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One-Dimensional Classical Massive Particle in the Ideal Gas

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Abstract. The motion of a one-dimensional massive particle under the action of collisions with points of the ideal gas is considered. It is shown that the normed displacement of the massive particle is represented asymptotically as the difference of random variables having limit Gauss distribution. Estimations of the diffusion coefficient not depending on the mass are found.

1. Description of the Model and Formulation of the Results

The probabilistic model of the Brownian particle was constructed more than fifty years ago. One of the first references is the classical paper of Wiener [11]. Since that time the Wiener measure became a subject of many deep investigations. Now it is a beautiful chapter of the theory of random processes which can be found practically in all text-books on the subject (see [4, 6]). It is surprising enough that a general mechanical model of the Brownian particle was not constructed so far. Only several particular cases were considered in [7, 8]. Moreover, it was discovered recently in direct experiments (see [5]) that for the Brownian particle the correlation functions for the velocity decay only as a power of time which shows that the representation of the displacement of the Brownian particle as a sum of independent or weakly dependent random variables is a crude approximation. It is worthwhile to mention also that after the discovery of Alder and Wainright (see [2]) such decay of correlations is typical for many problems of non-equilibrium statistical mechanics (see [1]).

The goal of this paper is to present several rigorous results concerning the asymptotic behaviour of a massive particle (m.p.) of mass M moving in one direction under the action of elastic collisions with particles of equal masses whose masses are taken to be equal to 1. It is assumed that the particles do not interact and their distribution is the equilibrium distribution of the ideal gas with density ϱ and inverse temperature β. The coordinate and velocity of the m.p. are denoted by $q_0, v_0, x_0=(q_0, v_0)$. A collection of equal particles is denoted by $X=\{x\}$, $x=(q, v)$, and $Y=(x_0, X)$. The phase space of all possible Y is denoted by Ω. For any subset

$A \subset R^2$ the intersection $Y \cap A$ is the set of all $(q, v) \in Y$ such that $(q, v) \subset A$, $|Y \cap A|$ is the cardinality of this set. We introduce the measure μ on Ω for which $d\mu = dq_0 \sqrt{\frac{\beta M}{2\pi}} e^{-\beta M v_0^2/2} dv_0 d\mathcal{P}(x)$, where $\mathcal{P}$ is the limit Gibbs measure on the phase space $\Omega(X)$ of all possible X. We recall that the projection of $\mathcal{P}$ to the configuration space of the ideal gas is the Poisson measure with parameter ϱ and the velocities of different particles have independent gaussian distribution with the density $\sqrt{\frac{\beta}{2\pi}} \exp\{-\frac{1}{2}\beta v^2\}$. The measure $\mathcal{P}$ is normed while the whole measure μ is infinite because the coordinate of m.p. can take arbitrary values with equal weight. The conditional measure μ_{q_0} arising under the specification of q_0 is $\sqrt{\frac{\beta M}{2\pi}} e^{-Mv_0/2} dv_0 d\mathcal{P}$ and finite. Especially, we mention the notation μ_0 for the conditional measure under the condition $q_0 = 0$. The subspace of Ω where $q_0 = 0$ is denoted by Ω_0.

The flow corresponding to the motion of the whole system is denoted by $\{T^t\}$. It preserves the infinite measure μ. In ergodic theory there are several constructions which permit reducing all problems for the flow $\{T^t\}$ to problems concerning flows or transformations preserving a probability measure. For example, we can consider points of the phase space just after a collision of m.p. and introduce the corresponding induced transformation (see e.g. [3]), or we can introduce relative coordinates $q - q_0$ and consider the action of the flow in the relative coordinates. The only result of these constructions which we need is a possibility to use ergodic theorems valid for flows preserving a finite measure.

Unfortunately nothing is known about ergodic properties of the flow $\{T^t\}$. We shall formulate now the results of this paper. Let us denote

$$T^t Y = Y(t) = (x_0(t), X(t)) = (q_0(t), v_0(t), X(t)).$$

In Sect. 2 we prove the following theorem.

Theorem 1. *For every $\varepsilon > 0$ and μ almost every Y one can find $t_0(Y, \varepsilon)$, such that for all $t > t_0(Y, \varepsilon)$*

$$|q_0(t) - q_0| \leqq t^{1/2 + \varepsilon}.$$

From this theorem we derive

Theorem 2. *For μ-almost every Y each particle of the ideal gas has finitely many collisions with* m.p.

In Sect. 3 we show a stronger result than Theorem 1.

Theorem 3. *Assume that $q_0 = 0$. Then with respect to μ_0*

$$q_0(t) = n^+(t) - n^-(t) + \xi(t),$$

where the random variables $n^\pm(t)/\sqrt{t}$ have the same gaussian limit distribution for $t \to \infty$, while $\xi(t)/\sqrt{t}$ converges in probability to zero.

This theorem shows that we may expect a non-trivial limit probability distribution for the normalized random variable $q_0(t)/\sqrt{t}$, where the normalization is the same as in the usual central limit theorem of probability theory. Recently Szasz and Toth proved that $\xi(t)/\sqrt{t}$ converges to zero in $\mathscr{L}^2(\Omega, \mu_0)$ (see [10]).

In Sect. 4 we present an analysis which shows that under the conditions of Theorem 3 $q_0(t)=\tilde{q}_0(t)+\tilde{\xi}(t)$, where $\tilde{\xi}(t)/\sqrt{t}$ converges in probability to zero while $\liminf E_{\mu_0}(\tilde{q}_0(t))^2 \geqq \text{const}\cdot t$, and thus limit distribution will be non-trivial.

2. An Estimation of the Displacement of the M.P.

This section contains the proofs of Theorems 1 and 2.

Proof of Theorem 1. Put

$$R(t)=t^{1/2+\varepsilon}, \qquad \delta q_0(t)=q_0(t)-q_0, \qquad d=d(t,Y)=q_0(t,Y)-\tfrac{1}{2}R(t)$$

and denote

$$\Delta(t,Y)=[d(t,Y), q_0(t,Y)], \qquad C^{(t)}=\{Y: \delta q_0(t,Y)>R(t)\}.$$

Introduce the random variable $\tau(Y)$ defined only for $Y\in C^{(t)}$, where

$$\tau(Y)=\max\{s: q_0(s,Y)=d(t,Y), 0\leqq s\leqq t\}.$$

Then $C^{(t)}=\bigcup\limits_{k=0}^{[t]-1} C_k^{(t)}$, $C_k^{(t)}=\left\{Y: \frac{tk}{[t]}<\tau(Y)\leqq\frac{t(k+1)}{[t]}\right\}$. It is sufficient to estimate the probability with respect to $\mu_0=\mu_{q_0}$, $q_0=0$ of each $C_k^{(t)}$. Let

$$N_-(t,Y)=\operatorname{card}\{(q,v)\in X\subset Y: q(t)\in\Delta(t,Y), v(t)<0\}=\operatorname{card}\{\mathcal{N}_-(t,Y)\},$$

$$K_-(t,Y)=\operatorname{card}\{(q,v)\in X\subset Y: q(t)<d(t), v(t)<0, q(s)=q_0(s)$$

for some $s: \tau\leqq s\leqq t$, i.e. a particle $(q,v)\in X$ had a collision with the massive particle in the time interval $[\tau,t]\}=\operatorname{card}\{\mathscr{K}_-(t,Y)\}$,

$$\begin{aligned}M_+(t,Y)&=\operatorname{card}\{(q,v)\in X\subset Y: q<q_0, q(s)=q_0(s) \text{ for some } s:\tau\leqq s\leqq t\}\\ &=\operatorname{card}\{\mathscr{M}_+(t,Y)\}\end{aligned}$$

Here $v(t)$ is the velocity of the particle (q,v) at the moment of t. We have the obvious inequality:

$$N_-(t,Y)\leqq M_+(t,Y)-K_-(t,Y). \tag{2.1}$$

Further $K_-(t,Y)\geqq K_-^{(1)}(t,Y)$ for $Y\in C_k^{(t)}$, where

$$K_-^{(1)}(t,Y)=\operatorname{card}\left\{(q,v)\in X\subset Y: q(t)<d(t), v(t)<0,\right.$$
$$\left. q(t)-\left(t-t\frac{(k+1)}{[t]}\right)v(t)\geqq d(t)\right\}.$$

Thus

$$N_-(t,Y)\leqq M_+(t,Y)-K_-^{(1)}(t,Y). \tag{2.2}$$

The random variable $K_-^{(1)}(t, Y)$ has Poisson distribution with parameter

$$\gamma_- = \frac{\varrho}{\sqrt{2\pi\beta}}\left(t - t\frac{k+1}{[t]}\right).$$

The estimation for $M_+(t, Y)$ will follow from the next lemma.

