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Eigenvalue Branches of the Schrödinger Operator $H-\lambda W$ in a Gap of $\sigma(H)$

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Abstract. The authors study the eigenvalue branches of the Schrödinger operator $H-\lambda W$ in a gap of $\sigma(H)$. In particular, they consider questions of asymptotic distribution of eigenvalues and bounds on the number of branches. They also address the completeness problem.

Introduction

Let $V(x)$, $W(x)$ be real bounded functions on $\mathbf{R}^\nu$ satisfing

(a) $$V(x) \geqq 1\,,$$

(b) $$\lim_{|x|\to\infty} W(x)=0\,.$$

Let H denote the self-adjoint operator $-\Delta+V$ on $L^2(\mathbf{R}^\nu)$.

This paper is devoted to the study of three questions concerning the eigenvalue branches of the family of Schrödinger operators $H\pm\lambda W$, in a gap of $\sigma(H)$:

(1) For $W\geqq 0$ we consider the asymptotics of the number of branches which cross an energy E in the gap and which emerge from below. To be more precise, we compute the number of branches of $H+\mu W$ which cross the level $E\in\mathbf{R}-\sigma(H)$ for $0<\mu<\lambda$, as $\lambda\to\infty$.

(2) When $W\geqq 0$ and $\operatorname{supp} W$ is contained in B_R, the ball of radius R, we prove a semi-classical phase-space type bound on the number of eigenvalue branches of the family $H+\lambda W$, $\lambda>0$, which cross a given level E in the gap. In particular, we show that the total number of such branches is finite and is bounded by the volume of the ball B_R,

$$\#\{\text{branches } E_j(\lambda) \text{ which cross } E\}\leqq C_0R^\nu\,,$$

where C_0 is independent of $W\in L^\infty(B_R)$, $W\geqq 0$, so long as $\operatorname{supp} W\subset B_R$.

(3) We address the "completeness problem" (cf. Deift and Hempel [DH]) for W which change sign; i.e., for *each* E in the gap, does there exist a $\lambda>0$ so that $E\in\sigma(H-\lambda W)$?

Problems involving eigenvalues in a spectral gap of a Schrödinger operator as above arise naturally in the investigation of impurity levels in the one-electron model of solids, and in particular in the theory of the color of crystals. We refer the reader to [BP, DH, H1, GHKSV], for example, for more information. Partial results on these questions have been obtained in [Kl, DH, H1, and GS].

One may think of questions 1–3 above in terms of the so-called generalized (or weighted) eigenvalue problem: given W and $E \notin \sigma(H)$, we seek $\lambda > 0$ and $u \in L^2(\mathbf{R}^\nu)$ so that

$$(H-E)u = \pm \lambda W u .$$

As E lies in a gap, $H-E$ is not a positive operator, and the eigenvalue problem is called "left indefinite". If, in addition, W changes sign, the problem is also called "right indefinite", and the existence and asymptotic distribution of (real) generalized eigenvalues no longer follow directly from classical methods; for further information on (left and right) indefinite problems, see [AM, DH, FL, GHKSV].

As a "folk theorem," the asymptotic distribution of eigenvalues is related to the rate of growth of certain volumes in phase space associated with the classical energy of the quantum system. The symbol of the operator H, viewed as a classical Hamiltonian, determines a region in phase space in which a classical particle with given energy is allowed to move. The uncertainty principle, however, demands that each bound state (eigenvector) requires a cube of volume $(2\pi)^\nu$ in phase space, and therefore the total number of bound states is approximately equal to this volume (see [RS, F]).

Define the eigenvalue distribution functions,

$$N_\pm(\lambda, H-E, W) := \#\{0 < \lambda_j < \lambda;\ E \in \sigma(H \mp \lambda_j W)\},$$

i.e., $N_\pm(\lambda, H-E, W)$ is the number of eigenvalue branches which cross E for $0 < \lambda_j < \lambda$ and emerge from above (respectively below). Hempel [H1, H2] has proven that for $0 \leqq W(x) \leqq c(1+|x|)^{-\alpha}$, $\alpha > 2$, the phase space volume correctly predicts the growth of $N_+(\lambda, H-E, W)$:

$$\begin{aligned}
&\lim_{\lambda\to\infty} N_+(\lambda, H-E, W)\lambda^{-\nu/2} \\
&\quad = (2\pi)^{-\nu} \lim_{\lambda\to\infty} \lambda^{-\nu/2}\,\mathrm{Vol}\{(x,p)\in\mathbf{R}^{2\nu};\ 0 < p^2 + V - E < \lambda W\} \\
&\quad = \frac{\omega_\nu}{(2\pi)^\nu} \int_{\mathbf{R}^\nu} (W(x))^{\nu/2} dx ,
\end{aligned}$$

where ω_ν is the volume of the unit ball in $\mathbf{R}^\nu$.

We will prove that if $W(x) \geqq 0$ and $W(x) \sim c|x|^{-\alpha}$ as $|x| \to \infty$ for some $c, \alpha > 0$, then

$$\lim_{\lambda\to\infty} N_-(\lambda, H-E, W)\lambda^{-\nu/\alpha} = \int_0^E d\varrho(t) \cdot \mathrm{Vol}\{y \in \mathbf{R}^\nu;\ -c|y|^{-\alpha} < t - E < 0\},$$

where $\varrho(\cdot)$ denotes the integrated density of states for H,

$$\varrho(E) := \lim_{\mathrm{Vol}(Q)\to\infty} (\mathrm{Vol}(Q))^{-1}\,\#\{\text{eigenvalues } E_j < E \text{ of } H \text{ in the cube } Q\}.$$

As we will see, this result is *not* in general in agreement with the corresponding phase space volume,

$$(2\pi)^{-\nu} \lim_{\lambda\to\infty} \lambda^{-\nu/\alpha} \operatorname{Vol}\{(x,p)\in\mathbf{R}^{2\nu};\ -\lambda W < p^2 + V - E < 0\}.$$

As

$$\int_0^E d\varrho(t)\cdot \operatorname{Vol}\{x\in\mathbf{R}^\nu;\ -c|x|^{-\alpha} < t < E < 0\}$$
$$= \lim_{\lambda\to\infty} \lambda^{-\nu/\alpha} \int_0^E d\varrho(t) \operatorname{Vol}\{x\in\mathbf{R}^\nu;\ -\lambda W(x) < t - E < 0\},$$

we see that the correct asymptotics for N_- are obtained by replacing $p^2+V\to t$ and $(2\pi)^{-\nu}dp\to d\varrho(t)$ in the phase space volume. The quantum states which contribute to N_- have bounded kinetic energy and so it is no surprise that the folk theorem fails; nevertheless, the phase space picture suggests useful bounds for related problems which can be made rigorous, as in problem (2) above, and also in [H1] where the author derives phase space bounds for $N_\pm(\lambda, H-E, W)$.

The paper is organized as follows:

In Sect. 1, we provide some notations and basic results on Birman-Schwinger kernels and on the exponential localization of eigenfunctions of $H-\lambda W$. Most propositions are stated without proof; more details may be found in [H1, H2].

In Sect. 2, we study the asymptotics of $N_-(\lambda, H-E, W)$ for $W\geqq 0$ with the prescribed asymptotic behavior $W(x)\sim c|x|^{-\alpha}$ as $|x|\to\infty$. We will also discuss briefly the situation where $W(x)$ satisfies different asymptotics as $|x|\to\infty$, for example $W(x)\sim e^{-\eta|x|}$, $\eta>0$.

Section 3 treats the case where $W\geqq 0$ is supported in a finite ball B_R. We present two entirely different approaches for obtaining the phase space estimate

$$\sup_{\lambda>0} N_-(\lambda, H-E, W) \leqq C_0 R^\nu, \tag{$*$}$$

the first based on exponential localization of eigenfunctions and the other using Dirichlet decoupling and trace estimates. The estimate ($*$) is crucial in solving the "completeness" problem of Sect. 4.

In Sect. 4 we consider $W=W_+-W_-$, $W_\pm\geqq 0$, and under mild and natural assumptions on the decay rate of W_-,

$$0\leqq W_-(x)\leqq c(1+|x|)^{-\alpha}, \qquad \alpha>2,$$

we prove that, for each E in the gap, there is indeed a $\lambda=\lambda(E)>0$ with $E\in\sigma(H-\lambda W)$. This result completes the work begun by Deift and Hempel [DH] and continued in [H1]; for a different approach to the "completeness" problem, see [GS].

1. Preliminaries

In this section, we introduce the approximating operators and present some of the theorems which we will use throughout the paper. In most cases, the proofs have been omitted, and the reader is referred to an appropriate source.

General Notation. If A is a self-adjoint operator, $\{P_\Delta(A), \Delta \text{ a Borel set}\}$ denotes its spectral decomposition.

First, let H denote the self-adjoint operator $-\Delta + V$ for $1 \leqq V \in L^\infty(\mathbf{R}^\nu)$, acting on $L^2(\mathbf{R}^\nu)$ with domain $H^2(\mathbf{R}^\nu)$. Our analysis of the operator H will rely upon comparisons with Schrödinger operators on bounded regions in $\mathbf{R}^\nu$, so we introduce:

Definition 1.1. Let $\Omega \subset \mathbf{R}^\nu$ be a domain with piecewise C^∞ boundary.

(1) The Dirichlet Laplacian, $-\Delta_\Omega^D$, acting in $L^2(\Omega)$ is the unique self-adjoint operator associated with the closure of the quadratic form $q(u, v) = \int \nabla \bar{v} \cdot \nabla u$ with domain $C_0^\infty(\Omega)$.

(2) The Neumann Laplacian, $-\Delta_\Omega^N$, acting in $L^2(\Omega)$ is the unique self-adjoint operator associated with the form $q(u, v) = \int \nabla \bar{v} \cdot \nabla u$ with domain $H^1(\Omega)$.

We also define the operators

$$H_n := -\Delta_{B_n}^D + V \tag{1.1}$$

for B_n the ball of radius $n > 0$, and note that $H_n \geqq -\Delta_{B_n}^D$.

The following estimate (see [H1, H2]) on the growth of the spectrum of $-\Delta_{B_n}^D$ is a simple consequence of Weyl's Law:

Proposition 1.2. *There exist constants $c_1, c_2, c_3 > 0$ so that*

$$c_1 n^\nu \mu^{\nu/2} - c_2 \leqq \dim P_{(-\infty, \mu)}(-\Delta_{B_n}^D) \leqq c_3 n^\nu \mu^{\nu/2} + c_2$$

for all $\mu > 0$ and $n > 0$.

The Birman-Schwinger Principle implies the following result.

Theorem 1.3. *Let T be a self-adjoint operator and $E \in \mathbf{R} - \sigma(T)$. Suppose $A \geqq 0$ is a bounded operator with $A(T-E)^{-1}$ compact. Then the Birman-Schwinger kernel $K := A^{1/2}(T-E)^{-1}A^{1/2}$ is compact and the following are equivalent:*

(1) E is an eigenvalue of $T - \lambda A$ of multiplicity m;

(2) λ^{-1} is an eigenvalue of K of multiplicity m.

Definition 1.4. (a) For K compact and $\lambda > 0$, define

$$n_+(\lambda, K) := \dim P_{(\lambda^{-1}, \infty)}(K)$$

$$n_-(\lambda, K) := \dim P_{(-\infty, -\lambda^{-1})}(K)$$

(b) Let T, A, E be as in Theorem 1.3. Then define

$$N_\pm(\lambda, T-E, A) := n_\pm(\lambda, A^{1/2}(T-E)^{-1}A^{1/2}), \qquad \lambda > 0.$$

By the Birman-Schwinger Principle, $N_\pm(\lambda, T-E, A)$ counts the number of (generalized) eigenvalues λ_i of the eigenvalue problem $(T-E)u_i = \pm \lambda_i A u_i$ which satisfy $0 < \lambda_i < \lambda$.

The advantage of introducing the Birman-Schwinger kernel in our context is that it permits the direct application of min-max methods to infer information about eigenvalues which lie in the gaps of $\sigma(H)$. For example, one may prove the following monotonicity property for $N_\pm$ (cf. [Kl, H1, H2]):

Theorem 1.5. *Let T be self-adjoint with $[E, E'] \subset \sigma(T)$ and let $A \geqq 0$ be a bounded operator with $A(T-E)^{-1}$ compact. Then for any $\lambda > 0$,*

$$N_+(\lambda, T-E, A) \leqq N_+(\lambda, T-E', A),$$
$$N_-(\lambda, T-E, A) \geqq N_-(\lambda, T-E', A).$$

The proof is a consequence of the fact that the eigenvalues of the Birman-Schwinger kernel are increasing with E,

$$\frac{\partial}{\partial E}(A^{1/2}(T-E)^{-1}A^{1/2}) = A^{1/2}(T-E)^{-2}A^{1/2} \geqq 0.$$

In addition, there is monotonicity with respect to A:

Proposition 1.6. *Let T be self-adjoint, $0 \in \varrho(T)$. Let A, B be bounded operators with AT^{-1} and BT^{-1} compact, satisfying $0 \leqq A \leqq B$, and let $\alpha_1 \geqq \alpha_2 \geqq \ldots > 0$, $\beta_1 \geqq \beta_2 \geqq \ldots > 0$ denote the positive eigenvalues of $A^{1/2}T^{-1}A^{1/2}$ and $B^{1/2}T^{-1}B^{1/2}$ respectively. Then*

$$\alpha_k \leqq \beta_k.$$

Proof. Let $A_\varepsilon := A + \varepsilon$, $B_\varepsilon := B + \varepsilon$ for $0 < \varepsilon \leqq 1$ and let $K_A(\varepsilon) := A_\varepsilon^{1/2}T^{-1}A_\varepsilon^{1/2}$ and $K_B(\varepsilon) := B_\varepsilon^{1/2}T^{-1}B_\varepsilon^{1/2}$. We denote the (min-max) eigenvalues of $K_A(\varepsilon)$ and $K_B(\varepsilon)$ by $\alpha_i(\varepsilon)$ and $\beta_i(\varepsilon)$ respectively. Now, for $\varepsilon > 0$, $A_\varepsilon^{1/2}$ (and $B_\varepsilon^{1/2}$) are continuous bijections and

$$\|A_\varepsilon^{1/2} - A^{1/2}\| \to 0$$

as $\varepsilon \to 0$ by the spectral theorem. Consequently, $\|K_A(\varepsilon) - K_A(0)\| \to 0$ as $\varepsilon \downarrow 0$, and $\alpha_i(\varepsilon) \to \alpha_i$ as $\varepsilon \downarrow 0$, for each fixed i. (Note that for $\varepsilon > 0$, K_A is no longer compact, but its spectrum in $(\gamma(\varepsilon), \infty)$ is discrete, where $\gamma(\varepsilon) > 0$ and $\gamma(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.) Similarly, $\beta_i(\varepsilon) \to \beta_i$ as $\varepsilon \downarrow 0$. Therefore, it is sufficient to show that

$$\beta_i(\varepsilon) \geqq \alpha_i(\varepsilon), \qquad 0 < \varepsilon \leqq \varepsilon_i$$

for any i fixed and some $\varepsilon_i > 0$.

