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Inequalities for the Schatten p-Norm. IV

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Abstract. We prove some inequalities for the Schatten p-norm of operators on a Hilbert space. It is shown, among other things, that if A, B, and X are operators such that $A+B \geqq |X|$ and $A+B \geqq |X^*|$, then $\|AX+XB\|_p^p + \|AX^*+X^*B\|_p^p \geqq 2\|X\|_{2p}^{2p}$ for $1 \leqq p < \infty$, and $\max(\|AX+XB\|, \|AX^*+X^*B\|) \geqq \|X\|^2$. Also, for any three operators A, B, and X,

$$\||A|X-X|B|\|_2^2 + \||A^*|X-X|B^*|\|_2^2 \leqq \|AX-XB\|_2^2 + \|A^*X-XB^*\|_2^2.$$

1. Introduction

In their work on free states of the canonical anticommutation relations, Powers and Størmer [9, Lemma 4.1] proved that if A and B are positive operators on a Hilbert space H, then $\|A^{1/2}-B^{1/2}\|_2^2 \leqq \|A-B\|_1$. Also, in studying the quasi-equivalence of quasifree states of canonical commutation relations, Araki and Yamagami [2, Theorem 1] proved that if A and B are operators on a Hilbert space H, then $\||A|-|B|\|_2 \leqq 2^{1/2}\|A-B\|_2$. This has been recently generalized so that $\||A|-|B|\|_2^2 + \||A^*|-|B^*|\|_2^2 \leqq 2\|A-B\|_2^2$ [7, Theorem 2].

The purpose of this paper, which is in the same spirit as those of [5–7], is to extend these inequalities to commutator versions and to show that in some cases the trace norm can be replaced by a general p-norm. In particular it will be shown that for positive operators A and B, $\|A^{1/2}-B^{1/2}\|_{2p}^2 \leqq \|A-B\|_p$ for $1 \leqq p \leqq \infty$.

Let H be a separable complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. Let $K(H)$ denote the closed two-sided ideal of compact operators on H. For any compact operator A, let $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A|=(A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. A compact operator A is said to be in the Schatten p-class C_p ($1 \leqq p < \infty$), if $\sum s_i(A)^p < \infty$. The Schatten p-norm of A is defined by $\|A\|_p = (\sum s_i(A)^p)^{1/p}$. This norm makes C_p into a Banach space. Hence C_1 is the trace class and C_2 is the Hilbert–Schmidt class. It is reasonable to let C_∞ denote the ideal of compact operators $K(H)$, and $\|\cdot\|_\infty$ stand for the usual operator norm.

If $A \in C_p$ ($1 \leqq p < \infty$) and $\{e_i\}$ is any orthonormal set in H, then $\|A\|_p^p \geqq \sum |(Ae_i, e_i)|^p$. More generally, if $\{E_i\}$ is a family of orthogonal projections satisfying $E_iE_j = \delta_{ij}E_i$, then $\|A\|_p^p \geqq \sum \|E_iAE_i\|_p^p = \|\sum E_iAE_i\|_p^p$, and for $p>1$ equality will hold if and only if $A = \sum E_iAE_i$. Moreover, if $\sum E_i = 1$ (the identity operator) and $p=2$, then $\|A\|_2^2 = \sum \|E_iAE_j\|_2^2$. One more fact that will be needed in the sequel is that if $A \in C_p (1 \leqq p \leqq \infty)$, then $\| A \|_p = \| A^* \|_p = \| |A^*| \|_p = \| |A| \|_p$. The reader is referred to [3] for further properties of the Schatten p-classes.

2. On the Powers–Stǿrmer Inequality

First we extend the Powers–Stǿrmer inequality for the usual operator norm.

Theorem 1. *If $A, B \in B(H)$ with $A + B \geqq \pm X$, where $X \in B(H)$ is self-adjoint, then $\| AX + XB \| \geqq \| X \|^2$.*

Proof. Since X is a self-adjoint operator, it follows that there exists a sequence $\{f_n\}$ of unit vectors in H such that $(Xf_n, f_n) \to t$ as $n \to \infty$, where $|t| = \| X \|$. But then,

$$\| Xf_n - tf_n \|^2 = \| Xf_n \|^2 + t^2 - 2t(Xf_n, f_n) \leqq 2t^2 - 2t(Xf_n, f_n).$$

Therefore $Xf_n - tf_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \| AX + XB \| &\geqq |((AX + XB)f_n, f_n)| \\ &= |(A(X - t)f_n, f_n) + (Bf_n, (X - t)f_n) + t((A + B)f_n, f_n)| \\ &\geqq |t|((A + B)f_n, f_n) - |(A(X - t)f_n, f_n) + (Bf_n, (X - t)f_n)| \\ &\geqq |t||(Xf_n, f_n)| - |(A(X - t)f_n, f_n) + (Bf_n, (X - t)f_n)|. \end{aligned}$$

Letting $n \to \infty$, we get that $\| AX + XB \| \geqq \| X \|^2$ as required.

