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Periodic Solutions in a Model of Pulsar Rotation

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Abstract. A rotating rigid body with ellipsoidal cavity filled with magnetic fluid is considered as a pulsar model. Dynamical equations for the pulsar model are derived and investigated, certain integrable cases are indicated. Three-parameter sets of periodic solutions integrable in terms of elliptic functions of the time variable are obtained. A formula is derived for the period of rotation and magneto-rotational oscillations of the pulsar.

Introduction

It is now generally acknowledged [1] that pulsars (neutron stars) have a solid envelope and a liquid core which has high conductivity (the liquid is plasma) and strong frozen magnetic fields; the liquid core contains the dominating part of the pulsar mass. "Starquakes" happen periodically in pulsars, and they can be observed as glitches of the pulsar rotation period. Asynchronous rotations of the pulsar core and envelope take place during the time intervals between two such phenomena. The relaxation time for the Vela pulsar (PSR 0833-45) is $\tau \approx 6$ years, while its rotation period is $P = 0.089$ s. Therefore the viscosity effects are negligible and an appropriate model of the pulsar core is that of the ideal incompressible magnetic fluid.

The model of the pulsar rotation which is proposed in the present work takes into account magnetic properties of the core and asynchronous rotation of the core and envelope (Sect. 1). The model is applicable for a finite time interval t, $P \ll t < \tau$, between two subsequent starquakes, where the energy losses to the viscous friction and electromagnetic radiation may be neglected.

The dynamical system describing the rotation of the pulsar model is derived in Sect. 2. It is a system of nine ordinary differential equations which are represented in a simple vector form, Eq. (2.12). This system has four first integrals J_k; J_1 is the total energy of the pulsar, J_2 is the total angular momentum squared, J_3 defines the magnitude of the frozen magnetic field, and J_4 is the scalar product of the curl of the fluid velocity by the magnetic intensity vector (Sect. 3).

The most important mathematical problem in the considered model is the existence of periodic solutions, as actually the pulsar rotation is periodic and the period is maintained with a high accuracy for a long time. It is shown in Sect. 4 that in every manifold of the first integral level $J_i = k_i\, k_4 = 0$ (for the domain of values of constants k_i) there are 12 closed trajectories of the dynamical system (2.12). These trajectories have been integrated explicitly in terms of elliptic functions of the time variable. Magneto-rotational oscillations for which the sign of the pulsar angular velocity periodically changes have been found for the present model. Such oscillations are in principle connected with the presence of magnetic fields. A formula is derived [Eq. (4.12)] expressing the minimal period of rotations and the magneto-rotational oscillations of the pulsar via its physical parameters. For the real values of the parameters the predicted period of rotation $T_0 \approx 1\, s$, which fully agrees with astrophysical data.

The dynamics of the pulsar model has some important mathematical properties: the dynamical system described by Eqs. (2.12) is a special case of Euler's equations in the space L^* which is dual to the Lie algebra associated with the group $E_3 \times SO(3)$, where E_3 is the group of motions of the three-dimensional Euclidean space. In the invariant manifolds corresponding to fixed values of the integrals $J_2 = k_2$, $J_3 = k_3$, $J_4 = k_4$ the system considered is of the Hamilton type, and its Hamiltonian is J_1. Some integrable cases are indicated in Sect. 3.

1. The Model of the Pulsar Rotation

The model of the pulsar dynamics is based on the following assumptions.

A. The pulsar envelope is absolutely rigid. Its liquid core has a constant density ϱ, and the fluid fills an ellipsoidal cavity with semi-axes d_1, d_2, d_3. The chosen reference frame S is fixed to the envelope, the origin of the frame is at the center of mass of the pulsar, and the coordinate axes are parallel to the principal axes of the ellipsoid. The center of the ellipsoid 0 has coordinates r^1, r^2, r^3 in the reference frame S.