By min-max, we have

$$\alpha_i(\varepsilon) = \inf_{O_{i-1}} \sup_{u \in O_{i-1}^\perp} \frac{(T^{-1}A_\varepsilon^{1/2}u, A_\varepsilon^{1/2}u)}{\|u\|^2},$$

with O_k denoting any k-dimensional subspace. Now the (non-singular) substitution

$$v := B_\varepsilon^{-1/2}A_\varepsilon^{1/2}u$$

transforms $(T^{-1}A_\varepsilon^{1/2}u, A_\varepsilon^{1/2}u)$ into $(T^{-1}B_\varepsilon^{1/2}v, B_\varepsilon^{1/2}v)$. Furthermore, the assumption $A \leqq B$ implies that $\|v\| \leqq \|u\|$; for, inserting $x = B_\varepsilon^{-1/2}y$ into $(A_\varepsilon^{1/2}y, y) \leqq (B_\varepsilon^{1/2}y, y)$, we get $\|A_\varepsilon^{1/2}B_\varepsilon^{-1/2}\| \leqq 1$, and taking adjoints we have $\|B_\varepsilon^{-1/2}A_\varepsilon^{1/2}\| \leqq 1$.

Finally, the condition $u \in O_{i-1}^\perp$ is equivalent to $v \in (B_\varepsilon^{1/2}A_\varepsilon^{-1/2}O_{i-1})^\perp$, and $B_\varepsilon^{1/2}A_\varepsilon^{-1/2}O_{i-1}$ ranges over all $(i-1)$-dimensional subspaces if O_{i-1} does.

Therefore, we obtain

$$\alpha_i \leqq \inf_{O_{i-1}} \sup_{v \in O_{i-1}^\perp} \frac{(T^{-1}B_\varepsilon^{1/2}v, B_\varepsilon^{1/2}v)}{\|v\|^2} = \beta_i(\varepsilon),$$

and we are done. $\left(\text{Note that as } \alpha_i(\varepsilon)>0, \ \sup_{v\in O_{i-1}^{\perp}} (T^{-1}A_\varepsilon^{1/2}u,\ A_\varepsilon^{1/2}u)\ \|u\|^2>0 \text{ also.}\right)$ □

Since we may replace T by $-T$ in Proposition 1.6, we have the following Corollary.

Corollary 1.7. *Under the assumptions of Theorem 1.6, we have*

$$N_\pm(\lambda, T, B) \geqq N_\pm(\lambda, T, A), \qquad \lambda>0.$$

The main technical device that we employ in this paper is to replace the operator H with approximating operators H_n acting on balls or cubes of size n, and compare their respective Birman-Schwinger kernels. If the Birman-Schwinger kernels are close enough, then the following simple lemma ([H1, H2]) assures us that the counting functions for the H_n will give a good approximation for the counting function $N_\pm(\lambda, H-E, W)$:

Lemma 1.8. *Let K and K' be compact self-adjoint operators. Let $0<\varepsilon\leqq 1$ be given, and suppose that for some $\lambda>0$ we have $\|K-K'\|<\varepsilon/2\lambda$. Then*

$$n_\pm((1+\varepsilon)\lambda, K') \geqq n_\pm(\lambda, K) \geqq n_\pm\left(\left(1-\frac{\varepsilon}{2}\right)\lambda, K'\right).$$

The essential ingredient in obtaining the bound necessary to apply Lemm 1.8 is the following statement of exponential localization for the operator H:

Proposition 1.9 (Hempel [H1, H2]). *Suppose that $M\subset \mathbf{C}$ is an open bounded set so that $\bar{M}\subset\sigma(H)$. Then there exist constants $c, \kappa>0$ so that for all $m\geqq\sqrt{\nu/2}$ and $n>m$ we have,*

$$\|\chi_m(H-z)^{-1}(1-\chi_n)\| \leqq c n^{\nu-1} e^{-\kappa(n-m)}$$

for all $z\in M$, where χ_m is the characteristic function of the ball (or cube) of radius m.

Proposition 1.9 gives exponential decay for the resolvent of H in the L^2-sense; for results on the pointwise exponential decay of the integral kernel $(H-z)^{-1}(x, y)$, see Simon [S3].

Consider the Birman-Schwinger kernels

$$K := W^{1/2}(H-E)^{-1}W^{1/2},$$
$$K_R := W_R^{1/2}(H-E)^{-1}W_R^{1/2},$$

where $W_R := W\cdot\chi_R$, and χ_R is the characteristic function of the cube $C_R := [-R, R]^\nu$. As a first application of Proposition 1.9, (see [H1, H2]) we have

Lemma 1.10. *Suppose W satisfies $0\leqq W(x)\leqq c(1+|x|)^{-\alpha}$, for constants c and $\alpha>0$. Then there exists a constant c_0 so that*

$$\|K-K_R\| \leqq c_0(1+R)^{-\alpha},$$

and there exists a $c_1>0$ so that for all $R\geqq R(\lambda,\varepsilon)=c_1(\lambda/\varepsilon)^{1/\alpha}$ we have

$$N_\pm(\lambda, H-E, W_R) \leqq N_\pm(\lambda, H-E, W) \leqq N_\pm((1+\varepsilon)\lambda, H-E, W_R).$$

(Note that the lower bound

$$N_{\pm}(\lambda, H-E, W_R) \leqq N_{\pm}(\lambda, H-E, W)$$

follows from monotonicity, Corollary 1.7.)

Finally, we introduce the localized operators $H_n^D := -\Delta_{C_n}^D + V$ and $H_n^N := -\Delta_{C_n}^N + V$ acting on $L^2(C_n)$ for the cube $C_n = [-n, n]^\nu$. Using standard truncation methods and Proposition 1.9, the resolvent kernel of H may be approximated by ther kernels of the localized operators H_n^D and H_n^N.

Proposition 1.11. *Let $b > a$ be so that $[a, b] \subset \varrho(H)$, and suppose $E \in (a, b)$. Then there exist $E_n^+ \in [E, b)$ and $E_n^- \in (a, E]$ so that $E_n^\pm \notin \sigma(H_n^D)$ and so that*

$$\|\chi_m[(H-E_n^\pm)^{-1} - (H_n^D - E_n^\pm)^{-1}]\chi_n\| < cn^{2\nu-1} e^{-\kappa(n-m)}$$

for $\sqrt{\nu/2} < m < n$, with $c, \kappa > 0$ independent of m, n. (A similar bound holds for H_n^N.)

Applying Lemma 1.8 again, the desired approximation by localized operators is achieved:

Lemma 1.12. *Let $a, b, E_n^\pm$ be as in Proposition 1.11, and let $\varepsilon > 0$ be given. Suppose W satisfies $0 \leqq W(x) \leqq c_0(1+|x|)^{-\alpha}$ for some $c_0, \alpha > 0$. Then, there exists $c_1 > 0$ so that if $R > c_1(\lambda/\varepsilon)^{1/\alpha}$ and $n = 2R$, we have*

$$N_{\pm}\left(\left(1-\frac{\varepsilon}{2}\right)\lambda, H_n^D - E_n^\mp, W_R\right) \leqq N_{\pm}(\lambda, H-E, W_R) \leqq N_{\pm}((1+\varepsilon)\lambda, H_n^D - E_n^\pm, W_R).$$

The following well-known estimate will be useful when W is compactly supported (see [S3] for a more general version):

Lemma 1.13. *Let $\Omega \subset \mathbf{R}^\nu$ be open, $U \in L^\infty(\Omega)$ a real valued function, and suppose $f \in H^2_{\mathrm{loc}}(\Omega)$ satisfies*

$$-\Delta f + Uf = 0.$$

Then for any $\psi \in C_0^\infty(\Omega; \mathbf{R}^\nu)$ we have:

$$\|\psi \cdot \nabla f\|^2 \leqq d(\psi)(1 + \|U_-|_{\operatorname{supp}\psi}\|_\infty)\, \|f|_{\operatorname{supp}\psi}\|^2,$$

where $U_- = \max(-U, 0)$ and

$$d(\psi) := \tfrac{1}{2}\|\Delta|\psi|^2)\|_\infty + \|\psi\|_\infty^2.$$

From the exponential decay of the resolvent (Proposition 1.9) and the above bound we obtain the following (technical) lemma:

Lemma 1.14. *Let $\Omega \subset \mathbf{R}^\nu$ be an open (possibly unbounded) set, and let $f \in H^2(\Omega)$ satisfy $(-\Delta + V - E)f = 0$. Furthermore, let $\varphi \in C^\infty(\Omega)$ and suppose $\operatorname{supp}\varphi \subset \Omega$ and $\Gamma := \operatorname{supp}\nabla\varphi$ is compact in Ω. Then, if K is a measurable subset of $\{x \in \Omega;\ \varphi(x) = 1\}$ we have*

$$\|\chi_K f\| \leqq \tilde{d}(\varphi)\|\chi_K(H-E)^{-1}\chi_\Gamma\| \cdot \|\chi_\Gamma f\|,$$

where $\tilde{d}(\varphi) := \|\Delta\varphi\|_\infty + 2(d(\nabla\varphi))^{1/2}(1+E)$ and $d(\varphi)$ is as in Lemma 1.13.

Finally, we shall need a restatement of exponential localization for compactly supported potentials. Choose $\varphi_1 \in C_0^\infty(B_1)$ so that $0 \leqq \varphi_1(x) \leqq 1$ and $\varphi_1(x) = 1$ for all $x \in B_{1/2}$, and define $\varphi_k(x) := \varphi_1(x/k)$. The following lemma is due to Hempel:

Lemma 1.15. *Suppose that* $[a,b] \cap \sigma(H) = \emptyset$. *Then, for* $R > 0$ *fixed, there exist constants* k_0, c, $\tilde{\kappa} > 0$ *with the following property: if* $0 \neq f \in D(H)$ *satisfies an equation*

$$(H - E)f = Uf,$$

with some $U \in L^\infty(\mathbf{R}^\nu)$, $\operatorname{supp} U \subset B_R$, *and* $E \in [a,b]$, *then we have*

$$\|(H - E - U)(\varphi_k f)\| < c e^{-\kappa k} \|\varphi_k f\|, \tag{1.2}$$

$$\|(1 - \varphi_k) f\| < c e^{-\kappa k} \|f\| \tag{1.3}$$

for $k \geqq k_0$.

Proof. We want to apply Lemma 1.14, making the identifications $\Omega := \mathbf{R}^\nu - \bar{B}_R$, $\varphi := 1 - \varphi_k$, $\Gamma := \operatorname{supp} \nabla\varphi_k \subset B_k - B_{k/2}$, and $K := \mathbf{R}^\nu - B_{2k}$. As $\operatorname{dist}(\Gamma, K) \geqq k$ and the constant $\tilde{d}(1 - \varphi_k) \leqq c_0 k^{-1}$, we obtain

$$\begin{aligned} \|f|_{\mathbf{R}^\nu - B_{2k}}\| &\leqq c_0 k^{-1} \|\chi_K (H - E)^{-1} \chi_\Gamma\| \cdot \|f\| \\ &\leqq c_1 k^{-1} (2k)^{\nu - 1} e^{-\kappa k} \|f\| \end{aligned}$$

for $k \geqq k_0$ by Proposition 1.9. As a consequence, there is a constant $c_1' > 0$ so that (letting $\tilde{\kappa} := \kappa/4$, $k_1 := 4k_0$)

$$\|f|_{\mathbf{R}^\nu - B_{k/2}}\| \leqq c_1' e^{-\tilde{\kappa} k} \|f\|, \qquad k \geqq k_1. \tag{1.4}$$

Using $(1 - \varphi_k)(H - E)f = 0$, $k > 2R$ (and applying Lemma 1.13 with $\psi := \nabla\varphi_k$) we have

$$\begin{aligned} \|(H - E)(1 - \varphi_k) f\| &\leqq 2\|\nabla\varphi_k \cdot \nabla f\| + \|f \Delta\varphi_k\| \leqq c_2 k^{-1} \|f|_{B_k - B_{k/2}}\| \\ &\leqq c_2 k^{-1} \|f|_{\mathbf{R}^\nu - B_{k/2}}\| \leqq c_3 e^{-\tilde{\kappa} k} \|f\|, \qquad k \geqq k_1 + 2R, \end{aligned} \tag{1.5}$$

by (1.4). As $\eta := \operatorname{dist}([a,b], \sigma(H)) > 0$, it follows from (1.5) that

$$\|(1 - \varphi_k) f\| \leqq \eta^{-1} (H - E)(1 - \varphi_k) f\| \leqq \eta^{-1} c_3 e^{-\tilde{\kappa} k} \|f\|,$$

for $k \geqq k_2$. Now (1.3) follows from (1.2) and the estimate

$$\begin{aligned} \|(H - U - E)(\varphi_k f)\| &\leqq \|(H - U - E) f\| + \|(H - U - E)(1 - \varphi_k) f\| \\ &\leqq \|(H - E)(1 - \varphi_k) f\|, \qquad k > 2R. \ \square \end{aligned}$$

2. Asymptotics for $N_-(\lambda, H - E, W)$

In this section, we calculate the asymptotic distribution of the negative coupling constants for non-negative potentials W with appropriate asymptotic decay properties.