Lemma 1. *There exist a constant $c_1>0$ and a subset $\mathscr{D}_t \subset \Omega$ such that*

$$\mu_0(\Omega\backslash\mathscr{D}_t) \leqq \exp(-c_1 t^{2\varepsilon})$$

and $M_+^{(1)}(t, Y) \geqq M_+(t, Y)$ for $Y \in \mathscr{D}_t$, where,

$$M_+^{(1)}(t, Y) = \operatorname{card}\left\{(q, v) \in X \subset Y : q\left(k\frac{t}{[t]}\right) < q_0\left(k\frac{t}{[t]}\right), v\left(k\frac{t}{[t]}\right) > 0,\right.$$
$$\left. q\left(k\frac{t}{[t]}\right) + \left(t - k\frac{t}{[t]}\right) v\left(k\frac{t}{[t]}\right) > q_0\left(k\frac{t}{[t]}\right) - t^{\varepsilon}\right\}. \tag{2.3}$$

Proof. The inequality $M_+(t, Y) \leqq M_+^{(1)}(t, Y)$ is obviously true if

$$q_0\left(k\frac{t}{[t]}\right) \leqq d(t) + t^{\varepsilon}.$$

Therefore assume that $q_0\left(k\frac{t}{[t]}\right) > d(t) + t^{\varepsilon}$. We recall that $q_0(\tau) = d(t)$ for $k\frac{t}{[t]} < \tau \leqq (k+1)\frac{t}{[t]}$, which implies in this case that

$$\max_{k\frac{t}{[t]} \leqq s \leqq (k+1)\frac{t}{[t]}} v_0(s) \geqq t^{\varepsilon}.$$

Denote $\mathscr{D}_t = \left\{Y : \max_{0 \leqq s \leqq t} v_0(s) \leqq t^{\varepsilon}\right\}$. The same arguments as in [9] show that

$$\mu_0\left\{Y : \max_{0 \leqq s \leqq t} v_0(s, Y) \geqq t^{\varepsilon}\right\} \leqq \exp\{-c_1 t^{2\varepsilon}\}$$

for a constant $c_1 > 0$. Q.E.D.

Let us estimate $\mu_0(C_k^{(t)} \cap \mathscr{D}_t)$. The random variable $M_+^{(1)}$ has Poisson distribution with parameter γ_+, where

$$\gamma_+ = \frac{\varrho\sqrt{\beta}}{\sqrt{2\pi}} \int_{-\infty}^{0} dq \int_{\max\left\{0, -\frac{q+t^{\varepsilon}}{t-(k+s)t/[t]}\right\}} e^{-\beta v^2/2} dv$$
$$\leqq \frac{\varrho\sqrt{\beta}}{\sqrt{2\pi}} \int_{-\infty}^{0} dq \int_{-\frac{q+t^{\varepsilon}}{t-(k+1)t/[t]}}^{\infty} e^{-\beta v^2/2} dv$$
$$= \frac{\varrho(t - t(k+1)/[t])}{\sqrt{2\pi\beta}} \exp\left(-\frac{\beta}{2}\frac{t^{2\varepsilon}}{t-(k+1)t/[t]}\right)$$
$$+ \frac{\varrho t^{\varepsilon}}{\sqrt{2\pi\beta}} \int_{-\frac{t^{\varepsilon}}{t-t(k+1)/[t]}}^{\infty} dv \leqq \frac{\varrho(t - t(k+1)/[t]}{\sqrt{2\pi\beta}} + \frac{t^{\varepsilon}\varrho}{\sqrt{\beta}}.$$

We put for a constant $c_2>0$

$$A_+=\left\{Y: M_+^{(1)}(t,Y)\geqq \frac{\varrho(t-t(k+1)/[t])}{\sqrt{2\pi\beta}}+\frac{t^\varepsilon\varrho}{\sqrt{\beta}}+c_2R(t)\right\},$$

$$A_-=\left\{Y: K_-^{(1)}(t,Y)\leqq \frac{\varrho(t-t(k+1)/[t]}{\sqrt{2\pi\beta}}-c_2R(t)\right\}.$$

From general properties of Poisson distribution it follows that for $C_3>0$, $C_3'>0$,

$$\mu_0(A_-)\leqq \exp\left\{-c_3\frac{R^2(t)}{t-(k+1)t/[t]}\right\}\leqq \exp\left\{-c_3'\frac{R^2(t)}{t}\right\}=\exp\{-c_3't^{2\varepsilon}\},$$

$$\mu_0(A_+)\leqq \mu_0\{Y: M_+^{(1)}(t,Y)\geqq \gamma_+ +c_2R(t)\}\leqq \exp\{-c_3't^{2\varepsilon}\}.$$

Thus

$$\mu_0\{C_k^{(t)}\cap\mathscr{D}_t\cap(A_+\cup A_-)\}\leqq \exp\{-c_4t^{2\varepsilon}\}$$

for a constant $c_4>0$. For $Y\in C_k^{(t)}\cap\mathscr{D}_t\cap\bar{A}_+\cap\bar{A}_-$,

$$N_-(t,Y)\leqq 2t^\varepsilon+2c_2R(t).$$

But $N_-(t,Y)$ has Poisson distribution with parameter $\frac{\varrho}{4}R(t)$. If c_2 is chosen sufficiently small then for a constant $c_5>0$,

$$\begin{aligned}&\mu_0\{C_k^{(t)}\cap\mathscr{D}_t\cap\bar{A}_+\cap\bar{A}_-\}\\ &\leqq \mu_0\left\{Y: N_-(t,Y)\leqq \frac{\varrho R(t)}{4}+2t^\varepsilon+\left(C_2-\frac{\varrho}{4}\right)R(t)\right\}\\ &\leqq \exp\{-c_5R(t)\}\leqq \exp\{-c_5t^{2\varepsilon}\}.\end{aligned}$$

Thus taking into account the estimations for $\mu_0(A_\pm)$, we get $\mu_0(C_k^{(t)})\leqq \exp\{-c_6t^{2\varepsilon}\}$ and

$$\mu_0(C_t)\leqq \sum_{k=0}^{[t]-1}\mu_0(C_k^{(t)})\leqq t\exp\{-c_6t^{2\varepsilon}\},$$

$c_6>0$ is a constant. From this inequality the statement of the theorem follows easily QED.

The arguments used during this proof can be called "balance arguments." Roughly speaking they show that the set of particles which would cross the point d from the left side in the free dynamics during the time interval $(\tau,t]$ consists of particles which lie at the final moment of time in the spatial interval $\Delta(t)$ and have positive velocities, and of particles which interacted with the m.p. during the same time interval. This idea will be used also in the proof of Theorem 3.

Proof of Theorem 2. Assume that with a positive μ_0-measure there exist particles which have infinitely many collisions with the m.p. For definiteness we consider particles which are on the left side of m.p. The subsequent velocities of any of these particles form a sequence of decreasing positive numbers tending to zero, because if it becomes negative then in view of Theorem 1 the particle eventually escapes to infinity.

Let us denote by $f(Y)$ and $g(Y)$ the velocity and position of the particle which lies on the left side of m.p., has infinitely many collisions with m.p. and is the nearest particle to m.p. having these two properties. Then it is easy to see that $f \geqq 0$ and

$$\int_0^T f(T^t(Y))dt \leqq \delta q_0(T) - g(Y).$$

From Theorem 1 $\lim_{T\to\infty} \frac{1}{T}(\delta q_0(T)) = 0$, μ_0 a.e. Thus

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(T^t(Y))dt = 0$$

a.e., i.e. $f=0$ μ_0 a.e., which is a contradiction QED.

3. An Expression for the Displacement of the M.P.

We start with several notations:

$$\begin{aligned}
\mathscr{A}^+(t, Y) &= \{(q, v) \in X \subset Y : q < q_0, q + tv > q_0\},\\
\mathscr{A}^-(t, Y) &= \{(q, v) \in X \subset Y : q(t) < q_0(t), q(t) - tv(t) > q_0(t)\},\\
\mathscr{B}(t, Y) &= \{(q, v) \in X \subset Y : q < q_0, q(\tau) = q_0(\tau) \text{ for some } \tau;\ 0 \leqq \tau \leqq t\},\\
A^+(t, Y) &= \operatorname{card}\{\mathscr{A}^+(t, Y)\},\\
A^-(t, Y) &= \operatorname{card}\{\mathscr{A}^-(t, Y)\},\\
B(t, Y) &= \operatorname{card}\{\mathscr{B}(t, Y)\}.
\end{aligned}$$

Here is a more precise formulation of Theorem 3.