B. The rotation of the pulsar envelope is represented with an orthogonal matrix $Q_1(t)$. The motion of the fluid in the cavity is described by magnetohydrodynamical equations [2] which are

$$\begin{gathered} \varrho dv/dt = -\operatorname{grad} p + (\operatorname{rot} H \times H)/4\pi - \varrho \operatorname{grad} \Phi\,, \\ \operatorname{div} v = 0\,, \qquad \partial H/\partial t = \operatorname{rot}(v \times H)\,, \qquad \operatorname{div} H = 0\,, \end{gathered} \tag{1.1}$$

where v is the velocity vector, p is the pressure, and H is the magnetic field intensity vector; Φ is the gravitational Newton potential inside the fluid. The motion of the fluid is a motion with homogeneous deformation [3, 4], and the transformation from the Lagrange coordinates a^k to the Euler coordinates x^i is

$$x^i = \sum_{k=1}^{3} (F^i_k a^k + (Q_1)^i_k r^k)\,, \qquad F = Q_1 D Q_2\,. \tag{1.2}$$

Here $Q_2(t)$ is an orthogonal matrix, $D_{ij} = d_i \delta_{ij}$; the Lagrange coordinates a^k lie within the unit sphere, $(a^1)^2 + (a^2)^2 + (a^3)^2 \leqq 1$.

C. The magnetic field H^i at the point with coordinates (1.2), is

$$H^i = \sum_{k,j=1}^{3} F^i_k h^k_j a^j , \tag{1.3}$$

where h^k_j is a constant skew-symmetrical matrix.

D. The electromagnetic field has a discontinuity at the fluid-envelope interface ϕ_1. On both sides of the interface the magnetic field is tangent to the ellipsoid surface and frozen in the medium. Magnetic lines of force from the envelope are closed in the surrounding vacuum.

We will show that all the necessary boundary conditions are fulfilled at the boundary of the ellipsoidal cavity, that is the discontinuity surface. Let H_n, H_τ, E_n, E_τ, v_n, v_τ be normal and tangent components of the magnetic field, electric field, and the fluid velocity at the surface. The boundary conditions known in magnetic hydrodynamics [2] are (thermal conductivity is neglected)

$$\{E_\tau\}=0\,, \quad \{E_n\}=4\pi\theta\,, \quad \{H_n\}=0\,, \quad \{H_\tau\}=4\pi c^{-1}(\mathbf{i}\times\mathbf{n})\,. \tag{1.4}$$

$$\{\varrho v_n\}=0\,, \quad \{s_n-(P\cdot\mathbf{v})\mathbf{n}+\varrho v_n(\varepsilon+\mathbf{v}^2/2)\}=0\,, \tag{1.5}$$

$$\{\varrho v_n\mathbf{v}-P\cdot\mathbf{n}-T\cdot\mathbf{n}\}=0\,, \tag{1.6}$$

$$\mathbf{s}=c(\mathbf{E}\times\mathbf{H})/4\pi\,, \quad P_{ij}=-p\delta_{ij}\,, \quad T_{ij}=(H_iH_j-H^2\delta_{ij}/2)/4\pi\,.$$

Here $\{X\}=X_+-X_-$ stands for the discontinuity of the quantity X at the interface, θ is the surface charge, $\mathbf{i}$ is the surface current, $\mathbf{n}$ is the normal vector to the surface, $\mathbf{s}$ is the vector of the electromagnetic energy flux density, P and T are matrices with components P_{ij}, T_{ij}; ε is the internal energy density of the fluid.

From the Eqs. (1.2), (1.3) we have $v_n=0$, $H_n=0$. In the approximation which is adopted in magnetic hydrodynamics $\mathbf{E}=-(\mathbf{v}\times\mathbf{H})/c$, so $E_\tau=0$. Consequently, conditions (1.4) are fulfilled and determine the surface current and charge density, in the envelope $H_n=0$, $E_\tau=0$. Conditions (1.5) are fulfilled, as $v_n=0$, $s_n=0$. Since $v_n=0$, $H_n=0$, conditions (1.6) lead to $\{p+H^2/8\pi\}=0$. The latter condition determines the pressure from the envelope, thus it is also fulfilled in the case of the absolutely rigid envelope.