The asymptotic behavior of the positive eigenvalues, $N_+(\lambda)$, has already been calculated by Hempel ([H1, H2]), in the case that $0 \leqq W(x) \leqq c(1 + |x|)^{-\alpha}$ for some $c > 0$ and $\alpha < 2$:

$$\lim_{\lambda \to \infty} N_+(\lambda, H - E, W) \lambda^{\nu/2} = \omega_\nu (2\pi)^{-\nu} \int_{\mathbf{R}^\nu} (W(x))^{\nu/2} dx,$$

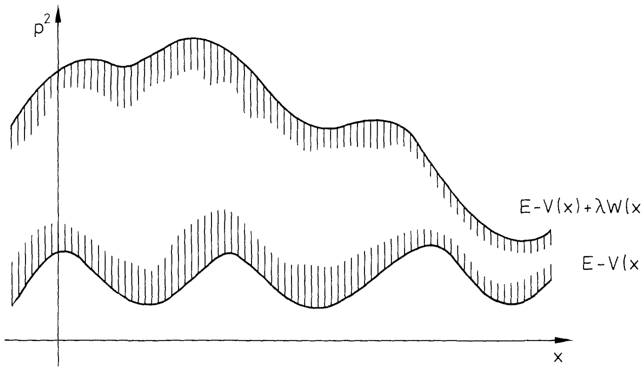

Fig. 1. The volume in phase space associated with N_+

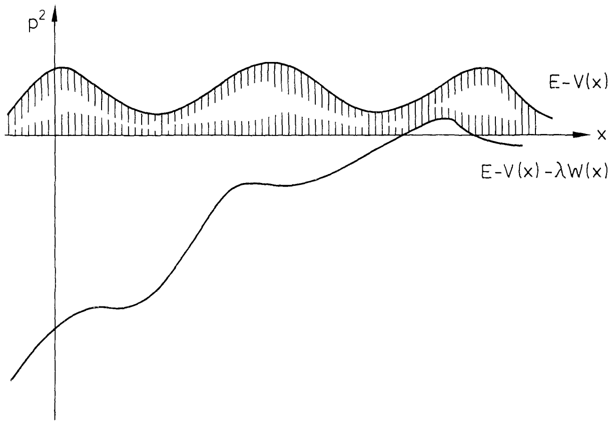

Fig. 2. The volume in phase space associated with N_-

where ω_ν is the volume of the unit ball in $\mathbf{R}^\nu$. If W satisfies $0 \leqq W(x) \leqq c(1+|x|)^{-\alpha}$ for $\alpha > \nu$, then this limit may be expressed in terms of the associated semi-classical phase space volume, (see Fig. 1):

$$N_+(\lambda) \sim (2\pi)^{-\nu} \operatorname{Vol}\{(x,p) \in \mathbf{R}^{2\nu}; 0 < p^2 + V(x) - E < \lambda W(x)\}$$

as $\lambda \to \infty$.

If one assumes that $W(x) \sim c|x|^{-\alpha}$ for some $c > 0$ and $\alpha > \nu$ (see Remark 2 below) as $|x| \to \infty$, and $V(x)$ is periodic with period module II, then the phase space volume associated with N_- is given by:

$$\begin{aligned}&\operatorname{Vol}\{(x,p) \in \mathbf{R}^{2\nu}; -\lambda W(x) < p^2 + V - E < 0\}\\ &\quad \sim \lambda^{\nu/\alpha} \operatorname{Vol}\{(x,p) \in \mathbf{R}^{2\nu}: -W(x) < p^2 + V(\lambda^{1/\alpha}x) - E < 0\}\end{aligned}$$

(see Fig. 2). Expanding the y-periodic function

$$\begin{aligned}f(x,y) &= ((E - V(y))_+)^{\nu/2} - ((E - V(y) - W(x))_+)^{\nu/2}\\ &= \sum_{k \in \Pi^*} \hat{f}(x,k) e^{2\pi i k \cdot y}\end{aligned}$$

in its Fourier series, one sees ([A]) that as $\lambda \to \infty$,

$$\operatorname{Vol}\{(x,p) \in \mathbf{R}^{2\nu};\ -W(x) < p^2 + V(\lambda^{1/\alpha}x) - E < 0\}$$
$$\to \int \left(\int_{\Pi} [((E - V(y))_+)^{\nu/2} - ((E - V(y) - W(x))_+)^{\nu/2}] dy \right) dx. \tag{2.0}$$

The folk-theorem then suggests that $\lim_{\lambda \to \infty} N_-(\lambda, H - E, W)\lambda^{-\nu/\alpha}$ exists and equals the right-hand side of (2.0). As we will see (Theorem 2.1 and calculation below) this limit does indeed exist, but is *not* equal to the above expression.

Assumption (A). *The integrated density of states for* H,

$$\varrho(E) = \lim_{\operatorname{Vol} Q \to \infty} (\operatorname{Vol}(Q))^{-1} \dim P_{(-\infty, E)}(H_Q)$$

for Q *a cube, exists independently of the boundary condition imposed on* H_Q.

This condition holds for almost periodic (and, in particular, periodic) potentials as well as for a wide class of random potentials – see [KM]. Note also that $\varrho(E)$ is a monotone function. We have:

Theorem 2.1. *Suppose* $W(x) \geqq 0$ *is a continuous function on* $\mathbf{R}^\nu$ *so that*

$$\lim_{|x| \to \infty} W(x)|x|^\alpha = c > 0,$$

for some $\alpha > 0$. *Then under assumption* (*A*) *we have:*

$$\lim_{\lambda \to \infty} N_-(\lambda, H - E, W)\lambda^{-\nu/\alpha} = \lim_{\lambda \to \infty} \lambda^{-\nu/\alpha} \iint d\varrho(t) \cdot \chi_{\{x \in \mathbf{R}^\nu;\ -\lambda W(x) < t - E < 0\}} dx$$
$$= \int_0^E d\varrho(t) \cdot \operatorname{Vol}\{y \in \mathbf{R}^\nu;\ -c|y|^{-\alpha} < t - E < 0\}.$$

Remark. A simple calculation shows that the two expressions for the limit of $N_-(\lambda, H - E, W)\lambda^{-\nu/\alpha}$ are equal, and therefore, as noted before, we see that the correct asymptotics are obtained by setting $p^2 + V(x) \to t$ and $(2\pi)^\nu dp \to d\varrho(t)$ in the classical phase space formula.

Proof. By hypothesis, given $\varepsilon > 0$ there exists R_0 so that for every $|x| \geqq R_0$ we have:

$$(1 - \varepsilon/2)c|x|^{-\alpha} \leqq W(x) \leqq (1 + \varepsilon)c|x|^{-\alpha}. \tag{2.1}$$

By Lemma 1.10, there is a c_1 so that if $R = R(\lambda) = c_1(\lambda/\varepsilon)^{1/\alpha}$, then

$$N_-((1+\varepsilon)\lambda, H - E, W_R) \geqq N_-(\lambda, H - E, W) \geqq N_-(\lambda, H - E, W_R),$$

where $W_R = W\chi_{C_R}$, and C_R denotes the cube $(-R, R)^\nu$. In what follows, we consider only λ sufficiently large so that $R(\lambda) \geqq R_0$.

Fix $\delta > 0$ with $(E - \delta, E + \delta) \subset \varrho(H)$. Applying Lemma 1.12 with $(a, b) = (E - \delta, E + \delta)$, and $n = 2R$, we localize H to the cubes C_n:

$$N_-((1+\varepsilon)\lambda, H_n^D - E_n^-, W_R) \geqq N_-(\lambda, H - E, W_R) \geqq N_-((1 - \varepsilon/2)\lambda, H_n^D - E_n^+, W_R),$$

where the E_n^+ lie in the interval $[E, E + \delta)$, and $E_n^- \in (E - \delta, E]$.

We shall first prove the following lower bound on $N_-(\lambda, H-E, W)$:

$$\liminf_{\lambda\to\infty} N_-(\lambda, H-E, W)\lambda^{-\nu/\alpha} \geqq \int_0^E d\varrho(t)\cdot \mathrm{Vol}\{y\in\mathbf{R}^\nu:\ -c|y|^{-\alpha}<t-E<0\}.$$

Now, as $(H_n^D+\lambda W_R)$ has purely discrete spectrum, its eigenvalue branches are globally defined, strictly monotonically increasing functions of λ. Thus, an eigenvalue branch of $(H_n^D+\lambda W_R)$ crosses the level E_n^+ at some $\lambda_j \leqq \lambda$ if and only if it lies *below* the level E_n^+ at $\lambda=0$ and *above* the level E_n^+ at λ. Therefore, we have

$$\begin{aligned} N_-(\lambda, H-E, W) &\geqq \dim P_{(-\infty, E_n^+]}(H_n^D) - \dim P_{(-\infty, E_n^+)}(H_n^D+(1-\varepsilon/2)\lambda W_R)\\ &\geqq \dim P_{(-\infty, E)}(H_n^D) - \dim P_{(-\infty, E+\delta)}(H_n^D+(1-\varepsilon/2)\lambda W_R). \quad (2.2)\end{aligned}$$

But, as $n=2c_1(\lambda/\varepsilon)^{1/\alpha}$,

$$\varlimsup_{\lambda\to\infty} \lambda^{-\nu/\alpha}\dim P_{(-\infty, E)}(H_n^D) = \mathrm{Vol}(C_{2c_1\varepsilon^{-1/\alpha}})\varrho(E), \quad (2.3)$$

so it remains to calculate the second term. By min-max,

$$\begin{aligned} \dim P_{(-\infty, E+\delta)}(H_n^D+(1-\varepsilon/2)\lambda W_R) &\leqq \dim P_{(-\infty, E+\delta)}(H_n^N+(1-\varepsilon/2)\lambda W_R)\\ &\leqq \dim P_{(-\infty, E+\delta)}(H_{C_n-C_R}^N)\\ &\quad + \dim P_{(-\infty, E+\delta)}(H_{C_R-C_m}^N+(1-\varepsilon/2)\lambda W_R)\\ &\quad + \dim P_{(-\infty, E+\delta)}(H_{C_m-C_{R_0}}^N+(1-\varepsilon/2)\lambda W_R)\\ &\quad + \dim P_{(-\infty, E+\delta)}(H_{R_0}^N+(1-\varepsilon/2)\lambda W_R), \quad (2.4)\end{aligned}$$

where

$$m := \nu^{-1/2}\left(\frac{(1-\varepsilon/2)^2 c\lambda}{E+\delta}\right)^{1/\alpha} < R$$

for ε sufficiently small, and $m>R_0$ for λ sufficiently large. Treating each term in (2.4) separately, we first have:

$$\dim P_{(-\infty, E+\delta)}(H_{R_0}^N+(1-\varepsilon)\lambda W_R) \leqq \dim P_{(-\infty, E+\delta)}(H_{R_0}^N) \leqq c_2 \quad (2.5)$$

for some $c_2>0$ as R_0 is a fixed constant. Also, if $x\in C_m-C_{R_0}$, then $R_0\leqq |x| \leqq \sqrt{\nu}m$ and, by (2.1), $(1-\varepsilon/2)\lambda W_R(x) \geqq E+\delta$, so:

$$\dim P_{(-\infty, E+\delta)}(H_{C_m-C_{R_0}}^N+(1-\varepsilon/2)\lambda W_R)=0, \quad (2.6)$$

The first term of the sum in (2.4) satisfies (cf. (2.3))

$$\varlimsup_{\lambda\to\infty} \lambda^{-\nu/\alpha}\dim P_{(-\infty, E+\delta)}(H_{C_n-C_R}^N) = \mathrm{Vol}(C_{2c_1e^{-1/}} - C_{c_1e^{-1/\alpha}})\varrho(E+\delta). \quad (2.7)$$

All that remains is to calculate the second term of the sum in (2.4). Let

$$p := \nu^{-1/2}\left(\frac{(1-\varepsilon/2)^2 c}{E+\delta}\right)^{1/\alpha}, \qquad q := c_1\varepsilon^{-1/\alpha} > p. \quad (2.8)$$

Given $s>0$, divide the region C_q-C_p into finitely many cubes $\{Q_j\}$ with each Q_j satisfying $\mathrm{Vol}\,Q_j \leqq s^\nu$. Denote the vertices of these cubes $\{x_k\}$. Then, $Q_j' := \lambda^{1/\alpha}Q_j$ are cubes which cover $C_R - C_m$. Denote their vertices $\{x'_k\}$. Note that $\operatorname{Vol} Q_j \to \infty$ as $\lambda \to \infty$.

For each j, let x_{k_j} be a vertex of Q_j for which $|x_{k_j}| \geqq |x|$ for all $x \in Q_j$. Then $x'_{k_j} := \lambda^{\nu/\alpha} x_{k_j}$ is a vertex of Q'_j for which $|x'_{k_j}| \geqq |x'|$ for all $x' \in Q'_j$. By Neumann bracketing,

$$\begin{aligned}
&\overline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} \dim P_{(-\infty, E+\delta)}(H^N_{C_R - C_m} + (1-\varepsilon/2)\lambda W_R) \\
&\leqq \overline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} \dim P_{(-\infty, E+\delta)}(H^N_{C_R - C_m} + (1-\varepsilon/2)^2 \lambda c |x|^{-\alpha}) \\
&\leqq \overline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} \sum_j \dim P_{(-\infty, E+\delta)}(H^N_{Q'_j} + (1-\varepsilon/2)^2 \lambda c |x'_{k_j}|^{-\alpha}) \\
&= \sum_j \operatorname{Vol} Q_j\ \overline{\lim_{\lambda\to\infty}}\ (\operatorname{Vol} Q'_j)^{-1} \dim P_{(-\infty, E+\delta-(1-\varepsilon/2)^2 c|x_{k_j}|^{-\alpha})}(H^N_{Q'_j}) \\
&= \sum_j \operatorname{Vol} Q_j \cdot \varrho(E+\delta-(1-\varepsilon/2)^2 c |x_{k_j}|^{-\alpha}) .
\end{aligned}$$

Taking $s \to 0$, (recall that $\varrho(\cdot)$ is monotone), we obtain:

$$\begin{aligned}
&\overline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} \dim P_{(-\infty, E+\delta)}(H^N_{C_R - C_m} + (1-\varepsilon/2)\lambda W_R) \\
&\leqq \int_{C_q - C_p} \varrho(E+\delta-(1-\varepsilon/2)^2 c|x|^{-\alpha}) dx . \qquad (2.9)
\end{aligned}$$

So, applying (2.3), (2.5), (2.6), (2.7), and (2.9) to (2.2) and (2.4), we obtain:

$$\begin{aligned}
\underline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} N_-(\lambda, H-E, W) \geqq{}& \operatorname{Vol}(C_{2q}) \cdot \varrho(E) - \operatorname{Vol}(C_{2q} - C_q) \cdot \varrho(E+\delta) \\
&- \int_{C_q - C_p} \varrho(E+\delta-(1-\varepsilon/2)^2 c|x|^{-\alpha}) dx ,
\end{aligned}$$

and taking first $\delta \to 0$,

$$\begin{aligned}
\underline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} N_-(\lambda, H-E, W) \geqq{}& \operatorname{Vol}(C_q) \cdot \varrho(E) - \int_{C_q - C_p} \varrho(E-(1-\varepsilon/2)^2 c|x|^{-\alpha}) dx \\
={}& \int_{C_q} (\varrho(E) - \varrho(E-(1-\varepsilon/2)^2 c|x|^{-\alpha}) dx
\end{aligned}$$

(note that $(1-\varepsilon/2)^2 c|x|^{-\alpha} > E$ if $|x| < p$), and then if $\varepsilon \to 0$, we obtain:

$$\underline{\lim_{\lambda\to\infty}}\ \lambda^{-\nu/\alpha} N_-(\lambda, H-E, W) \geqq \int_{\mathbf{R}^\nu} (\varrho(E) - \varrho(E - c|x|^{-\alpha})) dx$$

and the form of the limit in the statement of the theorem may be obtained by changing the order of integration.