Theorem 3.

$$\delta q_0(t) = \frac{1}{\varrho}(A^+(t, Y) - A^-(t, Y)) + \beta(t, Y),$$

where $\frac{1}{\sqrt{t}}\beta(t, \cdot) \to 0$ *in probability (with respect to* μ_0*) as* $t \to \infty$.

Proof. We shall show first that

$$A^+(t, Y) = \frac{\varrho}{2}\delta q_0(t, Y) + B(t, Y) + \alpha(t, Y), \tag{3.1}$$

where $\alpha(t, \cdot)/\sqrt{t} \to 0$ in probability as $t \to \infty$. In view of Theorem 1 we may consider only those Y and t for which $|\delta q_0(t, Y)| \leqq t^{1/2+\varepsilon}$ for a sufficiently small ε.

Assume that $\delta q_0(t) > 0$. The set $\mathscr{A}^+(t, Y) \backslash \mathscr{B}(t, Y) = \mathscr{L}(t, Y)$ consists of particles $(q, v) \in X$ such that

a$_1$) $q(t) \in [q_0, q_0(t)]$,

a$_2$) $v(t) > 0$,

a$_3$) $q(s) < q_0(s)$ for all $s \in [0, t]$.

Denote

$$\mathscr{E}(t, Y)=\{(q, v)\in X: v(t)>0, q(t)\in[q_0, q_0(t)]\},$$
$$E(t, Y)=\operatorname{card}\{\mathscr{E}(t, Y)\}.$$

The difference $\mathscr{E}_1(t, Y)=\mathscr{E}(t, Y)\backslash\mathscr{Z}(t, Y)$, $E_1(t, Y)=\operatorname{card}\{\mathscr{E}_1(t, Y)\}$ consists of particles with positive velocities which are at the moment t in the segment $[q_0, q_0(t)]$ and interacted with m.p. at a moment $s\in[0, t]$. We shall prove that $E_1(t,\cdot)/\sqrt{t}\to 0$ in probability.

For $(q, v)\in\mathscr{E}_1(t, Y)$, introduce τ_1 equal to the last moment of collision with m.p. in the interval $(0, t)$. We shall consider separately two cases.

Case 1. $\tau_1<t-t^{3/4}$. The number of particles with this property is denoted by $p_1(t, Y)$. Each of these particles moves freely in the time internal (τ_1, t) with a positive velocity and therefore, for the displacement $\delta q(\tau_1, t)$ of this particle during the interval (τ_1, t) we have

$$v(t)\leqq\frac{\delta q(\tau_1, t)}{t^{3/4}}\leqq\frac{\delta q_0(\tau_1, t)}{t^{3/4}}\leqq t^{1/2+\varepsilon-3/4}=t^{-1/4+\varepsilon}.$$

Also for these particles

$$q(t)\in[q_0, q_0(t)]\subset[q_0(t)-t^{1/2+\varepsilon}, q_0(t)],$$

Thus $p_1(t, Y)$ is not more than the number of particles $(q, v)\in X$ such that $q(t)\in[q_0(t)-t^{1/2+\varepsilon}, q_0(t)]$, $0\leqq v(t)\leqq t^{-1/4+\varepsilon}$, which has the Poisson distribution with parameter

$$\gamma\leqq\text{const}\cdot t^{1/2+\varepsilon}\cdot t^{-1/4+\varepsilon}=\text{const}\cdot t^{1/4+2\varepsilon}.$$

For sufficiently small ε we have $t^{-1/2}p_1(t,\cdot)\to 0$ in probability as $t\to\infty$.

Case 2. $\tau_1\geqq t-t^{3/4}$. The number of particles with this property is denoted by $p_2(t, Y)$. Let us put

$$\overline{\delta q_0}(t-t^{3/4}, t)=\max_{s\in[t-t^{3/4}, t]}|\delta q_0(s, t)|,$$

where $\delta q_0(s_1, s_2)$ is the displacement of the h.p. in the time interval (s_1, s_2). A simple stronger version of Theorem 1 gives that $\overline{\delta q_0}(t-t^{3/4}, t)\leqq t^{3/8+\varepsilon}$ for any $\varepsilon>0$, a.e. Y and sufficiently large t. We assume that we already deal with so large t. Thus

$$q(t)\in[q_0(t)-\overline{\delta q_0}(t-t^{3/4}, t), q_0(t)].$$

The number of particles for which $q(t)\in[q_0(t)-t^{3/8+\varepsilon}, q_0(t)]$ has the Poisson distribution with parameter $\gamma\leqq\text{const}\, t^{3/8+\varepsilon}$. Thus for sufficiently small ε we have $p_2(t,\cdot)/\sqrt{t}\to 0$ in probability as $t\to\infty$. Now we finally get

$$\frac{1}{\sqrt{t}}(E(t, Y)-\operatorname{card}\{\mathscr{A}(t, Y)\backslash\mathscr{B}(t, Y)\})\to 0, \tag{3.2}$$

and

$$\begin{aligned} A^+(t, Y) &= B(t, Y) + E_1(t, Y) - \operatorname{card}\{\mathscr{B}(t, Y)\backslash\mathscr{A}^+(t, Y)\} \\ &= B(t, Y) + E(t, Y) - \operatorname{card}\{\mathscr{B}(t, Y)\backslash\mathscr{A}^+(t, Y)\} \\ &\quad + \alpha_1(t, Y), \end{aligned}$$

where $\alpha_1(t, \cdot) = o(\sqrt{t})$ in probability. From the strong law of large numbers

$$E(t, Y) = \frac{\varrho}{2}\delta q_0(t, Y) + \alpha_2(t, Y),$$

where $\alpha_2(t, \cdot)/\sqrt{t} \to 0$. Now in order to complete the derivation of (4) we have to estimate $\operatorname{card}\{\mathscr{B}(t, Y)\backslash\mathscr{A}^+(t, Y)\}$. The set $\mathscr{B}(t, Y)\backslash\mathscr{A}^+(t, Y)$ consists of particles $(q, v) \in X$ with $q<0$, $v<0$ and of particles $(q, v) \in X$ with $q<q_0$, $v>0$, $q+tv<q_0$ and interacting with the m.p. during the interval $(0, t)$. The number of particles of the first (second) group is denoted by $p_3(t, Y)$ ($p_4(t, Y)$). For the particles of the first group $q \in [q_0 - t^{1/2+\varepsilon}, q_0]$. Again we introduce the moment τ_1 of the last collision with the h.p. If $\tau_1 < t^{3/4}$, then as above $q \in [q_0 - t^{3/4(1/2+\varepsilon)}, q_0]$ and the number of such particles is $o(\sqrt{t})$ in probability. If $\tau_1 > t^{3/4}$, then the velocity v of the particle must be small $0 < v < t^{-1/4+\varepsilon}$, which together with the inclusion $q \in [q_0 - t^{1/2+\varepsilon}, q_0]$ shows $p_3(t, Y) = o(\sqrt{t})$ in probability.

Now we shall estimate $p_4(t, Y)$. Let us define

$$\tau_2((q, v)) = \inf\{s : s \geqq 0, q(s) = q_0(s)\},$$

and put

$$\begin{aligned} n_1(t, Y) &= \operatorname{card}\{(q, v) \in X : \delta q_0(\tau_2) > -t^{1/2-\varepsilon}\}, \\ n_2(t, Y) &= \operatorname{card}\{(q, v) \in X : \delta q_0(\tau_2) \leqq -t^{1/2-\varepsilon}\}. \end{aligned}$$

We have $p_4(t, Y) = n_1(t, Y) + n_2(t, Y)$.