The electromagnetic field has a discontinuity at the outer vacuum-envelope interface ϕ_2. The electromagnetic field in the surrounding vacuum may be, for example, the field of a magnetic dipole. It is supposed that this field rotates together with the pulsar; radiation of electromagnetic waves is not taken into account. The conditions (1.4), (1.6) determine the surface current and the pressure in the envelope at the boundary (the surface charge is $\theta=0$, because due to infinite electric conductivity of the envelope one has $E=0$ and hence $E_n=4\pi\theta=0$). Boundary values of the magnetic field, determined on two surfaces ϕ_1 and ϕ_2, are matched by some magnetic field H_0 inside the envelope. The Maxwell equations inside the envelope lead to the condition that the magnetic field H_0 is frozen and to the determination of the volume current in the envelope $j=c(4\pi)^{-1}\operatorname{rot}H_0$, and no other additional constraints arise.

The dynamics of the pulsar model is considered during time intervals for which the fluid viscosity and the energy loses due to electromagnetic radiation may be neglected.

2. Dynamics of a Rigid Body with an Ellipsoidal Cavity Filled with Magnetic Fluid

The equation of motion (relative to the center of mass) for a rigid body having a cavity filled with magnetic fluid are equations of magnetic hydrodynamics (1.1) combined with the conservation law for the total angular momentum. Let us introduce the notations

$$\dot{Q}_1 = Q_1 A\,, \qquad \dot{Q}_2 = -BQ_2\,, \tag{2.1}$$

and use the isomorphism between three-dimensional vectors and skew-symmetrical 3×3 matrices in space R^3,

$$v^i \to V_{jk} = -\sum_{i=1}^{3} v^i \varepsilon_{ijk}\,, \tag{2.2}$$

where v^i are the vector components and V_{jk} are the matrix elements. Under this isomorphism the vector product $x\times y$ is corresponding to the commutator of the matrices, $[X, Y] = XY - Y X$. Skew-symmetrical matrices A, B are mapped to vectors with components A^i, B^i, $i = 1, 2, 3$.

The angular momentum of the fluid in the cavity (relative to the center of mass) is (the integral is over the cavity volume)

$$M_0^i = \varrho \int (x\times v)^i dx^1 dx^2 dx^3 = \sum_{j,k=1}^{3} \left(-\tfrac{1}{2}\varepsilon_{ijk} M_{jk} + (Q_1)^i_j I^0_{jk} A^k\right),$$

$$M = m_1(\dot{F}F^t - F\dot{F}^t) = m_1 Q_1 (D^2 A + AD^2 - 2DBD) Q_1^t\,,$$

$$I^0_{jk} = m\left(\delta_{jk} \sum_{\ell=1}^{3} (r^\ell)^2 - r^j r^k\right), \qquad m = 4\pi\varrho d_1 d_2 d_3/3\,, \qquad m_1 = m/5\,.$$

Here m is the total mass of the fluid, the superscript t stands for the matrix transposition, M_{jk} are elements of the matrix M.

The total angular momentum of the system (rigid body and fluid) has the following components in the reference frame S:

$$\begin{gathered} M^i = \sum_{k=1}^{3} I_{ik} A^k - \gamma_i B^i\,, \\ I_{ik} = g_i \delta_i^k + m_1^{-1}(I^0_{ik} + I^1_{ik})\,, \qquad g_i = d_j^2 + d_\ell^2\,, \qquad \gamma_i = 2d_j d_\ell\,, \end{gathered} \tag{2.3}$$

where I^1_{ik} is the inertia tensor of the rigid envelope in the reference frame S, $i, j, \ell = 1, 2, 3$. The conservation law for the total angular momentum looks like

$$\dot{\mathbf{M}} = \mathbf{M}\times\mathbf{A}\,. \tag{2.4}$$

The last three equations of magnetic hydrodynamics in (1.1) are fulfilled identically because of the definitions (1.2) and (1.3). Turning to transformation of the first equation in (1.1), note that in the case of the ideal incompressible fluid, gravitational forces are equivalent to a redefinition of the pressure, $p_1 = p + \varrho\Phi$, so they do not influence the dynamics of the model in view. For the motion with homogeneous deformation the effective pressure p_1 is a quadratic function of the coordinates,

$$p_1 = p_0(t) + \sum_{i,j=1}^{3} (p_{ij}(t) a^i a^j + p_i(t) a^i)\,,$$

where $p_{ij}(t)$ are components of a symmetrical matrix $P_0(t)$. Using this form of the pressure and substituting Eqs. (1.2), (1.3) in the first equation in (1.1) we conclude that this equation is equivalent to the following equations, a matrix one and a vector one,

$$\varrho\ddot{F} = -(F^{-1})^t P_0 + ((F^{-1})^t hF^tFh + Fh^2)/4\pi\,, \qquad p_i(t) = -\sum_{k,\ell=1}^{3} F_i^k(Q_1)_\ell^k r^\ell\,. \tag{2.5}$$