The proof of the upper bound

$$\limsup_{\lambda\to\infty} N_-(\lambda, H-E, W) \lambda^{-\nu/\alpha} \leqq \int_0^E d\varrho(t) \cdot \operatorname{Vol}\{y \in \mathbf{R}^\nu;\ -c|y|^{-\alpha} < t - E < 0\}$$

uses Dirichlet bracketing instead of Neumann bracketing, but is otherwise identical, and is left to the reader. $\square$

Remarks. 1. The condition on the asymptotic behavior of $W(x)$ may be weakened somewhat to allow for angular dependence. Without significantly changing the above proof, the condition $W(x)|x|^{-\alpha}\to c$ may be replaced by:

$$\lim_{t\to\infty} t^{\alpha}W(t\xi)=c(\xi)>0$$

uniformly for $|\xi|=1$. In this case,

$$\lim_{\lambda\to\infty} N_{-}(\lambda, H-E, W)\lambda^{-\nu/\alpha}=\int_{0}^{E} d\varrho(t)\cdot \mathrm{Vol}\{y\in\mathbf{R}^{\nu};\ -c(y/|y|)\,|y|^{-\alpha}<t-E<0\}.$$

2. Unlike the N_{+} result, where the decay rate α must satisfy $\alpha>2$, the above theorem for N_{-} holds for all $\alpha>0$. In addition, note that for the phase space volume to exist, the integrability condition $\alpha>\nu$ must be imposed. Thus, the asymptotic formulae for both N_{+} and N_{-} hold even when the phase space volume is not finite for finite values of λ.

3. The number of negative eigenvalues is (to first order) unaffected by the behavior of $W(x)$ on compact sets, as only the asymptotic form of W appears in the limiting expression.

4. Furthermore, we note that for $\alpha>2$, the number of negative eigenvalue grows more slowly than the number of positive eigenvalues. To understand this, recall that $\lambda_j<\lambda$ is counted in $N_{\pm}(\lambda)$ if $E\in\sigma(H\mp\lambda_j W)$, so one is counting how many eigenvalue branches of $H\mp\lambda W$ cross the level E. When we speak of positive λ, we are counting branches *pulled down* from higher energy bands by an attractive potential $-\lambda W$; for the negative λ, the branches are being *pushed up* from lower energy bands by a repulsive potential λW. But there should be "more" eigen-states in the (infinitely many) bands above E than there are in the (finitely many) bands below E, and hence it is not surprising that N_{+} grows faster than N_{-}.

5. Bounds on $N_{\pm}(\lambda)$ as well as the asymptotics for $N_{+}(\lambda)$ were proven in the one-dimensional case in [DH], and in $\nu>1$ dimensions in [H1, H2].

Now, we consider a one-dimensional example and show that the asymptotic limit of $N_{-}(\lambda, H-E, W)$ is not in agreement with the phase space volume. Define

$$A(E):=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\sqrt{(E-V(\lambda^{1/\alpha}x))_{+}}-\sqrt{(E-V(\lambda^{1/\alpha}x)-g(x))_{+}}\right]dx,$$

where $g(x):=|x|^{-\alpha}$, $\alpha>2$, (here $c=1$), and compare $A(E)$ with the actual leading order term for $N_{-}(\lambda)$ which we computed above,

$$B(E):=\int_{-\infty}^{\infty}\int_{0}^{E}\chi_{\{-g(x)<s-E<0\}}d\varrho(s)dx.$$

As noted in the introduction, using Fourier analysis $A(E)$ may be evaluated as

$$A(E)=\frac{1}{\pi}\int_{0}^{\infty}\int_{0}^{1}\left[\sqrt{(E-V(y))_{+}}-\sqrt{(E-V(y)-g(x))_{+}}\right]dy\,dx. \tag{2.10}$$

Consider the following periodic potential on $\mathbf{R}$:

$$V(x)=\begin{cases}0, & \text{for}\quad k<x\leqq k+\frac{1}{2},\ k\in\mathbf{N}\\ 1, & \text{for}\quad k+\frac{1}{2}<x\leqq k+1,\ k\in\mathbf{N}.\end{cases}$$

and let $H=-d^2/dx^2+V$ on $L^2(-\infty,\infty)$. It is well known that all the gaps in $\sigma(H)$ for this potential (Kronig-Penny model) are all open. Let $[E_0, E_1]$ be the lowest band in $\sigma(H)$, and suppose that $E>E_1$ lies in the first spectral gap. By Floquet theory, E_0 is the first periodic eigenvalue and E_1 the first anti-periodic eigenvalue. Since $E_0(H)\leqq E_0(-d^2/dx^2+1)=1$ and $E_1(H)\geqq E_1(-d^2/dx^2)=\pi^2$, we have $1\in[E_0,E_1]$ and $E>1=\|V\|_\infty$.

Formula (2.10) for $A(E)$ may be rewritten as:

$$\begin{aligned}A(E)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\left(\int_0^1(s-V(y))^{-1/2}\chi_{\{s>V(y)\}}dy\right)\chi_{\{-g(x)<s-E<0\}}ds\,dx\\&=\int_{-\infty}^{\infty}\int_0^{\infty}\chi_{\{-g(x)<s-E<0\}}dh(s)dx\end{aligned}\tag{2.11}$$

for $dh(s)=\frac{1}{2\pi}\left[\int_0^1(s-V(y))^{-1/2}\chi_{\{s>V(y)\}}dy\right]ds$. By direct calculation, we have

$$h(s)=\begin{cases}\frac{1}{2\pi}\sqrt{s}\,, & \text{for}\quad 0<s<1\\ \frac{1}{2\pi}(\sqrt{s}+\sqrt{s-1}), & \text{for}\quad 1<s.\end{cases}\tag{2.12}$$

$\left(\text{Note that } h(s)\sim\frac{\sqrt{s}}{\pi}\sim\varrho(s)\text{ for } s\gg 1.\right)$ Applying (2.12) to (2.11), we have:

$$\begin{aligned}A(E)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\chi_{\{E>s>E-g(x)\}}\frac{ds}{2\sqrt{s}}+\int_1^{\infty}\chi_{\{E>s>E-g(x)\}}\frac{ds}{2\sqrt{s-1}}\right]dx\\&=\frac{1}{2\pi}\int_0^E\left(\int_{-\infty}^{\infty}\chi_{\{|x|<(E-s)^{-1/\alpha}\}}dx\right)\frac{ds}{2\sqrt{s}}\\&\quad+\frac{1}{2\pi}\int_1^E\left(\int_{-\infty}^{\infty}\chi_{\{|x|<(E-s)^{-1/\alpha}\}}dx\right)\frac{ds}{2\sqrt{s-1}}\\&=\frac{1}{\pi}\int_0^E(E-s)^{-1/\alpha}\frac{ds}{2\sqrt{s}}+\frac{1}{\pi}\int_1^E(E-s)^{-1/\alpha}\frac{ds}{2\sqrt{s-1}}.\end{aligned}$$

Now the first term above,

$$\int_0^E(E-s)^{-1/\alpha}\frac{ds}{2\sqrt{s}}=E^{\frac{1}{2}-\frac{1}{\alpha}}\int_0^1(1-t)^{-1/\alpha}\frac{dt}{2\sqrt{t}}$$

is continuously differentiable in $E>0$ for all $\alpha>2$. Fixing γ with $1<\gamma<E_1$ we have that $\int_1^\gamma(E-s)^{-1/\alpha}(s-1)^{-1/2}ds$ is continuously differentiable for E near E_1 and

$$\int_\gamma^E(E-s)^{-1/\alpha}\frac{ds}{2\sqrt{s-1}}=E^{\frac{1}{2}-\frac{1}{\alpha}}\int_{\gamma/E}^1(1-t)^{-1/\alpha}\frac{dt}{2\sqrt{Et-1}}$$

is also C^1 near $E=E_1$. In particular, $A(E)$ is a C^1 function for E near E_1.

Now, consider $B(E)$. If $\Delta(\lambda)$ is the Hill's discriminant for $H-\lambda$, then for $s\in\sigma(H)$,

$$\frac{d\varrho}{ds}=\frac{1}{\pi}\frac{\dot{\Delta}(s)}{\sqrt{4-\Delta(s)^2}}$$

(see e.g. [M].) As $\varrho(s)=0$ in the gap (E_1,E),

$$B(E)=\int_{-\infty}^{\infty}\int_{0}^{E_1}\chi_{\{s>E-g(x)\}}d\varrho(s)dx=2\int_{E_0}^{E_1}(E-s)^{-1/\alpha}d\varrho(s).$$

For $E>E_1$ we differentiate to obtain:

$$\begin{aligned}\frac{dB}{dE}&=-\frac{2}{\alpha}\int_{E_0}^{E_1}(E-s)^{-1-1/\alpha}d\varrho(s)\\&=-\frac{2}{\pi\alpha}\int_{E_0}^{E_1}\frac{\dot{\Delta}(s)}{\sqrt{4-\Delta(s)^2}}(E-s)^{-1-1/\alpha}ds\\&\times\xrightarrow{E\downarrow E_1}-\frac{2}{\pi\alpha}\int_{E_0}^{E_1}\frac{\dot{\Delta}}{\sqrt{4-\Delta(s)^2}}(E_1-s)^{-1-1/\alpha}ds,\end{aligned}$$

by monotone convergence. But, $\dot{\Delta}(E_1)\neq 0$, as all gaps are open, so this integral is clearly infinite, so $B(E)$ is *not* C^1 near E_1, and thus $A(E)\neq B(E)$ for all E lying in the gap.

Finally, we remark that if $W(x)\sim ce^{-\eta|x|}$, then the asymptotic formula as $\lambda\to\infty$,

$$N_-(\lambda,H-E,W)\sim\int\left(\int d\varrho(t)\cdot\chi_{\{-\lambda W<t-E<0\}}\right)dx\sim\frac{\omega_\nu}{\eta^\nu}\varrho(E)(\log\lambda)^\nu,$$

where ω_ν is the volume of the unit ball in $\mathbf{R}^\nu$, is again valid. The proof (see [A]) follows the proof of Theorem 4.1 but with some modifications depending on whether $\eta\geqq\kappa$ or $\eta<\kappa$, where κ appears in Proposition 1.9 (κ is essentially the exponential decay rate of the Green's function $(H-E)^{-1}(x,y)$ as $|x-y|\to\infty$). It is an open conjecture that

$$N_-(\lambda,H-E,W)\sim\int\left(\int d\varrho(t)\cdot\chi_{\{-\lambda W<t-E<0\}}\right)dx$$

for all bounded $W(x)>0$, $W(x)\to 0$ as $|x|\to\infty$.

3. An Upper Bound on $N_-(\lambda)$ for W of Compact Support

The aim of this section is to prove the estimate

$$\sup_{\lambda>0}N_-(\lambda,H-E,W)\leqq cR^\nu,$$

provided $\operatorname{supp}W\subset B_R$, with a constant c independent of W and R. This result is suggested by the phase space formula: if $\operatorname{supp}W\subset B_R$, then

$$\iint\chi_{\{-\lambda W<p^2+V(x)-E<0\}}dp\,dx=\int d\varrho\int_{|x|\leqq R}\chi_{\{-\lambda W<p^2+V(x)-E<0\}}dx\leqq\text{const.}R^\nu.$$

The above estimate will also be of crucial importance in solving the completeness problem in Sect. 4.

We shall present two rather different proofs of this bound: the first proof is a refinement of the approach used in [H1] where a weaker result was obtained, while the second uses ideas from [DS] on decoupling via Dirichlet boundary conditions.

For later purposes, note that as $V \in L^\infty$, Proposition 1.2 provides the estimate

$$\dim P_{(-\infty, E+1)}(H_k) \leqq C_E k^\nu, \qquad k \geqq 1, \tag{3.1}$$

for some constant C_E.

Theorem 3.1. *Let* $1 \leqq V \in L^\infty(\mathbf{R}^\nu)$, $H := -\Delta + V$, *and* $E \in \mathbf{R} - \sigma(H)$. *Let* C_E *be as in* (3.1) *and* $C_0 := 2 \cdot 3^\nu \cdot C_E$. *Then there is a constant* R_0 *so that for all* $R \geqq R_0$,

$$N_-(\lambda, H-E, \chi_R) \leqq C_0 R^\nu. \tag{3.2}$$

Together with Corollary 1.7, this gives:

Corollary 3.2. *Suppose* $W(x) \geqq 0$ *is bounded with* $\operatorname{supp} W \subset B_R$. *Then*

$$N_-(\lambda, H-E, W) \leqq C_0 R^\nu, \qquad R \geqq R_0.$$

Proof of Theorem 3.1. Choose $a, b \in \mathbf{R}$ so that $a < E < b$ and $[a,b] \cap \sigma(H) = \emptyset$. Let $N_- = \sup\limits_{\lambda > 0} N_-(\lambda, H-E, \chi_R)$ and $N_R = \min(N_-, 2C_0 R^\nu)$. Denote by $\{e_j(\lambda)\}$ the eigenvalue branches of $(H + \lambda\chi_R)$ which cross the level E for some $\lambda > 0$. To be precise, for each branch e_j there exists an interval $I_j := [\alpha_j, \gamma_j]$ so that e_j is defined and continuous for $\lambda \in I_j$ and so that $e_j(\alpha_j) = a$ and $e_j(\gamma_j) = E$. In addition, we order the e_j so that $\gamma_1 \leqq \gamma_2 \leqq \dots$. We also define $\beta = \frac{1}{2}(E-a)$ and

$$E_i = E - \beta \sum_{l=1}^{i} l^{-2}, \qquad i = 1, 2, \dots. \tag{3.3}$$

Note that E_i is monotone decreasing and $E_i > a$ for all i.

Step 1. We organize the branches $e_1, \dots, e_{N_R}$ into disjoint collections.