Firstly we estimate $n_1(t, Y)$. If $q_0(\tau_2) > q_0 - t^{1/2-\varepsilon}$, then

$$q + tv \geqq q + \tau_2 v = q_0(\tau_2) > q_0 - t^{1/2-\varepsilon},$$

i.e.

$$q + tv \in [q_0 - t^{1/2-\varepsilon}, q_0].$$

The number of particles satisfying the last inclusion has the Poisson distribution with the parameter $\gamma \leqq \operatorname{const} t^{1/2-\varepsilon}$. This gives $n_1(t, \cdot) = o(\sqrt{t})$ in probability. In order to estimate $n_2(t, Y)$, we remark that from inequalities $\delta q_0(\tau_2) \leqq -t^{1/2-\varepsilon}$ and $\delta q_0(t) > 0$, we have $\delta q_0(\tau_2, t) \geqq t^{1/2-\varepsilon}$, which gives $(t-\tau_2)^{1/2+\varepsilon} > t^{1/2-\varepsilon}$. Here we again use a stronger version of Theorem 1. Thus $t - \tau_2 > t^{1-5\varepsilon}$ for sufficiently small ε. From another side

$$\begin{gathered} q + v\tau_2 + v(t-\tau_2) = q_0(\tau_2) + v(t-\tau_2) < q_0, \\ v(t-\tau_2) \leqq -\delta q_0(\tau_2) \leqq t^{1/2+\varepsilon}, \\ v \leqq t^{1/2+\varepsilon-1+5\varepsilon} = t^{-1/2+6\varepsilon}. \end{gathered}$$

This yields

$$n_2(t, Y) \leqq \operatorname{card}\{(q, v) \in X : q + tv \in [q_0 - t^{1/2+\varepsilon}, q_0],\ 0 \leqq v \leqq t^{-1/2+6\varepsilon}\}.$$

This immediately gives $n_2(t, \cdot) = o(\sqrt{t})$ in probability, and thus (4) is proven in case $\delta q_0(t) > 0$.

Now we assume $\delta q_0(t) \leqq 0$. Let us introduce

$$\mathscr{F}(t, Y) = \{(q, v) \in X : v > 0, q < q_0, q + tv > q_0(t)\}.$$

It is clear that $\mathscr{A}^+(t, Y) \subseteqq \mathscr{F}(t, Y)$ when $\delta q_0(t) \leqq 0$. We shall show that

$$\operatorname{card}\{\mathscr{B}(t, Y) \backslash \mathscr{F}(t, Y)\} = o(\sqrt{t})$$

in probability. We have

$$\mathscr{B}(t, Y) \backslash \mathscr{F}(t, Y) = \mathscr{G}(t, Y) \cup \mathscr{G}_+(t, Y),$$

where

$$\mathscr{G}_- = \{(q, v) \in X : (q, v) \in \mathscr{B}(t, Y), v < 0\},$$

$$\mathscr{G}_+ = \{(q, v) \in X : (q, v) \in \mathscr{B}(t, Y), v > 0, q + tv < q_0(t)\}.$$

For $(q, v) \in \mathscr{G}_\pm$, we put

$$\tau((q, v)) = \inf\{s : s \in [0, t], q(s) = q_0(s)\}$$

and

$$\mathscr{G}_- = \mathscr{G}_-^{(1)} \cup \mathscr{G}_-^{(2)},$$

$$\mathscr{G}_-^{(1)} = \{(q, v) : \tau((q, v)) < t^{3/4}\},$$

$$\mathscr{G}_-^{(2)} = \{(q, v) : \tau((q, v)) \geqq t^{3/4}\}.$$

For $(q, v) \in \mathscr{G}_-^{(1)}$,

$$q \geqq q_0(\tau) \geqq q_0 - \tau^{1/2+\varepsilon} \geqq q_0 - t^{3/4(1/2+\varepsilon)},$$

i.e. $q \in [q_0 - t^{3/8+3/4\varepsilon}, q_0]$, and therefore $\operatorname{card}\{\mathscr{G}_-^{(1)}(t, \cdot)\} = O(\sqrt{t})$ in probability. If $(q, v) \in \mathscr{G}_-^{(2)}$, then

$$q \geqq q + t^{3/4} v \geqq \min_{0 \leqq s \leqq t} q_0(s) \geqq q_0 - t^{1/2+\varepsilon}, \qquad |v| < t^{-1/4+\varepsilon}.$$

Using also $q \in [q_0 - t^{1/2+\varepsilon}, q_0]$, we easily have that

$$\operatorname{card}\{\mathscr{G}_-^{(2)}(t, \cdot)\} = o(\sqrt{t}),$$

and therefore $\operatorname{card}\{\mathscr{G}_-(t, \cdot)\} = o(\sqrt{t})$ in probability.

For the estimation of $\operatorname{card}\{\mathscr{G}_+\}$, we remark as before that

$$q + \tau v = q_0(\tau) \in [q_0(t) - t^{1/2+\varepsilon}, q_0(t)].$$

Again we decompose $\mathscr{G}_+ = \mathscr{G}_+^{(1)} \cup \mathscr{G}_+^{(2)}$, where

$$\mathscr{G}_+^{(1)} = \{(q, v) \in X : (q, v) \in \mathscr{G}_+, q + \tau v = q_0(\tau) < q_0(t) - t^{1/2-\varepsilon}\},$$

$$\mathscr{G}_+^{(2)} = \{(q, v) \in X : (q, v) \in \mathscr{G}_+, q + \tau v = q_0(\tau) \geqq q_0(t) - t^{1/2-\varepsilon}\}.$$

In the first case as above

$$q_0(\tau)=q+\tau v<q_0(t)-t^{1/2-\varepsilon}, \qquad \delta q_0(\tau,t)>t^{1/2-\varepsilon},$$
$$(t-\tau)^{1/2+\varepsilon}>t^{1/2-\varepsilon}, \qquad t-\tau>t^{1-4\varepsilon}.$$

Also

$$q_0(t)>q+tv>q+\tau v+vt^{1-4\varepsilon}=q_0(\tau)+vt^{1-4\varepsilon},$$

which gives $vt^{1-4\varepsilon}\leqq q_0(t)-q_0(\tau)$ and $0<v\leqq 2t^{1/2+\varepsilon}t^{-1+4\varepsilon}=2t^{-1/2+5\varepsilon}$. Using the inclusion

$$q+tv\in[q_0-t^{1/2+\varepsilon},q_0],$$

we easily get that $\mathrm{card}\{\mathscr{G}_+^{(1)}(t,\cdot)\}/\sqrt{t}\to 0$ in probability. If $(q,v)\in\mathscr{G}_+^{(2)}$, then

$$q+tv\in[q_0(\tau),q_0(t)]\subset[q_0(t)-t^{1/2-\varepsilon},q_0(t)]\subset[q_0-2t^{1/2+\varepsilon},q_0]. \tag{3.3}$$

We shall use the following lemma.

Lemma 3.1. *Let be given positive numbers $s>1$, $1>\alpha>\tilde{\beta}>0$, and the segment $\Delta_s=[-s,0]$. Put for every $t>0$ and a segment Δ,*

$$\eta(t,\Delta,Y)=\mathrm{card}\{(q,v)\in X: v>0, q+tv-q_0\in\Delta\}.$$

Then

$$\max_{\Delta\subset\Delta_s,|\Delta|<s^{\beta}}\eta(t,\Delta,Y)\leqq\xi(t,Y)s^{\alpha},$$

where $\xi(t,Y)$ is a stationary process (in t), $E(|\xi(t,Y)|)<\infty$.

This lemma will be proven later. Using (5) and Lemma 1 with $s^{\tilde{\beta}}=t^{1/2-\varepsilon}$, $s=2t^{1/2+\varepsilon}$, $s^{\alpha}=t^{1/2-\varepsilon}$, we get that $\mathrm{card}\{\mathscr{G}_+^{(2)}(t,\cdot)\}=o(\sqrt{t})$ in probability. Thus $\mathrm{card}\{\mathscr{B}(t,Y)\backslash\mathscr{F}(t,Y)\}=o(\sqrt{t})$ in probability.

The next step is to show that in probability,

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}\left[\mathrm{card}\{\mathscr{F}(t,Y)\backslash\mathscr{A}^+(t,Y)\}-\frac{\varrho}{2}|\delta q_0(t,Y)|\right]=0. \tag{3.4}$$

The set $\mathscr{F}(t,Y)\backslash\mathscr{A}^+(t,Y)$ consists of particles for which

$$q+tv\in[q_0(t,Y),q_0].$$

We shall use the following lemma.

Lemma 3.2. *Let us put for $t>0$, $\varepsilon>0$,*

$$\zeta_{\varepsilon,t}(Y)=\sup_L\frac{\left|\mathrm{card}\{(q,v)\in X: v>0, q+tv-q_0\in[-L,0]\}-\frac{\varrho}{2}L\right|}{L^{1/2+\varepsilon}}.$$

Then $E|\zeta_{\varepsilon,t}(Y)|$ is uniformly (in t) bounded.