Introduce the notation $K_0 = \dot{F}^tF - F^t\dot{F}$; evidently, $\dot{K}_0 = \ddot{F}^tF - F^t\ddot{F}$ is the antisymmetrical part of the matrix $F^t\ddot{F}$. The symmetrical part of this matrix determines the matrix $P_0(t)$,

$$2P_0 = -\varrho(F^t\ddot{F} + \ddot{F}^tF) + (2\pi)^{-1}hF^tFh + (4\pi)^{-1}(F^tFh^2 + h^2F^tF)\,. \tag{2.6}$$

Because of Eq. (2.5) we have

$$\varrho\dot{K}_0 = (4\pi)^{-1}(h^2F^tF - F^tFh^2)\,. \tag{2.7}$$

Using the definition (2.1) we get

$$K_0 = Q_2^tKQ_2\,, \qquad K = D^2B + BD^2 - 2DAD\,, \qquad F^tF = Q_2^tD^2Q_2\,. \tag{2.8}$$

By means of these formulae, Eq. (2.7) is transformed to an equivalent form,

$$\dot{K} = [K, B] + \kappa[Q_2h^2Q_2^t, D^2]\,, \qquad \kappa = (4\pi\varrho)^{-1}\,, \tag{2.9}$$

where the square brackets stand for the matrix commutator. With the notation $u = Q_2hQ_2^t$, we get, because of (2.1)

$$\dot{u} = [u, B]\,, \qquad [u^2, D^2] = [u, uD^2 + D^2u]\,. \tag{2.10}$$

The isomorphism (2.2) maps the skew-symmetrical matrix u to the vector with the components u^1, u^2, u^3, and the matrices K and w are mapped to vectors with the components

$$K^i = g_iB^i - \gamma_iA^i\,, \qquad w^i = \kappa g_iu^i\,, \qquad i,j,k = 1,2,3 \tag{2.11}$$

(no sum over i!). Equations (2.4), (2.9), and (2.10), rewritten in the vector notations (2.3), (2.11), are the complete set of equations describing the dynamics of a rigid body with an ellipsoidal cavity filled with the magnetic fluid,

$$\dot{\mathbf{M}} = \mathbf{M} \times \mathbf{A}\,, \qquad \dot{\mathbf{K}} = \mathbf{K} \times \mathbf{B} + \mathbf{u} \times \mathbf{w}\,, \qquad \dot{\mathbf{u}} = \mathbf{u} \times \mathbf{B}\,. \tag{2.12}$$

The above equations determine completely the time evolution of the matrix F, so Eq. (2.6) and the second equation in (2.5) enable one to get the matrix $P_0(t)$ and the coefficients $p_i(t)$, that is to say, to calculate the pressure inside the fluid (up to an inessential additive constant).

Equations (2.12) are a generalization of the classical equations describing motion of a body with a cavity filled with the ideal incompressible fluid [5]; in the present work they are derived for the first time. The classical case corresponds to the absence of the magnetic field, it is obtained from (2.12) if $\mathbf{u} = 0$.

3. First Integrals of the Dynamical System. The Integrable Cases

I. The most important first integral of the dynamical system (2.12) is that corresponding to the total energy E (with the constant gravitational energy excluded). It is the sum of the fluid kinetic energy E_1, the internal energy of the magnetic field E_2, and the kinetic energy of the rotation of the rigid body E_3,

$$E_1=\int \tfrac{1}{2}\varrho v^2 dx^1 dx^2 dx^3=\tfrac{1}{2}m_1 \operatorname{Tr}(\dot{F}\dot{F}^t)+\tfrac{1}{2}\sum_{i,k=1}^{3} I_{ik}^0 A^i A^k\,,$$
$$E_2=\int \frac{1}{8\pi} H^2 dx^1 dx^2 dx^3=\frac{1}{30}\operatorname{Tr}(h^t F^t F h)d_1 d_2 d_3\,, \tag{3.1}$$
$$E_3=\tfrac{1}{2}\sum_{i,j=1}^{3} I_{ij}^1 A^i A^j\,,\qquad E=E_1+E_2+E_3\,.$$