First, define $\lambda_0 := \gamma_1$,

$$S_0 := \{e_j : \lambda_0 \in I_j,\ E_1 \leqq e_j(\lambda_0) \leqq E,\ 1 \leqq j \leqq N_R\}$$

and $n_0 := \# S_0$. Now assume that $\lambda_0 < \dots < \lambda_{i-1}$, $S_0, \dots, S_{i-1}$, and $n_0, \dots, n_{i-1}$ have already been chosen. Let λ_i be the smallest $\lambda > \lambda_{i-1}$ so that $e_j(\lambda) = E_i$ for some $j \leqq N_R$. Let

$$S_i := \{e_j : \lambda_i \in I_j,\ E_{i+1} \leqq e_j(\lambda_i) \leqq E_i,\ 1 \leqq j \leqq N_R\}$$

and $n_i := \# S_i$. Clearly, after a finite number of steps, the above process will terminate, and monotonicity assures us that the S_i are disjoint. In fact, if we denote the number of the sets S_i by $J(R)$, we have

$$J(R) \leqq N_R \leqq 2C_0 R^\nu$$

and, as each branch $e_j(\cdot)$ meets each E_i at some $\lambda > 0$, the S_i must necessarily exhaust the branches $e_1, \dots, e_{N_R}$, i.e.,

$$N_R = \sum_{i=0}^{J(R)-1} b_i. \tag{3.4}$$

Step 2. We now consider an approximation of our problem, where H is replaced by H_k, for a suitable k (see (1.1)):

Let $\kappa>0$ be given by Lemma 1.15, $k:=3R>k_0$, $\varepsilon:=e^{-\kappa R}$. Letting $h_j^{(k)}$ denote the eigenvalue branches of the family $H_k+\lambda\chi_R$, $\lambda>0$, we define

$$S_i^{(k)}:=\{h_j^{(k)};\ E_{i+1}-\varepsilon<h_j^{(k)}(\lambda_i)<E_i+\varepsilon\},$$
$$d_i^{(k)}:=\#S_i^{(k)}, \qquad i=0,\dots,J(R)-1.$$

Clearly,

$$d_i^{(k)}=\dim P_{(E_{i+1}-\varepsilon,E_i+\varepsilon)}(H_k+\lambda_i\chi_R). \tag{3.5}$$

Consider $\{S_{2i}^{(k)}\}$. As $E_{2i+1}-E_{2i+2}=\beta(2i+2)^{-2}\geqq c'R^{-2\nu}$, for suitable c' and $R\geqq 1$, we can find $R_1>0$ so that

$$E_{2i+1}-E_{2i+2}\geqq 2\varepsilon=2e^{-\kappa R}, \qquad R\geqq R_1,$$

for each $2i\leqq J(R)-1$; as a consequence, none of the intervals $(E_{2i+1}-\varepsilon, E_{2i+2}+\varepsilon)$ intersect for $2i\leqq J(R)-1$. Therefore, by monotonicity, all branches $h_j^{(k)}$ in $\cup S_{2i}^{(k)}$ must be distinct, and hence:

$$\begin{aligned}\sum_{2i\leqq J(R)-1}\#S_{2i}^{(k)}=\sum_{2i\leqq J(R)-1}d_{2i}^{(k)}&\leqq\#\{h_j^{(k)};\ h_j^{(k)}(0)\leqq E+1\}\\ &\leqq\dim P_{(-\infty,E+1)}(H_k)\leqq 3^\nu C_E R^\nu\end{aligned} \tag{3.6a}$$

for $R\geqq R_1$, by (3.1). Similarly,

$$\sum_{2i+1\leqq J(R)-1}\#S_{2i+1}^{(k)}\leqq 3^\nu C_E R^\nu. \tag{3.6b}$$

In Step 3 below, we shall show that there exists $R_2>0$ so that (recall $k=3R$)

$$n_i\leqq d_i^{(k)}, \qquad i=0,\dots,J(R)-1, \qquad R\geqq R_2. \tag{3.7}$$

From (3.4), and (3.7) we obtain (with $R_0:=\max\{R_1,R_2\}$),

$$N_R=\sum_{0\leqq i\leqq J(R)-1}n_i\leqq\sum_{0\leqq i\leqq J(R)-1}d_i^{(k)}\leqq 2\cdot 3^\nu C_E R^\nu$$

for $R\geqq R_0$. As $N_R=\min\{N_2, 4\cdot 3^\nu C_E R^\nu\}$, it follows that $N_-\leqq 2\cdot 3^\nu C_E R^\nu$, $R\geqq R_0$, and we are finished.

Step 3. Suppose the statement of (3.7) were not true. Then, there exists a sequence of values of R tending to infinity, for which (3.7) is violated. We denote this sequence by $\mathscr{R}$. Then for $R\in\mathscr{R}$, we may find $0\leqq i(R)\leqq J(R)-1$ and $u_1,\dots,u_{d_i+1}$ (here, and in the sequel, we write $i=i(R)$, $d_i:=d_i^{(k)}$, $k=3R$) which are orthonormal eigenfunctions for $(H+\lambda_i\chi_R)$ with eigenvalues $e_1^i,\dots,e_{d_i+1}^i$ satisfying $E_{i+1}\leqq e_j^i\leqq E_i$, for $j=1,\dots,d_i+1$. Let $\varphi\in C_0^\infty(B_1)$ be so that $\varphi(x)=1$ for $x\in B_{1/2}$ and $0\leqq\varphi(x)\leqq 1$; define $\varphi_k(x):=\varphi(x/k)$. Consider the truncated functions, $\{\varphi_k u_j\}$. We show first that there is an $R_3>0$ so that the functions

$$\{\varphi_k u_j;\ j=1,\dots,d_i+1\}$$

are linearly independent, provided $R\geqq R_3$, $R\in\mathscr{R}$, Suppose $\sum_{j=1}^{d_i+1} a_j u_j \varphi_k = 0$ with a_j not all zero. Without loss, assume that $|a_j| \leqq a_1 = 1$. Then, taking the scalar product with u_1, we find for $R \geqq k_0/3$, $R \in \mathscr{R}$,

$$|(u_1, \varphi_k u_1)| \leqq \sum_{j=2}^{d_i+1} |a_j| \cdot |(\varphi_k u_1, u_j)| = \sum_{j=2}^{d_i+1} |a_j| \cdot |((1-\varphi_k)u_1, u_j)|$$
$$\leqq \sum_{j=2}^{d_i+1} \|(1-\varphi_k)u_1\| \leqq d_i c_1 e^{-3\kappa R} \leqq c_2 R^{\nu} e^{-3\kappa R}, \tag{3.8}$$

where we have used $u_1 \perp u_j$, the estimate (1.3) and

$$d_i + 1 := d_i^{(3R)} + 1 \leqq c_2 R^{\nu}. \tag{3.9}$$

But, by (1.3) again,

$$|(u_1, \varphi_k u_1)| = \int |u_1|^2 \varphi_k \geqq 1 - \|(1-\varphi_k)u_1\| \geqq 1 - c_1 e^{-3\kappa R}. \tag{3.10}$$

But clearly Eqs. (3.8) and (3.10) are incompatible for large values of R in the set $\mathscr{R}$. Thus, it must follows that, for some R_4, $\{u_j\varphi_k\}_{j=1,\dots,d_i+1}$ are independent, for $R \geqq R_4$, $R \in \mathscr{R}$.

Now, as the $\{u_j \varphi_k\}$ span a (d_i+1) dimensional space, it follows that there is a

$$v = \sum_{j=1}^{d_i+1} b_j u_j \varphi_k \neq 0,$$

which is perpendicular to $\operatorname{Ran} P_{(E_{i+1}-\varepsilon, E_i+\varepsilon)}(H_k + \lambda_i \chi_R)$. Let

$$\bar{E} = \tfrac{1}{2}(E_i + E_{i+1}), \qquad \delta = \tfrac{1}{2}(E_i - E_{i+1}).$$

By the spectral theorem and the choice of v we obtain (assuming $R \geqq R_4$, $R \in \mathscr{R}$),

$$\|(H_k + \lambda_i \chi_R - \bar{E})v\|^2 > (\delta + \varepsilon)^2 \|v\|^2. \tag{3.11}$$

On the other hand, applying (1.2) and (1.3), we have:

$$\begin{aligned}
&\|(H_k + \lambda_i \chi_R - \bar{E})v\| \\
&= \left\| \sum_{j=1}^{d_i+1} (H_k + \lambda_i \chi_R - e_j^i)(b_j u_j \varphi_k) + \sum_{j=1}^{d_i+1} (e_j^i - \bar{E}) b_j u_j \varphi_k) \right\| \\
&\leqq \sum_{j=1}^{d_i+1} |b_j| \, \|(H_k + \lambda_i \chi_R - e_j^i)(u_j \varphi_k)\| + \left\| \sum_{j=1}^{d_i+1} (e_j^i - \bar{E}) b_j u_j \right\| \\
&\quad + \left\| \sum_{j=1}^{d_i+1} (e_j^i - \bar{E}) b_j u_j (1-\varphi_k) \right\| \leqq \sum_{j=1}^{d_i+1} |b_j| \, \|(H + \lambda_i \chi_R - e_j^i)(u_j \varphi_k)\| \\
&\quad + \left(\sum_{j=1}^{d_i+1} (e_j^i - \bar{E})^2 |b_j|^2 \right)^{1/2} + \sum_{j=1}^{d_i+1} |b_j| \delta \|(1-\varphi_k)u_j\| \\
&\leqq \left(\sum_{j=1}^{d_i+1} |b_j| \right) c_1 e^{-\kappa k} + \delta \left(\sum_{j=1}^{d_i+1} |b_j|^2 \right)^{1/2} + \delta \left(\sum_{j=1}^{d_i+1} |b_j| \right) c_1 e^{-\kappa k} \\
&\leqq \left(\sum_{j=1}^{d_i+1} |b_j|^2 \right)^{1/2} [c_3 R^{\nu/2} e^{-3\kappa R} + \delta],
\end{aligned} \tag{3.12}$$

as $\sum_{j=1}^{d_i+1} |b_j| \leqq [(d_i+1) \sum |b_j|^2]^{1/2} \leqq c_2 R^{\nu/2} [\sum |b_j|^2]^{1/2}$ by (3.9).

Now we provide a bound for $\sum_{j=1}^{d_\iota+1} |b_j|^2$: We have

$$\begin{aligned}
\sum_{j=1}^{d_\iota+1} |b_j|^2 &= \sum_{j,l}^{d_\iota+1} \bar{b}_l b_j (u_l, u_j) \\
&= \sum_{l,j} \bar{b}_l b_j [((1-\varphi_k)u_l, u_j) + (\varphi_k u_l, (1-\varphi_k)u_j) + (\varphi_k u_l, \varphi_k u_j)] \\
&= \|v\|^2 + \sum_{j,l} \bar{b}_l b_j [((1-\varphi_k)u_l, u_j) + (\varphi_k u_l, (1-\varphi_k)u_j)] \\
&\leqq \|v\|^2 + 2c_1 e^{-\kappa k} \sum_{l,j}^{d_i+1} |b_l b_j| \\
&\leqq \|v\|^2 + 2c_1 R^\nu e^{-3\kappa R} \left(\sum_{j=1}^{d_\iota+1} |b_j|^2 \right).
\end{aligned}$$

Thus,

$$\sum_{j=1}^{d_i+1} |b_j|^2 \leqq \|v\|^2 (1 - 2c_1 R^\nu e^{-3\kappa R})^{-1},$$

or,

$$\left(\sum_{j=1}^{d_\iota+1} |b_j|^2 \right)^{1/2} \leqq \|v\| \, (1 + 4c_1 R^\nu e^{-3\kappa R}) \tag{3.13}$$

for $R \geqq R_5$, $R \in \mathscr{R}$ chosen sufficiently large that $2c_1 R_5^\nu e^{-3\kappa R} \leqq \frac{1}{2}$. Applying (3.13) to (3.12) yields:

$$\|(H_k + \lambda_i \chi_R - \bar{E})v\| \leqq \|v\| \, (\delta + c_4 R^\nu e^{-3\kappa R}) \leqq \|v\| \, (\delta + c_4 e^{-2\kappa R}) \tag{3.14}$$

for $R \geqq R_6 \geqq R_5$, $R \in \mathscr{R}$, with R_6 chosen large enough that $R_6^\nu e^{-3\kappa R_6} \leqq e^{-2\kappa R_6}$.

From (3.11) and (3.14) we obtain

$$\|v\| \, (\delta + e^{-\kappa R}) \leqq \|v\| \, (\delta + c_5 e^{-2\kappa R}),$$

which is incompatible for large R. The proof of Theorem 3.1 is now complete. □

We now present an entirely different approach for estimating $N_-(\lambda, H-E, \chi_R)$ using Dirichlet decoupling in the spirit of [DS]. We will obtain an estimate of the form (3.2), but with a different constant C_0, and for all $R \geqq 1$. Here we shall think of $\lambda \chi_R$, $\lambda \to \infty$ as a potential barrier whose repulsive effect is less than that of a Dirichlet boundary condition on ∂B_R in the sense that

$$H + \lambda \chi_R \leqq -\Delta^D_{\mathbf{R}^\nu - \bar{B}_R} + V, \qquad \lambda > 0.$$

We will need some definitions: write $H_R := -\Delta^D_{B_R} + V|_{B_R}$, and $H_{R,\infty} := -\Delta^D_{\mathbf{R}^\nu - \bar{B}_R} + V|_{(\mathbf{R}^\nu - \bar{B}_R)}$ so that $H_R \oplus H_{R,\infty}$ is just $-\Delta + V$ with a Dirichlet boundary condition on ∂B_R, and

$$H \leqq H_R \oplus H_{R,\infty}.$$

We define

$$J_R := H^{-1} - (H_R \oplus H_{R,\infty})^{-1}, \qquad \hat{J}_R := \mathrm{J}_R + H_R^{-1},$$

so that, in particular,

$$H^{-1} = H_{R,\infty} + \hat{J}_R,$$

and

$$0 \leqq J_R \leqq \hat{J}_R \leqq H^{-1}.$$

For $1 \leqq p < \infty$ let $\mathscr{B}_p$ be the p^{th} Schatten ideal, i.e., $\mathscr{B}_p$ is the class of all compact operators K for which $\sum \mu_j^p < \infty$, where the μ_j are the eigenvalues of the operator $|K| = (K^*K)^{1/2}$. (See, e.g. [S2].) The norm on $\mathscr{B}_p$ is given by

$$\|K\|_{\mathscr{B}_p} := (\sum \mu_j^p)^{1/p}.$$

The following properites of $\hat{J}_R$ are basic to our approach:

Lemma 3.3. *For $p > \nu/2$ we have $\hat{J}_R \in \mathscr{B}_p$ and there is a constant c_p, independent of R, so that*

$$\operatorname{trace}(\hat{J}_R^p) = \|\hat{J}_R\|_{\mathscr{B}_p}^p \leqq c_p R^\nu, \qquad R \geqq 1.$$

We defer the proof of this lemma to the end of the second and proceed to the second proof of Theorem 3.1.