The proof of Lemma 3.2 is an simple exercise and we omit the proof. Using it we have

$$\begin{aligned}\operatorname{card}\{\mathscr{F}(t,Y)\} &= \operatorname{card}\{\mathscr{B}(t,Y)\} + o(\sqrt{t})\\ &= \operatorname{card}\{\mathscr{A}^{+}(t,Y)\} + \operatorname{card}\{\mathscr{F}(t,Y)\backslash\mathscr{A}^{+}(t,Y)\} + o(\sqrt{t})\\ &= A^{+}(t,Y) + \frac{\varrho}{2}|\delta q_0(t,Y)|\,.\end{aligned}$$

This gives

$$A^{+}(t,Y) = B(t,Y) + \frac{\varrho}{2}\delta q_0(t,Y) + o(\sqrt{t})\,.$$

Now we can finish the proof of the theorem. Let us introduce the involution $\varphi:\Omega\to\Omega$, where $\varphi(Y)=Y'$ and Y' appears after changing the signs of all velocities. It is clear that $\varphi\circ T^t = T^{-t}\circ\varphi$. Now we remark that

$$\begin{aligned}A^{+}(t,\varphi\circ T^t(Y)) &= A^{-}(t,Y)\,,\\ B(t,\varphi\circ T^t(Y)) &= B(t,Y)\,,\\ \delta q_0(t,Y) &= -\delta q_0(t,\varphi\circ T^t(Y))\,.\end{aligned}$$

This gives

$$\begin{aligned}B(t,Y) &= A^{+}(t,Y) - \frac{\varrho}{2}\delta q_0(t,Y) + \alpha_1(t,Y)\\ &= B(t,\varphi\circ T^t(Y)) = A^{+}(t,\varphi\circ T^t(Y))\\ &\quad - \frac{\varrho}{2}\delta q_0(t,\varphi\circ T^t(Y)) + \alpha_1(t,\varphi\circ T^t(Y))\\ &= A^{-}(t,Y) + \frac{\varrho}{2}\delta q_0(t,Y) + \alpha_2(t,Y)\,,\end{aligned}$$

where $\alpha_1(t,\cdot)=o(\sqrt{t})$ in probability. Also

$$\alpha_2(t,\cdot) = \alpha_1(t,\varphi\circ T^t(\cdot)) = o(\sqrt{t})$$

in probability because $\varphi\circ T^t$ is measure-preserving. Now we have

$$\delta q_0(t,Y) = \frac{1}{\varrho}(A^{+}(t,Y) - A^{-}(t,Y)) + o(\sqrt{t})\,.$$

Putting $a^{\pm}(t,Y) = A^{\pm}(t,Y) - E(A^{\pm}(t,Y))$, and taking into account

$$E(A^{+}(t,Y)) = E(A^{-}(t,Y))$$

we get the statement of the theorem. Q.E.D.

Proof of Lemma 3.1. First we shall prove the lemma in a particular case $t=0$. It is sufficient to assume that s is an integer. Indeed, for $\Delta_s=[-s,0]$ with arbitrary $s\geqq 1$,

$$\begin{aligned}\sup_{\Delta\subset\Delta_s,|\Delta|<s^{\tilde\beta}}\eta(0,\Delta,Y)&\leqq\sup_{\Delta\subset\Delta_{[s]+1},|\Delta|\leqq s^{\tilde\beta}}\eta(0,\Delta,Y)\\&\leqq\sup_{\Delta\subset\Delta_{[s]+1},|\Delta|\leqq([s]+1)^{\tilde\beta}}\eta(0,\Delta,Y)\leqq\xi(Y)([s]+1)^{\alpha}\\&\leqq\xi^*(Y)\cdot s^{\alpha},\end{aligned}$$

where $\xi^*(Y)=\xi(Y)\sup\limits_{s\geqq 1}\frac{([s]+1)^{\alpha}}{s^{\alpha}}$.

Let us choose $\gamma:\tilde\beta<\gamma<\alpha$ and put

$$\Delta_s^{(j)}=[-(j+2)s^{\gamma},-js^{\alpha}],\qquad L(s)=1+[s^{1-\gamma}].$$

Then $\bigcup\limits_{j=0}^{L(s)}\Delta_s^{(j)}\supseteqq\Delta_s$. Also we remark that the Lebesgue number of the covering $\{\Delta_s^{(j)}\}$ is equal to s^{γ}. It means that if $|\Delta|<s^{\gamma}$ and $\Delta\subset\Delta_s$, then for some j, $0\leqq j\leqq L(s)$ we shall have $\Delta\subseteqq\Delta_s^{(j)}$. Denote

$$\eta_s^{(j)}(Y)=\operatorname{card}\{(q,v)\in X:q\in\Delta_s^{(j)},v>0\}.$$

Then $\eta_s^{(j)}$ has Poisson distribution with the parameter $\frac{1}{2}\varrho|\Delta_s^{(j)}|=\varrho s^{\gamma}$. Thus we have

$$\begin{aligned}\mu_0\left\{Y:\max_{0\leqq j\leqq L(s)}\eta_s^{(j)}\geqq s^{\alpha}\right\}&\leqq\sum_{j=0}^{L(s)}\mu_0\{\eta_s^{(j)}\geqq s^{\alpha}\}\\&\leqq\sum_{j=0}^{L(s)}\mu_0\{Y:\eta_s^{(j)}(Y)\geqq\varrho s^{\gamma}+(s^{\alpha}-\varrho s^{\gamma})\}\\&\leqq\sum_{j=0}^{L(s)}\exp\left\{-c\frac{(s^{\alpha}-\varrho s^{\gamma})^2}{\varrho s^{\gamma}}\right\}\leqq L(s)\exp\{-\tilde c\cdot s^{\delta}\}.\end{aligned}$$

Therefore

$$\sum_{s\in\mathbb{N}}\mu_0\left\{Y:\max_{0\leqq j\leqq jL(s)}\eta_s^{(j)}\geqq s^{\alpha}\right\}<\infty.$$

The Borel-Cantelli lemma can be applied and it gives the existence a.e. of $\xi(Y)$ such that for all integer s we shall have

$$\max_{0\leqq j\leqq L(s)}\eta_s^{(j)}(Y)\leqq\xi(Y)s^{\alpha},\qquad E(\xi(Y))<\infty.$$

Now let $\Delta\subset\Delta_s$, $|\Delta|\leqq s^{\tilde\beta}<s^{\gamma}$. Then there exists $j:j\in\{0,\dots,L(s)\}$ such that $\Delta\subset\Delta_s^{(j)}$. Therefore

$$\eta(0,\Delta,Y)\leqq\eta_s^{(j)}(Y)\leqq\xi(Y)s^{\alpha}.$$

Thus for $t=0$ the lemma is proven.

In the general case we introduce the flow $\{\overset{0}{T}{}^t\}$ in the space Ω such that $\overset{0}{T}{}^t(\{x_0,X\})=\{x_0,X'\}$, where X' appears from X under the action of the free dynamics. The flow $\{\overset{0}{T^t}\}$ preserves the measure μ_0. Now

$$\eta(t, \Delta, Y) = \eta(0, \Delta, \overset{0}{T^t}(Y)).$$

Therefore the statement of Lemma 1 is fulfilled for $t \geqq 0$ with a constant $\xi\left(\overset{0}{T^t}(Y)\right)$ instead of $\xi(Y)$. But $\xi\left(\overset{0}{T^t}(Y)\right)$ is a stationary random process. This completes the proof. Q.E.D.

4. An Estimation of the Diffusion Coefficient from Below

We prove in this section the following theorem.

Theorem 4. *For each $t > 0$ the normalized displacement $q_0(t)/\sqrt{t}$ has the following representation,*

$$\frac{q_0(t)}{\sqrt{t}} = \gamma(t) + \xi(t) + \zeta(t),$$

where the random variable $\gamma(t)$ has Gaussian distribution with the expectation 0 *and the variance $\sigma^2(\varrho, \beta) = \sqrt{\frac{\pi}{8}} \frac{1}{\varrho\sqrt{\beta}}$, not depending on M, $\zeta(t)$ is a random variable, independent of $\gamma(t)$, and $\xi(t) \to 0$ in μ_0 probability.*

Proof of Theorem 4 consists of several steps.

1°. Let us introduce the subset $\Gamma_0(t) = \bigcup_{i=1}^{4} \Gamma_0^{(i)}(t)$, where $\Gamma_0^{(i)}(t) \subset \mathbb{R}^2$, $t > 0$, $\varepsilon > 0$. Namely

$$\begin{aligned}
\Gamma_0^{(1)} &= \{(q, v) : -t^{1/2+\varepsilon} < q \leqq 0, -\infty < v < +\infty\},\\
\Gamma_0^{(2)} &= \{(q, v) : q \leqq -t^{1/2+\varepsilon}, -(q + t^{1/2+\varepsilon})/v < t\},\\
\Gamma_0^{(3)} &= \{(q, v) : (-q, -v) \in \Gamma_0^{(1)} \cup \partial\Gamma_0^{(1)}\},\\
\Gamma_0^{(4)} &= \{(q, v) : (-q, -v) \in \Gamma_0^{(2)} \cup \partial\Gamma_0^{(2)}\}.
\end{aligned}$$

Less formally if a particle $(q, v) \in \mathbb{R}^2 \setminus \Gamma_0(t)$, then it would remain outside the space interval $(-t^{1/2+\varepsilon}, t^{1/2+\varepsilon}]$ during the time interval $(0, t)$. By L_a, $a \in \mathbb{R}^1$ we denote the following transformation $\mathbb{R}^2 \to \mathbb{R}^2 : L_a(q, v) = (q, v + a)$, and $\Gamma_a(t) = L_a(\Gamma_0(t))$.