Writing these formulae in the notations of Sect. 2, we have

$$2H=2E/m_1=(\mathbf{M},\mathbf{A})+(\mathbf{K},\mathbf{B})+(\mathbf{u},\mathbf{w})$$
$$=\sum_{i,k=1}^{3}(I_{ik}A^iA^k-2\gamma_i A^i B^i+g_i(B^i)^2+\kappa g_i(u^i)^2)\,,$$
$$g_i=d_j^2+d_k^2\,,\qquad \gamma_i=2d_j d_k\,,\qquad i,j,k=1,2,3\,. \tag{3.2}$$

Evidently, $M^i=\partial H/\partial A^i$, $K^i=\partial H/\partial B^i$, $w^i=\partial H/\partial u^i$. It is easy to verify directly that the function $J_1=H$ is the first integral of the system (2.12). Other three first integrals of the system are

$$J_2=(\mathbf{M},\mathbf{M})\,,\qquad J_3=(\mathbf{u},\mathbf{u})\,,\qquad J_4=(\mathbf{K},\mathbf{u})\,. \tag{3.3}$$

The integral J_2 is, up to a factor, the total angular momentum squared, J_3 is the magnetic field intensity in the Lagrange coordinates, squared, and J_4 is the scalar product of the fluid velocity curl vector by the magnetic field vector $\mathbf{h}$. All three integrals (3.3) in combination determine the six-dimensional manifold $\mathscr{M}^6=T(S^6)\times S^2$, which is the product of the bundle tangent to the two-dimensional sphere by the two-dimensional sphere S^2.

The system (2.12) is a special case of Euler's equations [6] in the space L^* dual to the Lie algebra L which is the sum of the Lie algebra associated with the group of motions of the three-dimensional Euclidean space, E_3, and the Lie algebra of SO(3). The manifolds $\mathscr{M}^6$ are orbits of the co-adjoint representation of the Lie group $G=E_3\times \mathrm{SO}(3)$ in the space L^*, so the symplectic structure is determined in these manifolds in the standard manner [6]; in $\mathscr{M}^6$ the system (2.12) is of the Hamilton type, and its Hamiltonian is H.

In the case of a spherical cavity ($d_1=d_2=d_3$) the magnetic field produces no effect on the system dynamics, and equations (2.12) are reduced to the usual Euler equations describing the rotations of an effective rigid body. In the case where the rigid body and the cavity have an axial symmetry,

$$d_1=d_2\,,\qquad r^i=(0,0,r^3)\,,\qquad I_{ik}=I_i\delta_i^k\,,\qquad I_1=I_2\,,$$

Eqs. (2.12) have an additional first integral $J_5=M^3+K^3$; they are invariant under simultaneous rotations in the planes (M^1,M^2), (K^1,K^2), and (u^1,u^2). Therefore the system (2.12) at the common level of the first integrals, given in Eqs. (3.3), and of the additional integral J_5, is reduced after the factorization by this one-parameter group to a Hamiltonian system in a four-dimensional manifold, and the latter system is not integrable in general.

II. Let us consider the important case where the total angular momentum of the system is zero, $J_2=0$. We suppose that the center of mass is at the center of the ellipsoid ($r^i=0$), and the tensor of inertia of the rigid body is diagonal, $I^1_{ik}=I_i\delta^k_i$. Then because of Eq. (2.3) we have $A^i=\gamma_i B^i(g_i+m_1^{-1}I_i)^{-1}$, and the system (2.12) is reduced to

$$\begin{aligned}&\dot{\bar{\mathbf{K}}}=\bar{\mathbf{K}}\times\mathbf{B}+\mathbf{u}\times\mathbf{w}, \qquad \dot{\mathbf{u}}=\mathbf{u}\times\mathbf{B},\\ &B^i=\partial\bar{H}/\partial\bar{K}_i, \qquad w^i=\partial\bar{H}/\partial u^i, \qquad 2H=\sum_{i=1}^{3}(f_i^{-1}\bar{K}_i^2+\kappa g_i u_i^2),\\ &f_i=g_i-\gamma_i^2(g_i+I_i m_1^{-1})^{-1}.\end{aligned} \tag{3.4}$$