Proof of Theorem 3.1.

Step 1. Instead of considering H and $H_R \oplus H_{R,\infty}$ directly, we pass to their inverses: defining

$$\begin{aligned} K_R(\mu) &:= H^{-1} - (H + \mu\chi_R)^{-1}, \qquad \mu > 0, \\ B_R(\mu) &:= K_R(\mu)^{1/2}(H^{-1} - E^{-1})^{-1} K_R(\mu)^{1/2} \end{aligned} \tag{3.15}$$

(so that $K_R(\mu)$ and $B_R(\mu)$ are compact; note that $(H^{-1} - E^{-1})^{-1}$ is bounded) we shall show in this step that

$$N_-(\lambda, H - E, \chi_R) \leqq n_+(1; B_R(\lambda)), \qquad \lambda > 0. \tag{3.16}$$

We first observe as before that the eigenvalue branches of $H + \mu\chi_R$ are *strictly* monotonically increasing. We also note that the operators $K_R(\mu)$ depend monotonically on μ; we have

$$0 \leqq K_R(\mu) \leqq K_R(\mu') \leqq \hat{J}_R, \qquad 0 \leqq \mu \leqq \mu' \tag{3.17}$$

as $H + \mu\chi_R \leqq H + \mu'\chi_R \leqq H_{R,\infty}$ for $0 \leqq \mu \leqq \mu'$.

Now let $0 < \lambda_1 \leqq \lambda_2 < \dots$ denote the coupling constants where the branches of $H + \mu\chi_R$, $\mu > 0$ cross the level E. Suppose that m eigenvalue branches cross E at some $\bar{\mu} \in \{\lambda_j\}$. Then there are m eigenvalue branches of $H^{-1} - K_R(\mu) = (H + \mu\chi_R)^{-1}$ which cross the level E^{-1} at $\mu = \bar{\mu}$, i.e., E^{-1} is an eigenvalue of multiplicity m of $H^{-1} - K_R(\bar{\mu})$, and by the Birman-Schwinger Principle (and (3.15)), it follows that 1 is an eigenvalue of $B_R(\bar{\mu})$ of multiplicity m. Conversely, if 1 is an eigenvalue of $B_R(\mu')$ for some $\mu' > 0$ then $E \in \sigma(H + \mu'\chi_R)$ and has the same multiplicity. From (3.17) and Proposition 1.6 we conclude that the eigenvalue branches of $B_R(\mu)$ are non-decreasing functions of $\mu > 0$. Furthermore, they cannot be locally constant $= 1$, as the eigenvalue branches of $H + \mu\chi_R$ are *strictly* monotonically increasing.

As a consequence, at each $\mu=\lambda_j$ a (non-decreasing) eigenvalue branch of $B_R(\mu)$ crosses (strictly) the level 1, and we therefore see that

$$\#\{\lambda_j<\lambda\}\leqq n_+(1, B_R(\lambda)),$$

and (3.16) follows.

Step 2. By (3.17) we have $0\leqq K_R(\lambda)\leqq \hat{J}_R$, $\lambda>0$, and Proposition 1.6, applied to the Birman-Schwinger kernel $B_R(\lambda)$ in Eq. (3.16) yields

$$\begin{aligned} N_-(\lambda, H-E, \chi_R) &\leqq n_+(1, \hat{J}_R^{1/2}(H^{-1}-E^{-1})^{-1}\hat{J}_R^{1/2}) \\ &\leqq \|\hat{J}_R^{1/2}(H^{-1}-E^{-1})^{-1}\hat{J}_R^{1/2}\|_{\mathscr{B}_q}^q. \end{aligned} \tag{3.18}$$

Note that the right-hand side of (3.18) is independent of λ. Now we fix some $p\in\mathbf{N}$, $p>\nu/2$ and put $q:=2p$. As $\|AB\|_{\mathscr{B}_q}\leqq\|A\|_{\mathscr{B}_q}\cdot\|B\|$, for any $A\in\mathscr{B}_q$, and B bounded (see [S2]) we obtain from (3.18)

$$N_-(\lambda; H-E, \chi_R)\leqq \|\hat{J}_R^p\|_{\mathscr{B}_1}[\|(H^{-1}-E^{-1})^{-1}\|^q\cdot\|\hat{J}_R\|^p].$$

Since $\|(H^{-1}-E^{-1})^{-1}\|$ is independent of R and $\|\hat{J}_R\|\leqq\|H^{-1}\|\leqq 1$, there exists a constant c_1 independent of λ and R such that

$$N_-(\lambda, H-E, \chi_R)\leqq c_1\cdot\operatorname{trace}(\hat{J}_R^p)\leqq c_2 R^\nu$$

by Lemma 3.3 and we are done. $\square$

Proof of Lemma 3.4.

Step 1. Let $\tilde{V}:=V-1\geqq 0$, $\tilde{H}:=-\Delta+\tilde{V}$, $-\Delta':=-\Delta^D_{\mathbf{R}_\nu-\partial B_R}$, and $\tilde{H}':=-\Delta'+\tilde{V}$. Then, the integral kernels of the semi-groups $e^{-t\tilde{H}}$ and $e^{-\tilde{H}'}$ satisfy the estimate

$$0\leqq e^{-t\tilde{H}}(x,y)-e^{-t\tilde{H}'}(x,y)\leqq(2\pi t)^{-\nu/2}e^{-[(|x|-R)^2+(|y|-R)^2]/4t} \tag{3.19}$$

for $x, y\notin B_R$ and $t>0$. This inequality is proven in [S1] for the case $\tilde{V}=0$; however, by the Feynman-Kac formula, (3.19) still holds true if we include $\tilde{V}\geqq 0$.

Step 2. Applying the Laplace transform, we obtain

$$0\leqq J_R(x,y)=\int_0^\infty e^{-t}[e^{-t\tilde{H}}(x,y)-e^{-t\tilde{H}'}(x,y)]\,dt,$$

so that

$$J_R(x,y)\leqq c_1 e^{-[|x|+|y|-2R]}, \qquad x, y\notin B_{R+1}$$

with constant c_1 independent of R. As $H_R^{-1}(x,y)=0$ for $x\notin B_{R+1}$ or $y\notin B_{R+1}$ it follows that

$$\hat{J}_R(x,y)\leqq c_1 e^{-[|x|+|y|-2R]}, \qquad x, y\notin B_{R+1}.$$

On the other hand, for all x, y we have

$$0\leqq\hat{J}_R(x,y)\leqq(-\Delta+1)^{-1}(x,y),$$

so that

$$0\leqq\hat{J}_R^p(x,y)\leqq(-\Delta+1)^{-p}(x,y), \qquad x, y\in\mathbf{R}^\nu$$

for any positive integer p. Using the representation

$$(-\Delta+1)^{-p}(x,y)=c_2\int_0^\infty e^{-t}e^{-|x-y|^2/4t}t^{p-1-\nu/2}dt,$$

one easily shows that

$$\hat{J}_R^p(x,y)\leqq\begin{cases}c_3|x-y|^{-(\nu-2p)}e^{-\eta|x-y|}, & \text{for}\quad \nu>2p\\ c_4(1+|\log|x-y||)e^{-\eta|x-y|}, & \text{for}\quad \nu=2p\\ c_5e^{-\eta|x-y|}, & \text{for}\quad \nu<2p\end{cases}$$

for suitable constants c_3, c_4, c_5, and $\eta>0$.

Choosing $0<\eta'\leqq\eta$ we finally obtain

$$0\leqq\hat{J}_R(x,y)\leqq ce^{-\eta'[|x|+|y|-2R]},\qquad x,y\notin B_{R+1},\tag{3.20}$$

$$0\leqq\hat{J}_R(x,y)\leqq c|x-y|^{-\alpha}e^{-\eta'|x-y|},\qquad \forall x,y\in\mathbf{E}^\nu,\tag{3.21}$$

where $\alpha=\alpha(p)>0$ for $\nu\geqq 2p$ and $\alpha=\alpha(p)=0$ for $\nu<2p$. (Note that for $\nu=2p$ any $\alpha=\alpha(\nu/2)>0$ will do.)

Step 3. Here we show that for $R\geqq 1$ and any integer $p\geqq 1$,

$$0\leqq\hat{J}_R^p\leqq c(p)e^{-\eta'(|x|+|y|-2R)}R^\nu,\qquad x,y\notin B_{R+p}.$$

This is true for $p=1$. Assume by induction that the result is true for p. Then for $x,y\notin B_{R+p+1}$ we have

$$\begin{aligned}\hat{J}_R^{p+1}(x,y)&\leqq\int_{|z|<R+p}\hat{J}_R^p(x,z)\hat{J}_R(z,y)dz+\int_{|z|\geqq R+p}\hat{J}_R^p(x,z)\hat{J}_R(z,y)dz\\ &\leqq c_6\int_{|z|<R+p}\frac{e^{-\eta'(|x|+|y|-2R-2p)}}{|x-z|^{\alpha(p)}|y-z|^{\alpha(p)}}dz+c_7\int_{|z|\geqq R+p}e^{-\eta'(|x|+|y|+2|z|-4R)}dz\\ &\leqq c_8e^{-\eta'(|x|+|y|-2R)}R^\nu,\end{aligned}$$

where we have used (3.21) for the integral over $|z|<R+p$ and (3.20) and the induction hypothesis for the integral over $|z|\geqq R+p$. This completes the induction.

Step 4. The estimates in Step 3 imply that for $p>\nu/2$, $\hat{J}_R^p$ has a (continuous) kernel satisfying

$$0\leqq\hat{J}_R^p(x,y)\leqq\begin{cases}ce^{-\eta|x-y|}, & \text{for}\quad x,y\in\mathbf{R}^\nu\\ ce^{-\eta(|x|+|y|-2R)}R^\nu, & \text{for}\quad x,y\notin B_{R+p}.\end{cases}$$

As $\hat{J}_R^p$ is positive as an operator, we conclude that it is trace class and satisfies the estimate $\operatorname{trace}(\hat{J}_R^p)\leqq c_pR^\nu$, for $R\geqq 1$. This completes the proof of the Lemma. □

We do not provide *lower* bounds for $N_-(\lambda)$, but we wish to mention some results concerning the extreme cases, where $\operatorname{supp}W$ is either very small or very large.

If the support of W is very small, the phase space volume calculation suggests that there may be *no* negative eigenvalues at all: $N_-(\lambda)=0$ for all $\lambda>0$. This is true if the dimension is at least two:

Theorem 3.4 (Hempel [H1]). *Let $\nu \geqq 2$, $1 < V \in L^\infty(\mathbf{R}^\nu)$, and $H = -\Delta + V$ on $L^2(\mathbf{R}^\nu)$. Suppose $E \in \mathbf{R} - \sigma(H)$. Then there is a $\delta > 0$ independent of W so that*

$$N_-(\lambda, H-E, W) = 0$$

for all non-negative $W \in L^\infty(\mathbf{R}^\nu)$ with support in B_δ.

Combining the above theorem with Theorem 3.1 we have

Corollary 3.5. *Let $\nu \geqq 2$, $1 \leqq V \in L^\infty(\mathbf{R}^\nu)$, and $H = -\Delta + V$ on $L^2(\mathbf{R}^\nu)$. Suppose $E \in \mathbf{R} - \sigma(H)$. Then, there is a $c_1 > 0$ so that*

$$N_-(\lambda, H-E, W) \leqq c_1 R^\nu$$

for all non-negative $W \in L^\infty(\mathbf{R}^\nu)$ with support in B_R, $R \geqq 0$.

Remark. In dimension $\nu = 1$, the theorem above does not in general hold. The question of whether $\sup_{\lambda > 0} N_-(\lambda) > 0$ or not depends crucially upon the *location* of $\operatorname{supp} W$ in relation to the zeros of the Green's function $(H-E)^{-1}(x,x)$. For more details, see [H1; Theorem 8.1 and 8.2].

Remark. Theorem 3.1 depends critically on the fact that E lies in a gap of $\sigma(H)$. For example if $E > 0$ and $H = -\Delta_{B_n}^D$, then (3.2) cannot hold for any constant c_0; for details see [A], and also [Ki].

Conversely, one might expect that if $\operatorname{supp} W$ is "large enough," that there will be infinitely many negative eigenvalues. In fact, one has:

Proposition 3.6 (Alama[A]). *Suppose $V \in L^\infty(\mathbf{R}^\nu)$ is periodic, and $H = -\Delta + V$. Suppose $E \in \mathbf{R} - \sigma(H)$, with $E > \mu = \inf \sigma(H)$, and W a continuous function which satisfies $W(x) > 0$ for all $x \in \mathbf{R}^\nu$, and $W(x) \to 0$ as $|x| \to \infty$. Then, $\sup_\lambda N_-(\lambda) = \infty$.*

4. Completeness in $\mathbf{R}^\nu$

Up to this point, the assumption $W(x) \geqq 0$ was fundamental for our investigations. We shall now use results and methods of the preceding sections to study the eigenvalue problem

$$(H-E)u = \lambda W u,$$

where

$$W = W_+ - W_-, \qquad W_\pm \geqq 0,$$

restricting our efforts to the problem of completeness, as formulated in [DH]:

Question. For any given $E \in \mathbf{R} - \sigma(H)$, does there exist a real λ such that $E \in \sigma(H - \lambda W)$?

To be more precise, we say the triple (H, W, S), where S is a subset of $\mathbf{R}$, is *complete* if for any $E \in \mathbf{R} - \sigma(H)$ there exists a $\lambda \in S$ so that $E \in \sigma(H - \lambda W)$.

So we merely ask if there exists at least *one* eigenvalue branch of $H - \lambda W$ which crosses the level E. The paper of Deift and Hempel [DH] is entirely devoted to this question; the basic ν-dimensional result in [DH], Theorem 1, asserts under rather general conditions that the triple $(H, W, \mathbf{R}_+)$ is *essentially complete*, i.e., the eigenvalue branches of $H-\lambda W$, $\mu>0$ cover the spectral gaps of H with the possible exception of one level E_0 per gap, which is dubbed an "exceptional level." While [DH] has no results on completeness in $\mathbf{R}^\nu$, $\nu \geqq 2$, the paper contains three theorems on completeness in the ODE case. Subsequent progress was made by Hempel [H1] when $W_- = \min(-W, 0)$ has compact support, and by Gesztesy and Simon [GS], in the case where W has compact support.

Our main result reads as follows.