For any locally finite set of particles W the functions $f_1(a) = \chi_{\Gamma_a}(w)$, $f_2(a) = \chi_{\mathbb{R}^2 \setminus \Gamma_a}(w)$ are right-continuous.

2°. Let $\Omega^{(f)}$ be the phase space of the system, consisting of a finite number of particles, including m.p. A point $F \in \Omega^{(f)}$ has the form $F = \{v_0, v_1, \ldots, v_j, q_1, \ldots, q_j\}$, where (q_i, v_i) are parameters of a particle $x_i = (q_i, v_i) \in F$, $j + 1 = \kappa(F)$ is the total number of particles.

Then $\Omega^{(f)}=\bigcup_{k=1}^{\infty}\Omega_k^{(f)}$, where $\Omega_j^{(f)}$ consists of F for which $\kappa(F)=j$. We introduce a non-normed measure m on $\Omega^{(f)}$ whose restriction to $\Omega_{j+1}^{(f)}$ has the form

$$dm(F)=\frac{\varrho^j}{j!}\left(\frac{\beta}{2\pi}\right)^{\frac{j+1}{2}}\sqrt{M}\exp\left\{-\frac{\beta}{2}Mv_0^2-\frac{\beta}{2}\sum_{i=1}^{j}v_i^2\right\}\prod_{i=1}^{j}dq_i\prod_{i=0}^{j}dv_i.$$

By $\mathscr{P}$ we denote as before the limit Gibbs measure corresponding to the density ϱ and inverse temperature β, which is a normed probability measure on $\Omega(X)$. We have the natural mapping of the direct product of two measure spaces

$$\varphi:(\Omega^{(f)},m)\otimes(\Omega(x),\mathscr{P})\to\Omega_0,$$

where $\varphi(F,X)=F\cup X$, which is defined a.e. with respect to $m\times\mathscr{P}$.

Lemma 4.1. *Let $A\subset\mathbb{R}^2$ be a Borel subset containing the line $q=0$. Then for any $f\in L^1(\Omega_0,\mu_0)$,*

$$\int f(Y)d\mu_0(Y)=\int_{\Omega^{(f)}}\int_{\Omega(x)}dm(F)d\mathscr{P}(X)\chi_{\mathbb{R}^2\setminus A}(x)\chi_A(F)f(\varphi(F,x)),\tag{4.1}$$

where χ is the indicator of the corresponding set.

Proof. For any set $A\subset\mathbb{R}^2$ for the conditional measure $d\mu_0(Y_A|Y_{\mathbb{R}^2\setminus A})$, provided that $Y_{\mathbb{R}^2\setminus A}=Y\cap(\mathbb{R}^2\setminus A)$ is fixed, we have

$$d\mu_0(Y_A|Y_{\mathbb{R}^2\setminus A})=\frac{1}{\Xi(A)}dm(Y_A),$$

where the partition function $\Xi(A)=\exp\left(\frac{\varrho\sqrt{\beta}}{\sqrt{2\pi}}\int_A e^{-\frac{v^2\beta}{2}}dv\,dq\right)$. The induced measure $d\mu_0(Y_{\mathbb{R}^2\setminus A})$ is the measure of the ideal gas in $\mathbb{R}^2\setminus A$. But $\Xi^{-1}(A)d\mu_0(Y_{\mathbb{R}^2\setminus A})$ is precisely the measure on the space of X, for which $X\cap A=\emptyset$ because $\Xi^{-1}(A)$ is the probability for the measure $\mathscr{P}$ that none of particles is in A QED.

3°. From (4.1) we have for any Γ_a,

$$\int f(Y)d\mu_0(Y)=\int_{\Omega^{(f)}}\int_{\Omega(x)}dm(F)d\mathscr{P}(x)\chi_{\Gamma_a}(F)\cdot\chi_{\mathbb{R}^2\setminus\Gamma_a}(x)\,f(\varphi(F,X)).$$

We shall show that a similar formula is valid when a is a function of $Y=\varphi(F,X)$.

Lemma 4.2. *Let $a=a(Y)$ be a measurable function of Y. Then*

$$\int f(Y)d\mu_0(Y)=\int_{\Omega^{(f)}}\int_{\Omega(X)}dm(F)d\mathscr{P}(x)\chi_{\Gamma_{a(\varphi(F,x))}}(F)\cdot\chi_{\bar{\Gamma}_{a(\varphi(F,X))}}(x)f(\varphi(F,X)).\tag{4.2}$$

Proof. Assume that $a(Y)$ takes values $a_1,a_2,\dots,a_n,\dots$ on subsets $C_1,C_2,\dots,C_i\subseteqq\Omega_0$. Then

$$f(Y)=\sum_i f(Y)\chi_{C_i}(Y),$$

and

$$\int f(Y)d\mu_0(Y)=\sum_i \int f(Y)\chi_{C_i}(Y)d\mu_0(Y).$$

For each i we apply (4.1) to $f(Y)\chi_{C_i}(Y)$. This gives

$$\int f(Y)\chi_{C_i}(Y)d\mu_0(Y)=\int_{\Omega^{(f)}}\int_{\Omega(x)} dm(F)d\mathscr{P}(X)$$
$$\cdot\chi_{\Gamma_{a_i}}(F)\chi_{\mathbb{R}^2\setminus\Gamma_{a_i}}f(\varphi(F,X))\chi_{C_i}(\varphi(F,X)),$$
$$\int f(Y)d\mu_0(Y)=\sum_i\int f(Y)\chi_{C_i}(Y)d\mu_0(Y)$$
$$=\int_{\Omega^{(f)}}\int_{\Omega(X)} dm(F)d\mathscr{P}(x)\sum_i\chi_{\Gamma_{a_i}}(F)\chi_{\mathbb{R}^2\setminus\Gamma_{a_i}}(x)\chi_{C_i}(\varphi(F,X))\cdot f(\varphi(F,X)),$$

i.e. we have (4.2) for functions $a(Y)$ taking a finite or countable number of values.

Let be $a(Y)$ be any measurable bounded function, $\|a\|_\infty<\infty$. We may find a sequence of a function $a_n(Y)$ such that each a_n takes only not more than a countable number of values, $a_n(Y)\downarrow a_1(Y)$ as $n\to\infty$ and $\|a_n\|_\infty\leqq\text{const}<\infty$. Then

$$\int f(Y)d\mu_0(Y)=\lim_{n\to\infty}\int_{\Omega^{(f)}}\int_{\Omega(X)} dm(F)d\mathscr{P}(X)\chi_{\Gamma_{a_n(\varphi(F,X))}}(F)$$
$$\cdot\chi_{\mathbb{R}^2\setminus\Gamma_{a_n(\varphi(F,X))}}f(\varphi(F,X))$$
$$=\int_{\Omega^{(f)}}\int_{\Omega(X)} dm(F)d\mathscr{P}(X)\chi_{\Gamma_{a(\varphi(F,X))}}(F)\chi_{\mathbb{R}^2\setminus\Gamma_{a(\varphi(F,X))}}(X)f(\varphi(F,X)).$$

The last equality is true because the functions $\chi_{\Gamma_a}(F)$ and $\chi_{\mathbb{R}^2\setminus\Gamma_a}(X)$ are right-continuous, and the sequence of functions

$$\chi_{\Gamma_{a_n(\varphi(F,X))}}(F)\cdot\chi_{\mathbb{R}^2\setminus\Gamma_{a_n(\varphi(F,X))}}(X)$$

is uniformly integrable if $\|a_n\|_\infty\leqq\text{const}<\infty$. Let $a(Y)$ be any measurable function. Then (4.2) holds for such f that

$$\operatorname{supp}(f)\subset\{Y:|a(Y)\leqq N\}=A_N$$

for some N. But $A_N\uparrow\Omega_0$ and this proves the result.