Corollary 1. *If $A, B \in B(H)$ with $A + B \geqq \pm X$, where $X \in B(H)$ is self-adjoint such that $AX + XB = 0$, then $X = 0$.*

Next we establish the corresponding inequality for a general p-norm.

Theorem 2. *If $A, B \in B(H)$ with $A + B \geqq \pm X$, where $X \in B(H)$ is self-adjoint, then $\| AX + XB \|_p \geqq \| X \|_{2p}^2$ for $1 \leqq p \leqq \infty$.*

Proof. Of course the $p = \infty$ case is the content of Theorem 1. Now assume that $1 \leqq p < \infty$ and $AX + XB \in C_p$ (otherwise we have nothing to prove). Hence $AX + XB$ is compact. If $\pi: B(H) \to B(H)/C_\infty$ is the quotient map of $B(H)$ onto the Calkin algebra $B(H)/C_\infty$, then we have $\pi(A)\pi(X) + \pi(X)\pi(B) = 0$ and $\pi(A) + \pi(B) \geqq \pm \pi(X)$. Applying Corollary 1 now implies that $\pi(X) = 0$, in other words X is compact. (Recall that the Calkin algebra is a B^*-algebra and so it is representable as an operator algebra.) But it is known that a compact self-adjoint operator is diagonalizable, hence $Xe_n = t_n e_n$, where $\{e_n\}$ is an orthonormal basis for H. Therefore,

$$\begin{aligned} \| AX + XB \|_p^p &\geqq \sum |((AX + XB)e_n, e_n)|^p \\ &= \sum |(AXe_n, e_n) + (Be_n, Xe_n)|^p = \sum |t_n((A + B)e_n, e_n)|^p \\ &\geqq \sum |t_n|^p |(Xe_n, e_n)|^p = \sum |t_n|^{2p} = \| X \|_{2p}^{2p}. \end{aligned}$$

As a Corollary of Theorem 2, we obtain the Powers–Stǿrmer inequality [9, Lemma 4.1] and extend it to other p-norms (including the usual operator norm).

Corollary 2. *If $A, B \in B(H)$ are positive, then $\| A - B \|_{2p}^2 \leqq \| A^2 - B^2 \|_p$ for $1 \leqq p \leqq \infty$.*

Proof. Let $X = A - B$, and then apply Theorem 2.

The above theorems can be generalized further by removing the restriction on X.

To accomplish this we first recall the following lemma which has appeared in [7].

Lemma. *If* $A, B \in B(H)$ *and* $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ *is defined on* $H \oplus H$, *then* $|T| = \begin{pmatrix} |B| & 0 \\ 0 & |A| \end{pmatrix}$. *Moreover,* $\|T\|_p^p = \|A\|_p^p + \|B\|_p^p$ *for* $1 \leqq p < \infty$ *and* $\|T\| = \max(\|A\|, \|B\|)$.

Theorem 3. *If* $A, B,$ *and* $X \in B(H)$ *with* $A + B \geqq |X|$ *and* $A + B \geqq |X^*|$, *then* $\|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p \geqq 2\|X\|_{2p}^{2p}$ *for* $1 \leqq p < \infty$, *and max* $(\|AX + XB\|, \|AX^* + X^*B\|) \geqq \|X\|^2$.

Proof. On $H \oplus H$, let $T = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $S = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$. Then Y is self-adjoint and by the lemma, we have $|Y| = \begin{pmatrix} |X^*| & 0 \\ 0 & |X| \end{pmatrix}$. From $A + B \geqq |X|$ and $A + B \geqq |X^*|$, we obtain that $T + S \geqq |Y|$. Since Y is self-adjoint, it follows that $T + S \geqq |Y| \geqq \pm Y$. Applying Theorem 2 to the operators T, S and Y we get $\|TY + YS\|_p \geqq \|Y\|_{2p}^2$ for $1 \leqq p \leqq \infty$. But $TY + YS = \begin{pmatrix} 0 & AX + XB \\ AX^* + X^*B & 0 \end{pmatrix}$. Now using the lemma, the proof can be completed as that of Theorem 1 in [7].

Corollary 3. *If* A, $X \in B(H)$ *with* $A + A^* \geqq |X|$ *and* $A + A^* \geqq |X^*|$, *then* $\|AX + XA^*\|_p \geqq \|X\|_{2p}^2$ *for* $1 \leqq p \leqq \infty$.

Proof. This follows from Theorem 3 applied to A and A^* with the observation that $\|AX + XA^*\|_p = \|AX^* + X^*A^*\|_p$ for $1 \leqq p \leqq \infty$.

Remarks. (1) If A is a positive operator and X is a self-adjoint operator such that $A \geqq \pm X$, then it need not be true that $A \geqq |X|$. For example, consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which act on a two-dimensional Hilbert space.

