \quad \beta_0 = -\mathrm{Im}(dz^1 \wedge dz^2 \wedge dz^3), \\ \alpha_1 &= \mathrm{Re}(dz^1 \wedge dz^2 \wedge d\bar{z}^3), \quad \beta_1 = \mathrm{Im}(dz^1 \wedge dz^2 \wedge d\bar{z}^3). \end{aligned} \tag{V.5}$$

Then the classical period map is

$$\tilde{\Omega}(s) = (1, s, -i, is) \in \tilde{D} \subset P\mathbb{C}^3, \tag{V.6}$$

or, multiplying by X^0 and putting $X^1 = sX^0$,

$$\tilde{\Omega}(s) = (X^0, X^1, -iX^0, iX^1) \equiv \left(X^0, X^1, -\frac{i}{2}\frac{\partial F}{\partial X^0}, -\frac{i}{2}\frac{\partial F}{\partial X^1}\right) \in \tilde{D} \subset P\mathbb{C}^3 \tag{V.7}$$

with $F(X^0, X^1) = (X^0)^2 - (X^1)^2$. This is "minimal coupling" of one vector multiplet to $N = 2$ SUGRA. In this case the "natural" coordinate s is also adapted to the contact structure and so the period map has the simple canonical form of Eq. (V.7).

For the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold of ref. [40] the vector multiplets correspond to the three (2, 1) harmonic forms

$$dz^1 \wedge dz^2 \wedge d\bar{z}^3, \quad dz^1 \wedge d\bar{z}^2 \wedge dz^3, \quad d\bar{z}^1 \wedge dz^2 \wedge dz^3, \tag{V.8}$$

and then

$$\varepsilon(s^1, s^2, s^3) = (dz^1 + s^1 d\bar{z}^1) \wedge (dz^2 + s^2 d\bar{z}^2) \wedge (dz^3 + s^3 d\bar{z}^3). \tag{V.9}$$

Unfortunately these natural coordinates are not adapted to the symplectic structure. Therefore we use our formula Eq. (IV.8) for the Kähler potential which holds in arbitrary complex coordinates. Then

$$G = -\ln[-i\int \varepsilon(s) \wedge \bar{\varepsilon}(\bar{s})] = -\ln\left[\prod_{A=1}^{3}(1-|s^A|^2)\right] = -\sum_{A=1}^{3}\ln(1-|s^A|^2) \quad \text{(V.10)}$$

so the Kähler manifold is $[SU(1,1)/U(1)]^3$, in agreement with ref. [40].

The other Calabi–Yau spaces we want to discuss are the complete intersections of N polynomials $P^1, P^2, \_, P^N$ in $P\mathbb{C}^{N+3}$.

To compute the Kähler potential G we have to evaluate

$$\int_X \varepsilon(s) \wedge \bar{\varepsilon}(\bar{s}), \quad \text{(V.11)}$$

where $\varepsilon(s)$ is the (3, 0) holomorphic form written as a (holomorphic) function of the moduli parameters s^i. [The particular choice of $\varepsilon(s)$ is irrelevant as long as it is holomorphic]. We can choose as coordinates s^i some of the coefficients in the polynomials $P^\alpha(Z, s)$ defining the space.

The idea is to convert the integral over X in Eq. (V.11) into an integral over all $P\mathbb{C}^{N+3}$. We can write

$$e^{-G(s,\bar{s})} = -i \int_{P\mathbb{C}^{N+3}} \varepsilon(s) \wedge \bar{\varepsilon}(\bar{s}) \prod_{\alpha=1}^{N} \delta^{(2)}(P_s^\alpha) d^2 P_s^\alpha. \quad \text{(V.12)}$$

From the definition of $\varepsilon(s)$ as a residue, one has the identity [19]

$$\varepsilon(s) \wedge dP_s^1 \wedge \cdots \wedge dP_s^N = \text{const.}\, \varepsilon_{A_1 A_2 \cdots A_{N+4}} Z^{A_1} dZ^{A_2} \wedge \cdots \wedge dZ^{A_{N+4}}, \quad \text{(V.13)}$$

so we get

$$e^{-G(s,\bar{s})} = \text{const.} \int_{P\mathbb{C}^{N+3}} |\varepsilon_{A_1 A_2 \cdots A_{N+4}} Z^{A_1} dZ^{A_2} \wedge \cdots \wedge dZ^{A_{N+4}}|^2 \prod_{\alpha=1}^{N} \delta^{(2)}[P_s^\alpha(Z)]. \quad \text{(V.14)}$$

The explicit computation of this expression implies the integrals of higher transcendental functions, so it seems of little practical use.

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