4°. Let us take the C^∞ function $f_1(z)$ with a compact support and put $f(Y)=f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)$, $a(Y)=\frac{q_0(t)}{t}$. Then from (4.2)

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)d\mu_0(Y)=\int_{\Omega^{(f)}}\int_{\Omega(X)} dm(F)d\mathscr{P}(X)\chi_{\Gamma_{a(\varphi(F,X))}}(F)$$
$$\cdot\chi_{\mathbb{R}^2\setminus\Gamma_{a(\varphi(F,X))}}(X)\,f_1\left(\frac{q_0(t,\varphi(F,X))}{\sqrt{t}}\right),$$
$$a(\varphi)=\frac{1}{t}q_0(t,\varphi).\tag{4.3}$$

The natural flow in $\Omega^{(f)}$ generated by the dynamics of particles of F is denoted by $\{\hat{T}^t\}$, $\hat{q}(t)$ is the coordinate of m.p. at t, $\hat{a}(F)=\frac{1}{t}\hat{q}(t)$. We shall show that with μ_0

probability tending to 1 as $t\to\infty$:

$$a(\varphi(F,X))=\hat{a}(F),$$

$$\chi_{\Gamma_{a(\varphi(F,X))}(t)}(F)\chi_{\mathbb{R}^2\setminus\Gamma_{a(\varphi(F,X))}}(X)f_1(a(\varphi(F,X))\sqrt{t})$$
$$=\chi_{\Gamma_{\hat{a}(F)}}(F)\cdot\chi_{\mathbb{R}^2\setminus\Gamma_{\hat{a}(F)}}(X)f_1(\hat{a}(F)\sqrt{t}).$$

Using this equality we get from (4.3)

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)d\mu_0(Y)=\int_{\Omega(f)}\chi_{\Gamma_{\hat{a}(F)}}(F)f_1(\hat{a}(F)\sqrt{t})dm(F)$$
$$\int_{\Omega(X)}\chi_{\mathbb{R}^2\setminus\Gamma_{\hat{a}(F)}}(X)d\mathscr{P}(X)+\varepsilon_1(t),\tag{4.5}$$

where $\varepsilon_1(t)\to 0$ as $t\to\infty$. In fact, one can get more precise information about the order of decay of $\varepsilon_1(t)$. Further,

$$\int_{\Omega(X)}\chi_{\mathbb{R}^2\setminus\Gamma_{\hat{a}(F)}}(X)d\mathscr{P}(X)=\exp\left\{-\frac{\sqrt{\beta}\varrho}{\sqrt{2\pi}}\int_{\Gamma_{\hat{a}(F)}}e^{-\frac{\beta v^2}{2}}dv\,dq\right\},$$

and

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)d\mu_0(Y)=\int_{\Omega(f)}\chi_{\Gamma_{\hat{a}(F)}}(F)f_1(\hat{a}(F)\sqrt{t})dm(F)$$
$$\cdot\exp\left\{-\frac{\sqrt{\beta}\varrho}{\sqrt{2\pi}}\int_{\Gamma_{\hat{a}(F)}(t)}\exp\left\{-\frac{\beta v^2}{2}\right\}dv\,dq\right\}+\varepsilon_1(t).\tag{4.6}$$

5°. Now we shall prove (4.4). The equality

$$\frac{1}{t}q_0(t,\varphi(F,X))=\frac{1}{t}\hat{q}_0(t,F)$$

holds if the dynamics of m.p. is the same for the flow $\{T^t\}$ and for the flow $\{\hat{T}^t\}$ during the time interval $(0,t)$. This will happen if $|q_0(s)|<t^{1/2+\varepsilon/2}$ for all $s\in[0,t]$. Indeed, in this case $|a|=\left|\frac{q_0(t)}{t}\right|\leqq t^{-1/2+\varepsilon/2}$, and if $X\subset\mathbb{R}^2\setminus\Gamma_a$, then it consists of particles which do not reach $[-\frac{1}{2}t^{1/2+\varepsilon},\frac{1}{2}t^{1/2+\varepsilon}]$, and therefore do not interact with m.p. during the time interval $[0,t]$. Thus $q_0(t)=\hat{q}(t)$ for $F=Y\cap\Gamma_a$. Q.E.D.

6°. In this subsection we shall investigate the behaviour of

$$\Delta(\Gamma_a)=\int_{\Gamma_a}\exp\left\{-\frac{\beta v^2}{2}\right\}dv\,dq.$$

We have $\Gamma_a\Delta\Gamma_0=S_1\cup S_2$ (see Fig. 1), and $\Delta(\Gamma_a)=\Delta(\Gamma_0)+\Delta(S_2)-\Delta(S_1)$. Further,

$$\Delta(S_1)=\left[ta\int_a^\infty e^{-\frac{\beta v^2}{2}}dv+\int_0^a tve^{-\frac{\beta v^2}{2}}dv\right],$$

$$\Delta(S_2)=\left[t\cdot a\int_0^\infty e^{-\frac{\beta v^2}{2}}dv+\int_0^a t(a-v)e^{-\frac{\beta v^2}{2}}dv\right],$$

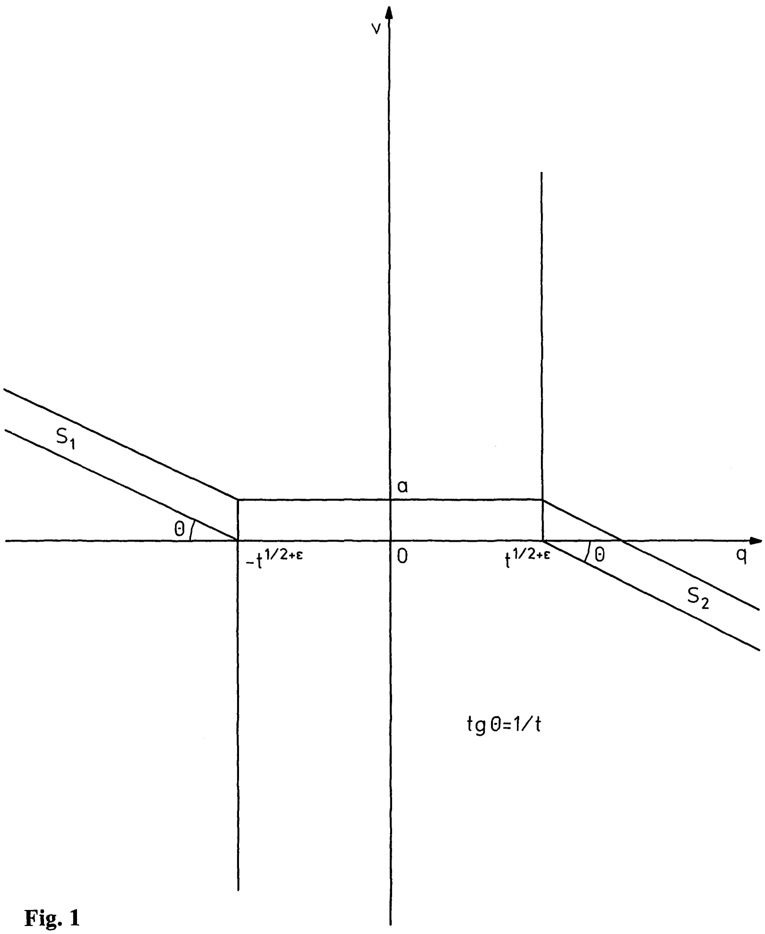

Fig. 1

and therefore

$$\Delta(S_2)-\Delta(S_1)=ta\int_0^a e^{-\frac{\beta v^2}{2}}dv+t\int_0^a (a-2v)e^{-\frac{\beta v^2}{2}}dv$$
$$=2ta\int_0^a e^{-\frac{\beta v^2}{2}}dv-2t\int_0^a ve^{-\frac{\beta v^2}{2}}dv$$
$$=2ta^2+\frac{2t}{\beta}\left(e^{-\frac{\beta a^2}{2}}-1\right)+\delta(t,a),$$

where $|\delta(t,a)|\leqq \mathrm{const}|a|^3\cdot t$. For $|a|<t^{-1/2+\varepsilon/2}$, we have

$$|\delta(t)|\leqq \mathrm{const}\, t^{-1/2+\frac{3\varepsilon}{2}}.$$

This yields

$$\Delta(S_2)-\Delta(S_1)=2t\left(a^2-\frac{a^2}{2}\right)+\delta_1(t,a), \quad |\delta_1(t,a)|\leqq ct^{-1/2}+\frac{3\varepsilon}{2}.$$

The substitution of the last expression into (4.6) gives

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right) d\mu_0(Y)=\exp\left(-\varrho\sqrt{\frac{\beta}{2\pi}}\Delta(\Gamma_0)\right)$$
$$\cdot\int_{\Omega^{(f)}} \chi_{\Gamma_{\hat{a}(F)}}\cdot f_1(\hat{a}(F)\sqrt{t})\exp\left\{-t(\hat{a}(F))^2\varrho\sqrt{\frac{\beta}{2\pi}}\right\}\cdot$$
$$\cdot\, dm(F)\cdot(1+\varepsilon_2(t)), \tag{4.7}$$

where $\varepsilon_2(t)\to 0$, $t\to\infty$.