Equations (3.4) are analogous to the classical Kirchhoff equations describing the motion of a rigid body having three symmetry planes in the ideal incompressible fluid. It is known from the theory of the Kirchhoff equations that under the Clebsch conditions [7],

$$f_1(g_2-g_3)+f_2(g_3-g_1)+f_3(g_1-g_2)=0, \tag{3.5}$$

the system (3.4) has the additional first integral

$$J=\bar{K}_1^2+\bar{K}_2^2+\bar{K}_3^2+\kappa f_2(g_1-g_3)u_1^2+\kappa f_1(g_2-g_3)u_2^2, \tag{3.6}$$

and is therefore completely integrable.

We shall show that for any magnitude of semi-axes of the ellipsoidal cavity, d_1, d_2, d_3, there exists a two-parameter family of values of the rigid body inertia tensor I_i for which relation (3.5) holds; that is to say the system dynamics is integrable at the level $J_2=0$. Put $d_3>d_1>d_2$, and introduce the following notations:

$$\begin{aligned}&\beta_i=1+m_1^{-1}I_i g_i^{-1}>1, \qquad x_1=d_1d_3^{-1}, \qquad x_2=d_2d_3^{-1}, \qquad x_2<x_1<1,\\ &\alpha_1=2x_2(1+x_1^2)^{-1}, \qquad \alpha_2=2x_1(1+x_2^2)^{-1}, \qquad \alpha_3=2x_1x_2(x_1^2+x_2^2)^{-1}.\end{aligned} \tag{3.7}$$

We get from Eq. (3.4) that $f_i=g_i(1-\alpha_i^2\beta_i^{-1})$. After substitution of (3.7) and a simple transformation, Eq. (3.5) is reduced to the form

$$\frac{x_1^2(1-x_1^2)}{\beta_2(1+x_1^2)}+\frac{x_2^2(x_2^2-1)}{\beta_1(x_2^2+1)}+\frac{x_1^2x_2^2(x_1^2-x_2^2)}{\beta_3(x_1^2+x_2^2)}=0. \tag{3.8}$$

In view of this equation, we have $\beta_1>0$ for two arbitrary parameters $\beta_2, \beta_3>0$ and at $0<x_2<x_1<1$. The solutions of this equation admit the transformation $\beta_i\to L\beta_i$, so we get the two-parameter solution with $\bar{\beta}_i=L\beta_i>1$ if L is large enough. Then the corresponding components of the inertia tensor of the rigid body, I_i, are found from Eq. (3.7). In particular, Eq. (3.8) has solutions for which $x_1\approx x_2\approx 1$ and $\beta_1\approx\beta_2\approx\beta_3$, and the necessary conditions $I_i<I_j+I_k$ are also fulfilled in this case for large L.

The familiar integrable case found by S.A. Chaplygin [8] for the Kirchhoff equations does also belong to the system considered. Under the conditions

$$f_1=f_2=2f_3, \qquad g_3=\tfrac{1}{2}(g_1+g_2), \qquad J_4=0, \tag{3.9}$$

Eqs. (3.4) have an additional first integral,

$$J_5=((K_2^2-K_1^2)f_3^{-1}+\kappa(d_1^2-d_2^2)u_3^2)^2+4f_3^{-2}K_1^2K_2^2,$$

and the system is therefore completely integrable. After substitution of Eqs. (3.4), (3.7) it is not difficult to see that Eqs. (3.9) have a three-parameter family of solutions d_i, I_k satisfying all the necessary conditions.

4. Periodic Solutions

I. For periodic rotations in the pulsar model the matrices $Q_1(t)$, $Q_2(t)$ are periodic functions of the time variable which have identical periods. Closed trajectories of the system (2.12) are corresponding to such solutions; inversely, if the closed trajectories of the system (2.12) form a three-dimensional set, it has an everywhere dense subset corresponding to periodic rotations $Q_1(t)$ which leads to periodic variations of the external electromagnetic field of the pulsar.