(2) If the assumptions $A + B \geqq |X|$ and $A + B \geqq |X^*|$ are strengthened so that $A \geqq |X^*|$ and $B \geqq |X|$, then following the proofs of Theorems 1, 2, and 3, we obtain that $\|AX + XB\|_p \geqq 2\|X\|_{2p}^2$ for $1 \leqq p \leqq \infty$. In this case the operators, T, S in the proof of Theorem 3 should be taken as $T = S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. It should be also noticed that if the roles of X and X^* are interchanged, that is if $A \geqq |X|$ and $B \geqq |X^*|$, then such inequality may not be true. For example, consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which act on a two-dimensional Hilbert space.

3. On the Araki–Yamagami Inequality

In [1, Lemma 5.2], Araki proved that if A and B are self-adjoint operators in $B(H)$, then $\||A| - |B|\|_2 \leqq \|A - B\|_2$. A commutator version of this result is also true, namely $\||A|X - X|B|\| \leqq \|AX - XB\|_2$ for any $X \in B(H)$. This has been recently obtained in a more general setting where A and B are normal operators [8, Corollary 2]. For general operators A and B, Araki and Yamagami [2, Theorem 1], proved that $\| |A| - |B| \|_2 \leqq 2^{1/2} \| A - B \|_2$. This also has been extended so that $\| |A| - |B| \|_2^2 + \| |A^*| - |B^*| \|_2^2 \leqq 2 \| A - B \|_2^2$ [7, Theorem 2].

In this section we establish a commutator version of this Araki–Yamagami type inequality.

Theorem 4. *If A, B, and $X \in B(H)$, then*

$$\| |A| X - X |B| \|_2^2 + \| |A^*| X - X |B^*| \|_2^2 \leqq \| AX - XB \|_2^2 + \| A^*X - XB^* \|_2^2.$$

Proof. On $H \oplus H$, let $T = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$, and $Y = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$. Then T and S are self-adjoint. Thus $\| |T| Y - Y |S| \|_2 \leqq \| TY - YS \|_2$. Simple calculations and the lemma show that

$$|T| Y - Y |S| = \begin{pmatrix} |A^*| X - X |B^*| & 0 \\ 0 & |A| X - X |B| \end{pmatrix} \quad \text{and} \quad TY - YS = \begin{pmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{pmatrix}.$$

Since $\| |T| Y - Y |S| \|_2^2 = \| |A| X - X |B| \|_2^2 + \| |A^*| X - X |B^*| \|_2^2$ and $\| TY - YS \|_2^2 = \| AX - XB \|_2^2 + \| A^*X - XB^* \|_2^2$, it follows that $\| |A| X - X |B| \|_2^2 + \| |A^*| X - X |B^*| \|_2^2 \leqq \| AX - XB \|_2^2 + \| A^*X - XB^* \|_2^2$.

Corollary 4. *If N, $M \in B(H)$ are normal, then for any $X \in B(H)$, $\| |N| X - X |M| \|_2 \leqq \| NX - XM \|_2$.*

Proof. Since N and M are normal operators, the spectral theorem implies that $|N| = |N^*|$ and $|M| = |M^*|$, and the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class [10, Theorem 1] implies that $\| NX - XM \|_2 = \| N^*X - XM^* \|_2$. Now the result follows by Theorem 4.

Inspired by the results of this section and by the fact that every operator $A \in B(H)$ has a normal dilation in $B(H \oplus H)$, we obtain the following extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class [10, Theorem 1].

Theorem 5. *If A, B, and $X \in B(H)$, then*

$$\| AX - XB \|_2^2 + \| A^*X \|_2^2 + \| XB^* \|_2^2 = \| A^*X - XB^* \|_2^2 + \| AX \|_2^2 + \| XB \|_2^2.$$

Proof. On $H \oplus H$, let $N = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}$, $M = \begin{pmatrix} B & B^* \\ B^* & B \end{pmatrix}$, and $Y = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$. Then N and M are normal [4, p. 123], and so by the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class we have $\| NY - YM \|_2 = \| N^*Y - YM^* \|_2$. But

$$NY - YM = \begin{pmatrix} AX - XB & -XB^* \\ A^*X & 0 \end{pmatrix} \quad \text{and} \quad N^*Y - YM^* = \begin{pmatrix} A^*X - XB^* & -XB \\ AX & 0 \end{pmatrix}.$$

Since

$$\|NY - YM\|_2^2 = \|AX - XB\|_2^2 + \|A^*X\|_2^2 + \|XB^*\|_2^2,$$

and

$$\|N^*Y - YM^*\|_2^2 = \|A^*X - XB^*\|_2^2 + \|AX\|_2^2 + \|XB\|_2^2,$$

we have the required result.

Remark. If in Theorem 5, A and B are assumed to be normal operators, then we retain the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class, because in this case we have

$$\|AX\|_2 = \|A^*X\|_2 \quad \text{and} \quad \|XB\|_2 = \|XB^*\|_2.$$

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