7°. We shall introduce new coordinates in $\Omega^{(f)}$ which will be very useful for further calculations. Denote by $\hat{\alpha}$ the measurable partition induced by the function $\hat{a}(F)$. An element $C_z^{(\hat{\alpha})}$ of this partition consists of all F for which $\hat{a}(F)=z$. It is easy to see that $C_z^{(\hat{\alpha})}=L_z C_0^{(\hat{\alpha})}$ (for the definition of L_z see 1°).

In the space $\Omega_{0,j}^{(f)}$ of all $F\in C_0^{(\hat{\alpha})}$ with $\kappa(F)=j+1$, the element $C_0^{(\hat{\alpha})}$ is mod0, a countable union of open subsets of $(2j)$-dimensional planes. Here we use essentially the one-dimensional character of the dynamics. Each L_a shifts points $F\in\Omega_{0,j}^{(f)}$ along the normal lines to these subsets. It means that for any $F\in\Omega_{0,j}^{(f)}$ one can find in a unique way $a\in\mathbb{R}^1$ and $V\in C_0^{(\hat{\alpha})}$ such that $F=L_aV$. We shall use a, V as new coordinates in $\Omega_j^{(f)}$.

The next problem is to express the measure $dm(F)$ in coordinates a, V. It follows from what has been said above that

$$\prod_{i=1}^{j} dq_i \prod_{i=0}^{j} dv_i = da\cdot dV,$$

where dV is the Lebesgue measure on each $C_0^{(\hat{\alpha})}$. Further,

$$H(F)=H(v_0,v_1,\ldots,v_j,q_1,\ldots,q_j)=\frac{Mv_0^2}{2}+\tfrac{1}{2}\sum_{i=1}^{j} v_i^2$$
$$=\frac{M(v_0-a)^2}{2}+\tfrac{1}{2}\sum_{i=1}^{j}(v_i-a)^2+aM(v_0-a)$$
$$+a\sum_{i=1}^{j}(v_i-a)+\frac{M+j}{2}a^2.$$

The point $(v_0-a, v_1-a,\ldots,v_j-a,q_1,\ldots,q_j)=V\in C_0^{(\hat{\alpha})}$. Thus

$$H(F)=\frac{M+j}{2}\left(a+\frac{1}{M+j}\sum_{i=1}^{j}(v_i-a)\right)^2-\left[\frac{1}{M+j}\sum_{i=1}^{j}(v_i-a)\right]^2+H(V).$$

Let us put $E(V)=\dfrac{-1}{M+\kappa(V)}\sum_{i=1}^{\kappa(V)} v_i$. Then

$$H(F)=\frac{M+\kappa(V)}{2}(\hat{a}(F)-E(V))^2+H(V)-(E(V))^2.$$

Now we can write

$$dm(F)=\exp\{-\beta H(F)\}\prod_{i=1}^{j} dq_i \prod_{i=0}^{j} dv_i \cdot c(j)$$
$$=\exp\left\{-\beta\left[\frac{M+\kappa(V)}{2}(a-E(V))^2\right.\right.$$
$$\left.\left.+H(V)-(E(V))^2\right]\right\} da\, dV\cdot c(\kappa)\,.$$

This formula shows that the conditional distribution of $\hat{a}(F)\sqrt{t}$ for any fixed V is gaussian with the expectation $\sqrt{t}E(V)$ and the variance $\frac{M+\kappa(V)}{t}$. We shall show later that

$$\lim_{t\to\infty}\frac{M+\kappa(V)}{t}=\lim_{t\to\infty}\frac{\kappa(V)}{t}=\lim_{t\to\infty}\frac{1}{t}\Delta(\Gamma_0(t))\varrho\sqrt{\frac{\beta}{2\pi}}=\sigma_1$$

in μ_0-probability, where σ_1 is a constant depending only on β, ϱ and not depending on M. In fact one can prove the convergence with μ_0-probability 1, but for our goals it is sufficient to have a weaker result. Returning to (4.7) we can write

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)d\mu_0(Y)=\exp\left\{-\varrho\sqrt{\frac{\beta}{2\pi}}\Delta(\Gamma_0)\right\}$$
$$\cdot\int_{\Omega(f)}\chi_{\Gamma_{\hat{a}(F)}}(F)f_1(\hat{a}(F)\sqrt{t})\exp\left\{-t(\hat{a}(F))^2\varrho\sqrt{\frac{\beta}{2\pi}}\right\}dm(F)$$
$$+\varepsilon_3(t)=\exp\left\{-\varrho\sqrt{\frac{\beta}{2\pi}}\Delta(\Gamma_0)\right\}\sum_{j=0}^{\infty}\int f_1(a\sqrt{t})$$
$$\cdot\exp\left\{-(a\sqrt{t})^2\varrho\sqrt{\frac{\beta}{2\pi}}\right\}\exp\left\{-\beta\left[\frac{M+j}{2t}(a\sqrt{t}-\sqrt{t}E(V))^2\right.\right.$$
$$\left.\left.+H(V)-(E(V))^2\right]\right\}\chi_{\Gamma_0}(V)\cdot c(j)\cdot da\, dV+\varepsilon_3(t)$$
$$=\exp\left\{-\varrho\sqrt{\frac{\beta}{2\pi}}\Delta(\Gamma_0)\right\}\sum_{j=0}^{\infty}c(j)\int f_1(a\sqrt{t})\exp\left\{-\varrho\sqrt{\frac{\beta}{2\pi}}(a\sqrt{t})^2\right\}$$
$$\cdot\exp\left\{-\beta\left[\frac{\varrho}{\sqrt{2\pi\beta}}(a\sqrt{t}-\sqrt{t}E(V))^2+H(V)-(E(V))^2\right]\right\}\chi_{\Gamma_0}(V)da\,dV$$
$$+\varepsilon_n(t),\quad c(j)=\frac{\varrho^j}{j!}\left(\sqrt{\frac{\beta}{2\pi}}\right)^{j+1}\sqrt{M}\,.$$

Here $\varepsilon_3(t)\to 0$, $\varepsilon_1(t)\to 0$ as $t\to\infty$. Putting $z=a\sqrt{t}$, we have

$$\int f_1\left(\frac{q_0(t,Y)}{\sqrt{t}}\right)d\mu_0(Y)=\frac{\sqrt{\sigma}}{\sqrt{2\pi}}\int f_1(z)\exp\left\{-\frac{\sigma}{2}(z-E(V))^2\right\}$$
$$\cdot dG(V)+\varepsilon_n(t),\quad \sigma=4\varrho\sqrt{\frac{\beta}{2\pi}},$$

where G is the distribution function of the random variable $\sqrt{t}E(V)$. The last expression gives the statement of the theorem.

8°. We have to prove the existence of the limit $\lim_{t\to\infty}\frac{\kappa(V)}{t}$. It is easy to see that for $|a|\leqq t^{-1/2+\varepsilon/2}$, $|\kappa(V)-\kappa(F)|\leqq t^{1/2+\varepsilon}$ with a big μ_0 probability. Again by the same reasons

$$|\kappa(F)-|Y\cap\Gamma_0\|=\|Y\cap\Gamma_a|-|Y\cap\Gamma_0\|\leqq t^{1/2+\varepsilon}.$$

The existence of the limit $\lim_{t\to\infty}\frac{|Y\cap\Gamma_0(t)|}{t}$ is obvious.

5. Concluding Remarks

In the paper [10] Szasz and Toth got the estimation of Theorem 4 by a different method. From our considerations it follows also that if the limit probability distribution for $\frac{q_0(t)}{\sqrt{t}}$ exists then it is absolutely continuous and its density is analytic.

The results of [10] also shows that if the limiting variance $\lim_{t\to\infty}\frac{E(q_0^2(t))}{t}$ does not depend on M, then the limit probability distribution of $\frac{q_0(t)}{\sqrt{t}}$ is Gaussian, with the same variance. The authors of the present paper believe that it is more plausible that the variance depends on M.

Also all arguments of this paper can be applied to the case where m.p. interacts with particles of the ideal gas through a short-range repelling potential.

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