It will be seen that under the conditions $I_{ik}=(g_i+I_i)\delta_{ik}$ and $J_4=0$ there are 12 closed trajectories of the system (2.12) in the open set of the level surfaces for the first integrals $J_i=k_i$.

For $I_{ik}=(g_i+I_i)\delta_{ik}$ the integral $J_1=H$ is

$$\begin{gathered}2J_1=\sum_{i=1}^{3}(a_iM_i^2+2c_iM_iK_i+b_iK_i^2+\kappa g_iu_i^2),\\ a_i=g_is_i,\qquad c_i=\gamma_is_i,\qquad b_i=(g_i+I_i)s_i,\qquad s_i=((g_i+I_i)g_i-\gamma_i^2)^{-1}.\end{gathered}\tag{4.1}$$

At the level $J_4=0$ the system (2.12) has three invariant submanifolds $V_k^4: u_k=M_i=M_j=K_i=K_j=0$ $(i,j,k=1,2,3)$. In the manifold V_1^4 Eqs. (2.12) and the integrals in Eqs. (3.2) and (3.3) are written as

$$\begin{gathered}\dot K_3=\kappa(g_2-g_1)u_1u_2,\qquad \dot u_1=u_2B_3,\qquad \dot u_2=-u_1B_3,\qquad \dot M_3=0,\\ 2J_1=a_3M_3^2+2c_3M_3K_3+b_3K_3^2+\kappa g_1u_1^2+\kappa g_2u_2^2,\\ J_2=M_3^2,\qquad J_3=u_1^2+u_2^2,\qquad B_3=c_3M_3+b_3K_3.\end{gathered}\tag{4.2}$$

The level surface for the integral $J_2=k_2$ contains two components $M_3=\varepsilon k_2^{1/2}$, $\varepsilon=\pm 1$. In each component the manifold given by the integrals $J_1=k_1$, $J_3=k_3$ is the intersection of an ellipsoid ($J_1=k_1$) and a cylinder ($J_3=k_3$) having a common axis K_3; it either consists of two closed trajectories of the system (4.2), or is empty (the number of these closed trajectories is equal for both components of the manifold, $M_3=\varepsilon k_2^{1/2}$). The total number of closed trajectories in three invariant submanifolds V_k^4 depends on the relation between the quantities J_1, J_2, J_3; this number is 12, 8, 4 or zero, and there are exactly 12 closed trajectories for $2J_1>J_2\cdot\max(a_i-c_i^2/b_i)+\kappa J_3\max(g_i)$.

The closed trajectories describe the pulsar rotation around a fixed axis. The maximal number of such trajectories (4 for every one of three axes) is associated with two possible directions of the pulsar total angular momentum and two possible directions of the rotation of the liquid core with respect to the envelope.

After the substitution $K_3=(B_3-c_3M_3)/b_3$, Eqs. (4.2) acquire the form of the classical Euler's equations,

$$\dot B_3=-\omega u_1u_2,\qquad \dot u_1=u_2B_3,\qquad \dot u_2=-u_1B_3,\tag{4.3}$$

where $\omega = \kappa b_3(g_1 - g_2)$. Let us calculate the period of the closed trajectories for the system (4.2) and (4.3). The integrals of the system (4.3) are

$$\ell = B_3^2 + \omega u_1^2, \qquad J_3 = u_1^2 + u_2^2. \tag{4.4}$$

Let $d_2 > d_1$, then $\omega = \kappa b_3(d_2^2 - d_1^2) > 0$. Expressing B_3 and u_2 via u_1 by means of (4.4) and substituting them into (4.3) we obtain

$$\dot{u}_1 = ((J_3 - u_1^2)(\ell - \omega u_1^2))^{1/2}. \tag{4.5}$$

Solutions of the equations of this type as known [9] are

$$u_1 = (\ell/\omega)^{1/2} \operatorname{sn}(\tau), \qquad \tau = (\omega J_3)^{1/2}(t - t_0), \tag{4.6}$$

where $\operatorname{sn}\tau$ is Jacobi's elliptic function corresponding to the parameter $k^2 = \ell/(\omega J_3)$. Putting the result (4.6) into Eq. (4.4), we get

$$B_3 = \ell^{1/2} \operatorname{cn}\tau, \qquad u_2 = J_3^{1/2} \operatorname{dn}\tau. \tag{4.7}$$