Theorem 4.1. *Let* $1 \leqq V \in L^\infty(\mathbf{R}^\nu)$ *and* $H = -\Delta + V$. *Suppose* $W \in L^\infty(\mathbf{R}^\nu)$ *satisfies:*

(1) $W(x) \to 0$ *as* $|x| \to \infty$;

(2) *There exist constants* $c>0$ *and* $\alpha>2$ *so that*

$$0 \leqq W_-(x) \leqq c(1+|x|)^{-\alpha}. \tag{4.1}$$

(3) *There exists* ϱ, $\eta>0$ *so that* $W(x) \geqq \eta$ *for all* $x \in B_\varrho$.

Then $(H, W, \mathbf{R}_+)$ *is complete.*

Our proof uses the original strategy of [DH] to consider approximating problems

$$(\tilde{H}_n - E) f_n = \lambda_n W f_n, \tag{4.2}$$

where the operators $\tilde{H}_n$ act in $L^2(B_n)$ and have a spectral gap around E. Following rather closely the proof of [H1: Theorem 9.1], we show that we can find solutions of (4.2) with $0<\lambda_n \leqq \text{const}$. Then, letting n tend to infinity, we arrive at a solution of $(H-E)u = \lambda(E) W u$.

In order to solve (4.2) with $0<\lambda_n \leqq \Lambda_0$, we simply show that $\tilde{H}_n - \Lambda_0 W$ has more eigenvalues below E than $\tilde{H}_n$, for some Λ_0 independent of n. Here, the main difficulty arises from the competition between the potential well $-\lambda W_+$ and the potential barrier λW_-, as λ increases, and we have to employ the estimate (3.2) to control the repulsive effect of the barrier created by λW_-.

For the proof of Theorem 4.1 we need several definitions and lemmas which we present first, postponing their proof to the end of this section.

As always, we assume $1 \leqq V \in L^\infty(\mathbf{R}^\nu)$, $H = -\Delta + V$ and $E \in \mathbf{R} - \sigma(H)$, in the sequel. Also, we fix numbers $a' < a < b < b'$ so that $[a', b'] \subset \varrho(H)$ and $E \in (a, b)$. For $n \geqq 1$, let again $H_n = -\Delta_n^D + V|_{B_n}$ and let

$$\Pi_n := P_{(a', b')}(H_n) \tag{4.3}$$

the projection on the subspace spanned by the eigenfunctions of H_n associated with the eigenvalues in the interval (a', b'). Let $\varphi \in C_0^\infty(B_{5/6})$ so that $0 \leqq \varphi(x) \leqq 1$ for all x, and $\varphi(x) = 1$ for $x \in B_{1/2}$; define $\varphi_n \in C_0^\infty(B_n)$ by

$$\varphi_n(x) := \varphi(x/n).$$

Let $\psi_n(x) := 1 - \varphi_n(x)$ and consider the operators

$$\tilde{H}_n := H_n + c_0 \psi_n \Pi_n \psi_n, \tag{4.4}$$

where $c_0 := b' - a'$; compare the slightly different definition of $\hat{H} = H_n + c\Pi_n$ in [DH]. Clearly, $\tilde{H}_n$ is self-adjoint on $D(\tilde{H}_n) = D(H_n)$, and $\tilde{H}_n \geqq H_n$. The basic spectral properties of $\tilde{H}_n$ are a consequence of the following two lemmas which exploit the fact that eigenfunctions of H_n associated with eigenvalues in (a', b') are exponentially localized near ∂B_n.

Lemma 4.2. *Let Π_n be defined as in* (4.3). *Then there exist constants $\tilde{\kappa} > 0$ and $n_0 \in \mathbf{N}$ so that*

$$\|\Pi_n \varphi_n\| \leqq e^{-\tilde{\kappa} n}, \qquad n \geqq n_0 .$$

Lemma 4.3. *There is an n_1 so that*

$$\sigma(\tilde{H}_n) \cap [a', b'] = \emptyset$$

for all $n \geqq n_1$.

Remark. The point of Lemma 4.3 is that we have an (essentially) n-independent spectral gap of $\tilde{H}_n$ around E. In contrast to the operators $\hat{H}_n$ used in [DH], the non-local part $c_0 \psi_n \Pi_n \psi_n$ of $\tilde{H}_n$ is now restricted to $B_n - B_{n/2}$; this fact will be crucial later on, as Dirichlet-Neumann bracketing is applicable to *local* operators only.

We need yet another pair of operators with Dirichlet boundary conditions: For $0 < R < n/2$, let $H_{R,n} := -\Delta^D_{B_n - B_R} + V|_{(B_n - B_R)}$, and

$$\tilde{H}_{R,n} := H_{R,n} + c_0 \psi_n \Pi_n \psi_n . \tag{4.5}$$

Now, since $\psi_n \Pi_n \psi_n|_{L^2(B_{n/2})} = 0$, and $H_n \leqq H_R \oplus H_{R,n}$, it follows that

$$\tilde{H}_n \leqq H_R \oplus H_{R,n} + c_0 \psi_n \Pi_n \psi_n = H_R \oplus \tilde{H}_{R,n} \tag{4.6}$$

for $0 < R < n/2$; this direct decomposition would not have been possible with the operators $\hat{H}_n$ in [DH]. Writing

$$M_n := \dim P_{(-\infty, E)}(\tilde{H}_n) , \tag{4.7}$$

$$M_{R,n} := \dim P_{(-\infty, E_1)}(\tilde{H}_{R,n}) , \tag{4.8}$$

where $E_1 := (a + E)/2$, we have the following estimate.

Lemma 4.4. *Let R_0 be as in Theorem 3.1, M_n, $M_{R,n}$ as in* (4.7), (4.8). *Then for any $R \geqq R_0$, there exists $n(R) \geqq 2R$ such that*

$$M_{R,n} \geqq M_n - k_0 R^\nu , \qquad n \geqq n(R) ,$$

with a constant k_0 independent of R and n.

Remark. Lemma 4.4 says that taking the ball B_R out of B_n (and introducing an additional Dirichlet boundary condition on ∂B_R) will shift at most a finite number (less that $k_0 R^\nu$) of eigenvalues of $\tilde{H}_n$ beyond the level E_1. Although this result is very intuitive, its proof is the hardest step in obtaining Theorem 4.1.

Proof of Theorem 4.1. Step 1. In this step, we solve the approximating problems (4.2) with uniformly bounded coupling constants $\lambda_n > 0$, for n large.

By Eq. (4.6), we have for $n > 2R$,

$$\tilde{H}_n - \lambda W \leqq (H_R - \lambda W|_{B_R}) \oplus (\tilde{H}_{R,n} - \lambda W|_{B_n - B_R}) .$$

Discarding the annular region $B_R - B_\varrho$, we first estimate $H_R - \lambda W|B_R \leqq H_\varrho - \lambda\eta$, for $\lambda > 0$, as $W|B_\varrho \geqq \eta$ by hypothesis (3) of Theorem 4.1 (without restriction, we assume

$R \geqq \max(R_0, \varrho)$ in the sequel.) Next, on the region $B_n - B_R$ we use the bound (4.1) on W_- to the effect that

$$\tilde{H}_{R,n} - \lambda W|_{B_n - B_R} \leqq \tilde{H}_{R,n} + \lambda c(1+R)^{-\alpha}.$$

Now we tie R to $\lambda > 0$ by setting

$$R = R(\lambda) := \max[(c\lambda/(E - E_1))^{1/\alpha}, R_0]$$

(recall that $E_1 := (a+E)/2$); in particular, we have

$$\tilde{H}_{R,n} - \lambda W|_{B_n - B_R} \leqq \tilde{H}_{R,n} + E - E_1 .$$

Consequently, we obtain for $\lambda > 0$,

$$\tilde{H}_n - \lambda W|_{B_n} \leqq (H_\varrho - \lambda\eta) \oplus (\tilde{H}_{R,n} + E - E_1)$$

for $n \geqq 2R(\lambda)$, whence

$$\dim P_{(-\infty, E)}(\tilde{H}_n - \lambda W|_{B_n}) \geqq \dim P_{(-\infty, E)}(H_\varrho - \lambda\eta) + \dim P_{(-\infty, E_1)}(\tilde{H}_{R,n}),$$

for $\lambda > 0$, $R = R(\lambda)$, and $n > 2R$. By Proposition 1.2, we can find constants $c_1, c_2 > 0$ so that

$$\dim P_{(-\infty, E)}(H_\varrho - \lambda\eta) \geqq \dim P_{(-\infty, E - \|V\|_\infty + \lambda\eta)}(-\Delta_\varrho^D) \geqq c_1 \lambda^{\nu/2} - c_2 .$$

By Lemma 4.4, we have a constant k_0 such that

$$\dim P_{(-\infty, E_1)}(\tilde{H}_{R,n}) \geqq M_n - k_0 R^\nu, \qquad n > n(R) \geqq 2R,$$

so that

$$\dim P_{(-\infty, E)}(\tilde{H}_n - \lambda W|_{B_n}) \geqq M_n - k_0 R^\nu + c_1 \lambda^{\nu/2} - c_2$$

for $R = R(\lambda)$ and $n > n(R) \geqq 2R$. As $R \sim \lambda^{1/\alpha}$, with $\alpha > 2$, it is clear that we can find a $\Lambda_0 > 0$ so that

$$c_1 \Lambda_0^{\nu/2} - c_2 - k_0 R(\Lambda_0)^\nu \geqq 1,$$

whence

$$\dim P_{(-\infty, E)}(\tilde{H}_n - \Lambda_0 W|_{B_n}) \geqq \dim P_{(-\infty, E)}(\tilde{H}_n) + 1$$

for $n \geqq n_0 := n(R(\Lambda_0))$. Therefore, regular perturbation theory implies that an eigenvalue branch of the family $\tilde{H}_n - \mu W|_{B_n}$, $\mu > 0$ must have crossed the level E at some $\mu = \lambda_n \in (0, \Lambda_0]$ for $n \geqq n_0$. In other words, for $n \geqq n_0$ there exist $\lambda_n \in (0, \Lambda_0]$ and $f_n \in D(\tilde{H}_n)$, $\|f_n\| = 1$, such that (4.2) holds.

Step 2. (Convergence step). As $0 < \lambda_n \leqq \Lambda_0$, we may suppose that

$$\lambda_n \to \lambda_E, \qquad n \to \infty$$

for some $\lambda_E \geqq 0$. Furthermore, as $\tilde{H}_n \geqq H_n \geqq -\Delta_n^D$, we have

$$\|\nabla f_n\|^2 \leqq (\tilde{H}_n f_n, f_n) \leqq E + \Lambda_0 \|W\|_\infty .$$

Extending f_n by zero outside B_n it follows by Rellich's compactness theorem that there exists $f \in H^1(\mathbf{R}^\nu)$ so that (a subsequence of) $f_n \to f$ weakly in $H^1(\mathbf{R}^\nu)$ and strongly in L^2_{loc}. As W decays and $\lambda_n \to \lambda_E$, we see that

$$\|\lambda_n W f_n - \lambda_E W f\| \to 0 .$$

As $(a,b)\cap\sigma(\tilde{H}_n)=\emptyset$, $n\geqq n_1$, by Lemma 4.3 and $E\in(a,b)$, we have a constant $\gamma>0$ so that, for $n\geqq n_1$,

$$\|\lambda_n Wf_n\|=\|(\tilde{H}_n-E)f_n\|\geqq \operatorname{dist}(E,\sigma(\tilde{H}_n))\geqq\gamma,$$

whence $\|\lambda_E Wf\|=\lim_{n\to\infty}\|\lambda_n Wf\|>0$, and it follows that $f\neq 0$ and $\lambda_E\neq 0$. To conclude the proof, let $g\in C_0^\infty(\mathbf{R}^\nu)$ and $r>0$ with $\operatorname{supp} g\subset B_r$. Then for $n>2r$ we have $\tilde{H}_n g=Hg$, and hence, for $n>2r$,

$$\begin{aligned}0&=((\tilde{H}_n-E-\lambda_n W)f_n,g)=(f_n,(\tilde{H}_n-E)g)-(\lambda_n Wf_n,g)\\&=(f_n,(H-E)g)-(\lambda_n Wf_n,g),\end{aligned}$$

so that

$$(f,(H-E-\lambda_E W)g)=0\,,\qquad g\in C_0^\infty(\mathbf{R}^\nu)$$

as $f_n\to f$ weakly and $\lambda_n Wf_n\to\lambda_E Wf$ strongly. By the essential self-adjointness of $H|_{C_0^\infty(\mathbf{R}^\nu)}$, it is clear that $f\in D(H)$ and $(H-E)f=\lambda_E Wf$ and we are done.□

It remains to prove Lemmas 4.2–4.4.

Proof of Lemma 4.2. Let u_{ni}, $i=1,\ldots,i_n$, denote a complete set of (normalized) eigenfunctions of H_n associated with eigenvalues E_{ni} in the interval (a',b'). By Proposition 1.2 and using $V\geqq 1$, we see that $i_n\leqq c_1 n^\nu$. Defining a sequence of cut-off functions $\zeta_n\in C_0^\infty(B_n)$ by

$$\zeta_n:=j_{1/2}*\chi_{n-1}\,,$$

(where j_ε, $\varepsilon>0$ denotes the standard Friedrichs mollifier and χ_n is the characteristic function of B_n), and applying Lemma 1.14 (with $\Gamma_n:=\operatorname{supp}\nabla\zeta_n\subset B_n-B_{n-2}$), we obtain

$$\begin{aligned}\|\chi_{5n/6}u_{ni}\|&\leqq\tilde{d}(\zeta_n)\|\chi_{5n/6}(H-E_{ni})^{-1}\chi_{\Gamma_n}\|\leqq c\|\chi_{5n/6}(H-E_{ni})^{-1}(1-\chi_{n-2})\|\\&\leqq c' n^{\nu-1}e^{-\kappa(n-2-5n/6)}\leqq c'' e^{-\tilde{\kappa}n},\qquad n\geqq n_0\end{aligned}$$

by Proposition 1.9 (here we have also used that $\tilde{d}(\zeta_n)\leqq c$, and that $[a',b']\cap\sigma(H)=\emptyset$, so that the estimate in Proposition 1.9 is uniform for $E_{ni}\in(a',b')$).

Now, as

$$\|\Pi_n\varphi_n f\|=\left\|\sum_{i=1}^{i_n}(u_{ni}\varphi_n,f)u_{ni}\right\|\leqq\sum_{i=1}^{i_n}\|u_{ni}\varphi_n\|\,\|f\|\leqq\sum_{i=1}^{i_n}\|u_{ni}\chi_{5n/6}\|\,\|f\|,$$

for $f\in L^2(\mathbf{R}^\nu)$, we obtain

$$\|\Pi_n\varphi_n\|\leqq i_n c'' e^{-\tilde{\kappa}n}\leqq c_1 n^\nu c'' e^{-\tilde{\kappa}n},$$

for $n\geqq n_0$, and the result follows. □

Proof of Lemma 4.3. From the definition of Π_n it is immediate that $H_n+(b'-a')\Pi_n$ has no eigenvalues in the interval (a',b'). Expanding

$$\Pi_n=\varphi_n\Pi_n+\psi_n\Pi_n\varphi_n+\psi_n\Pi_n\psi_n\,,$$

we obtain

$$\|\Pi_n-\psi_n\Pi_n\psi_n\|\leqq 2\|\Pi_n\varphi_n\|\leqq c' e^{-\tilde{\kappa}n}$$

for $n \geqq n_0$ by the preceding Lemma 4.2. Therefore, the distance between the spectra of $\tilde{H}_n = H_n + (b' - a')\psi_n \Pi_n \psi_n$ and $H_n + (b' - a')\Pi_n$ cannot exceed $(b' - a')c'e^{-\bar{\kappa}n}$. As $H_n + (b' - a')\Pi_n$ has no spectrum in (a, b), and $[a, b] \subset (a', b')$, there exists some n_1 such that $\tilde{H}_n$ has no spectrum in (a, b) for $n \geqq n_1$, and the lemma is proven. □

In order to reduce Lemma 4.4 to Theorem 3.1, we employ two different types of approximation: first, we use the fact that the eigenvalues of $\tilde{H}_n + \mu\chi_R$ converge (from below) to the eigenvalues of $\tilde{H}_{R,n}$, as $\mu \to \infty$. Second, we obtain information on the spectrum of $\tilde{H}_n + \mu\chi_R$ in (a, b) by comparison with the operators $H + \mu\chi_R$, which have been studied in Sect. 3; this will be the object of Lemma 4.5.

Proof of Lemma 4.4. Recall that $E_1 = (a + E)/2$ and let $E_2 := (a + E_1)/2$, $E_3 := (a + E_2)/2$. We approximate the operators $\tilde{H}_{R,n}$ by $\tilde{H}_n + \mu\chi_R$, $\mu \to \infty$: as $\tilde{H}_n + \mu\chi_R$ converge to $\tilde{H}_{R,n}$ in norm resolvent sense (see e.g., [Ka2, Ka1; Chap. 8, Theorem 3.5]), we have

$$M_{R,n} = \dim P_{(-\infty, E_1)}(\tilde{H}_{R,n}) \geqq \lim_{\mu \to \infty} \dim P_{(-\infty, E_2)}(\tilde{H}_n + \mu\chi_R). \tag{4.9}$$

By regular perturbation theory, the eigenvalues of $\tilde{H}_n + \mu\chi_R$, $\mu > 0$, form smooth branches which are strictly increasing functions of μ. (We note that in a gap of $\sigma(\tilde{H}_n)$ all eigenvalue branches of $\tilde{H}_n + \mu\chi_R$ have positive derivative.)

Therefore, whenever a branch crosses the level E_3, the number of eigenvalues below E_3 is diminished by 1, so we have

$$N_-(\mu, \tilde{H}_n - E_3, \chi_R) = \dim P_{(-\infty, E_3)}(\tilde{H}_n) - \dim P_{(-\infty, E_3]}(\tilde{H}_n + \mu\chi_R), \tag{4.10}$$

for $\mu > 0$. As

$$\dim P_{(-\infty, E_3)}(\tilde{H}_n) = \dim P_{(-\infty, E)}(\tilde{H}_n) = M_n,$$

we see by (4.10) that

$$\dim P_{(-\infty, E_2)}(\tilde{H}_n + \mu\chi_R) \geqq \dim P_{(-\infty, E_3]}(\tilde{H}_n + \mu\chi_R) = M_n - N_-(\mu, \tilde{H}_n - E_3, \chi_R).$$

Returning to (4.9), we therefore obtain

$$M_{R,n} \geqq M_n - \lim_{\mu \to \infty} N_-(\mu, \tilde{H}_n - E_3, \chi_R) \geqq M_n - k_0 R^\nu$$

for $n \geqq n(R)$ by Lemma 4.5 below (applied with $A := E_3$), and we are done. □

The aim of Lemma 4.5 is to show that the estimate of Theorem 3.1 on $N_-(\mu, H - E, \chi_R)$ still holds true if we replace H by $\tilde{H}_n$, for n sufficiently large. A related problem has been studied by Kirsch [Ki]; one should note, however, that the eigenvalues counted by $N_-(\mu, \tilde{H}_n - E, \chi_R)$ must travel the whole distance from the gap edge to the level E, while in the Kirsch paper no gap is present and eigenvalues sitting just below E must move only a "little bit."

Lemma 4.5. *Let $A \in (a, b) \subset \varrho(H)$ be fixed. For $R \geqq R_0$, there exists $n(R)$ and k_0 independent of R and n so that*

$$\sup_{\mu > 0} N_-(\mu, \tilde{H}_n - A, \chi_R) \leqq k_0 R^\nu$$

for $n \geqq n(R)$.

Proof. Step 1. By Theorem 3.1 we have a constant k_0 depending on a only so that

$$\sup_{\mu>0} N_-(\mu, H-a, \chi_R) \leqq k_0 R^\nu . \tag{4.11}$$

Since the eigenvalue branches of $H+\mu\chi_R$ for $\mu>0$ are monotonically increasing, these branches will either eventually cross the level b or they will asymptotically approach some level $E' \geqq b$. By (4.11), only a finite number of branches can cross the level a, and so there may only be a finite number of such asymptotic levels in $[a,b]$. Consequently, there exists a $\Lambda>0$ and constants α, β with $a \leqq \alpha < \beta \leqq \Lambda$ so that the interval (α, β) is free of eigenvalues of $H+\mu\chi_R$ for $\mu>\Lambda$. (Of course, α, β, Λ all depend on R.)

Let $\Lambda_0 := (\alpha+\beta)/2$. Our aim (in Step 1) is to show that, for n sufficiently large, no eigenvalue branch of $\tilde{H}_n + \mu\chi_R$ crosses Λ_0 at a $\mu>\Lambda$, i.e., there exists $n_0(R)$ so that for all $n \geqq n_0(R)$:

$$\sup_{\mu>0} N_-(\mu, \tilde{H}_n - \Lambda_0, \chi_R) \leqq N_-(\Lambda, \tilde{H}_n - \Lambda_0, \chi_R) . \tag{4.12}$$

To prove (4.12), suppose for a contradiction that there exist $\mu_j > \Lambda$, $n_j \in \mathbf{N}$, $n_j \geqq j$, and $f_j \in D(\tilde{H}_{n_j})$ satisfying $\|f_j\| = 1$ and

$$(\tilde{H}_{n_j} - \Lambda_0) f_j = -\mu_j \chi_R f_j \tag{4.13}$$

for all $j \in \mathbf{N}$. Let the cut-off functions φ_j and $\psi_j := 1-\varphi_j$ be as before; in particular, $\varphi_{j/2}\psi_j = 0$. Using $n_j \geqq j$ and

$$\begin{aligned}\tilde{H}_{n_j}(\varphi_{j/2} f_j) &= H_{n_j}(\varphi_{j/2} f_j) = \varphi_{j/2} H_{n_j} f_j - 2\nabla\varphi_{j/2} \cdot \nabla f_j - \Delta\varphi_{j/2} f_j \\ &= \varphi_{j/2} \tilde{H}_{n_j} f_j - 2\nabla\varphi_{j/2} \cdot \nabla f_j - \Delta\varphi_{j/2} f_j ,\end{aligned}$$

by the definition of $\tilde{H}_n$, we obtain from (4.13)

$$\|(\tilde{H}_{n_j} + \mu_j\chi_R - \Lambda_0)(\varphi_{j/2} f_j)\| \leqq 2\|\nabla\varphi_{j/2} \cdot \nabla f_j\| + \|\Delta\varphi_{j/2}\|_\infty \|f_j\| \leqq c_1 j^{-1} \tag{4.14}$$

for all j sufficiently large, using Lemma 1.13.

Now, $\varphi_{j/2} f_j + \psi_{j/2} f_j = f_j$ together with (4.13) implies for j large,

$$\begin{aligned}\|(\tilde{H}_{n_j} - \Lambda_0)(\psi_{j/2} f_j)\| &= \|(\tilde{H}_{n_j} + \mu_j\chi_R - \Lambda_0)(\psi_{j_2} f_j)\| \\ &= \|(\tilde{H}_{n_j} + \mu_j\chi_R - \Lambda_0)(\varphi_{j/2} f_j)\| \leqq c_1 j^{-1}\end{aligned}$$

by (4.14). Since $(a,b) \cap \sigma(\tilde{H}_{n_j}) = \emptyset$ for j sufficiently large, and $\Lambda_0 \in (a,b)$ is independent of n_j, there exists $\gamma>0$ so that

$$\|\psi_{j/2} f_j\| = \gamma \|(\tilde{H}_{n_j} - \Lambda_0)(\psi_{j/2} f_j)\| \to 0$$

as $j \to \infty$, whence $\|\varphi_{j/2} f_j\| \to 1$ as $j \to \infty$. Therefore, returning to (4.14), we finally obtain

$$\|(\tilde{H}_{n_j} + \mu_j\chi_R - \Lambda_0)(\varphi_{j/2} f_j)\| < 2c_1 j^{-1} \|\varphi_{j/2} f_j\|$$

for j sufficiently large, and it follows (noting again that $\tilde{H}_{n_j}(\varphi_{j/2} f_j) = H(\varphi_{j/2} f_j)$), that the operator $H+\mu_j\chi_R$ has an eigenvalue in the interval $(\Lambda_0 - 2c_1 j^{-1}, \Lambda_0 + 2c_1 j^{-1})$ for j sufficiently large. But, for j sufficiently large, this contradicts the fact that no eigenvalue branch of $H+\mu\chi_R$, $\mu>\Lambda_0$ lives in the interval (α, β). This concludes the proof of (4.12).

Step 2. Now let A_0 be as in Step 1, and let $E_n^- \in (a, A_0)$ be as in Proposition 1.11. Then, by the second resolvent equation, we have (recall that $E_n^- \in \sigma(\tilde{H}_n)$ for $n \geqq n_1$ by Lemma 4.3),

$$(\tilde{H}_n - E_n^-)^{-1} = (H_n - E_n^-)^{-1} - (\tilde{H}_n - E_n^-)^{-1}(c\psi_n \Pi_n \psi_n)(H_n - E_n^-)^{-1},$$

which gives

$$\begin{aligned}
&\chi_R[(\tilde{H}_n - E_n^-)^{-1} - (H - E_n^-)^{-1}]\chi_R \\
&\quad = \chi_R[(H_n - E_n^-)^{-1} - (H - E_n^-)^{-1}]\chi_R - \chi_R(\tilde{H}_n - E_n^-)^{-1}(c\psi_n \Pi_n \psi_n)\chi_n(H - E_n^-)^{-1}\chi_R \\
&\qquad - \chi_R(\tilde{H}_n - E_n^-)^{-1}(c\psi_n \Pi_n \psi_n)\chi_n(H_n - E_n^-)^{-1}\chi_R \\
&\qquad + \chi_R(\tilde{H}_n - E_n^-)^{-1}(c\psi_n \Pi_n \psi_n)\chi_n(H - E_n^-)^{-1}\chi_R,
\end{aligned}$$

where we have used $\Pi_n \psi_n = \Pi_n \chi_n \psi_n = \Pi_n \psi_n \chi_n$. By Proposition 1.11,

$$\|\chi_R[(H_n - E_n^-)^{-1} - (H - E_n^-)^{-1}]\chi_R\| \leqq \|\chi_n[(H_n - E_n^-)^{-1} - (H - E_n^-)^{-1}]\chi_R\| \to 0$$

as $n \to \infty$, and by Proposition 1.9,

$$\|\psi_n(H - E_n^-)^{-1}\chi_R\| \to 0$$

as $n \to \infty$. As $\|(\tilde{H}_n - E_n^-)^{-1}\| \leqq \operatorname{dist}(E_n^-, \sigma(\tilde{H}_n))^{-1} \leqq c_3$, and $|\varphi_n| \leqq 1$, $\|\Pi_n\| \leqq 1$, it follows that

$$\|\chi_R[(\tilde{H}_n - E_n^-)^{-1} - (H - E_n^-)^{-1}]\chi_R\| \to 0 \tag{4.15}$$

as $n \to \infty$.

Step 3. By (4.12), we have an $n_0(R) > 0$ so that, for $n > n_0(R)$,

$$\sup_{\mu > 0} N_-(\lambda, \tilde{H}_n - A_0, \chi_R) \leqq N_-(A_0, \tilde{H}_n - A_0, \chi_R) \leqq N_-(A_0, \tilde{H}_n - E_n^-, \chi_R)$$

as $E_n^- \leqq A_0$, (E_n^- as in Step 2). By (4.15), we can find $n(R) \geqq n_0(R)$, so that the norm difference of the Birman-Schwinger kernels $\chi_R(\tilde{H}_n - E_n^-)^{-1}\chi_R$ and $\chi_R(H - E_n^-)^{-1}\chi_R$ is less than $1/(2A_0)$, provided $n \geqq n(R)$. By Lemma 1.8, this implies that

$$N_-(A_0, \tilde{H}_n - E_n^-, \chi_R) \leqq N_-(2A_0, H - E_n^-, \chi_R) \leqq \sup_{\mu > 0} N_-(\mu, H - a, \chi_R) \leqq k_0 R^\nu,$$

with the constant k_0 from (4.11). Finally, monotonicity implies that

$$\sup_{\mu > 0} N_-(\mu, \tilde{H}_n - A, \chi_R) \leqq \sup_{\mu > 0} N_-(\mu, \tilde{H}_n - A_0, \chi_R) \leqq k_0 R^\nu$$

for $n \geqq n(R)$, and we are done. □

An interesting problem which arises from Lemma 4.4 is the question of whether the same sort of bound holds for Schrödinger operators other than H, and in particular for $-\Delta$. In fact, the bound in Lemma 4.4 does *not* hold for general operators, but only in our "gap" situation; for the Laplacian, we have (see [A]; cf. also [Ki]):

Proposition 4.6. *In dimension $\nu = 2$, we have*

$$\sup_{n > R} [\dim P_{(-\infty, E)}(-\Delta_{B_n}) - \dim P_{(-\infty, E)}(-\Delta_{B_n - B_R})] = \infty$$

for each fixed $R > 0$.

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