The period of the elliptic functions presented in Eqs. (4.6) and (4.7) is given by the expression

$$T = 4(\omega J_3)^{-1/2} \int_0^{\pi/2} (1 - k^2 \sin^2\alpha)^{-1/2} d\alpha. \tag{4.8}$$

This is the period of the closed trajectories for the system (4.2) and (4.3).

II. Let us find the magnitude of the period, T, which appears in the models of the real pulsars, that is for $d_1 \approx d_2 \approx d_3 \approx R$ and at constant J_3, I_3, ϱ. The function T in Eq. (4.8) attains the minimal value, T_m, at $k = \ell = 0$; in other words, for small oscillations taking place in a vicinity of the axis $u_2(B_3 = u_1 = 0)$, thereby $T_m = 2\pi(\omega J_3)^{-1/2}$. Asymptotically, such oscillations are ($\ell \ll 1$)

$$u_1 = \ell^{1/2} \sin(\omega^{1/2} u_2^0 (t - t_0)), \qquad B_3 = (\ell\omega)^{1/2} \cos(\omega^{1/2} u_2^0 (t - t_0)), \qquad u_2 = u_2^0. \tag{4.9}$$

After the substitution of Eqs. (4.1) and (2.3) into Eq. (4.7) we get for $k \ll 1$,

$$T_m = 2\pi(4\pi\varrho/J_3)^{1/2} K(d_1, d_2, I_3),$$
$$K = (d_1^2 + d_2^2 + I_3)^{-1/2}(d_2^2 - d_1^2 + I_3(d_1^2 + d_2^2)(d_2^2 - d_1^2)^{-1})^{1/2}, \qquad I_3 = m_1^{-1}(I_3^0 + I_3^1).$$

The function K attains its maximum K_m at

$$d_2^2 - d_1^2 = (I_3(d_1^2 + d_2^2))^{1/2}, \qquad K_m = 2^{1/2}(I_3(d_1^2 + d_2^2))^{1/4}(d_1^2 + d_2^2 + I_3)^{-1/2}. \tag{4.10}$$

Hence we obtain the minimal value of the period

$$T_0 = 4\pi^{3/2}(2\varrho/J_3)^{1/2}(I_3(d_1^2 + d_2^2))^{1/4}(d_1^2 + d_2^2 + I_3)^{-1/2}. \tag{4.11}$$

For the real pulsars we have [1]: $d_1 \approx d_2 \approx d_3 \approx R \sim 10^6$ cm, the matter density in the liquid core is $\varrho \sim 10^{14}$ g/cm^3, the matter density in the envelope is $\varrho_1 \sim 10^8$ g/cm^3, the envelope thickness is $r \sim 10^4$ cm, the magnetic field intensity at the pulsar surface is $|H| \sim 10^{12}$ Gs (all the numerical values are presented with an accuracy up to an order of magnitude). According to the definition in (1.3), the maximal magnitude of the magnetic field intensity at the surface of the ellipsoidal cavity is given by the formula $|H| = R|h|$. The definition $u = Q_2 h Q_2^t$ leads to $J_3 = |u|^2 = |h|^2$, so $J_3^{1/2} = |H|/R$. If $r \ll R$ the interia tensor of the envelope is

$$I_{ik}^1 = \tfrac{8}{3}\pi R^4 r \varrho_1 \delta_{ik}.$$

It is natural to assume that the center of mass of the pulsar is near the center of the ellipsoidal cavity, and $|r^i| < R(2r\varrho_1/R\varrho)^{1/2}$, then $|I^0_{ik}| = \gamma |I^1_{ik}|$, where $\gamma < 1$ [cf. Eq. (2.3)]. Putting into Eq. (4.11) the corresponding expression for the relevant component of the interia tensor,

$$I_3 = m_1^{-1}(I_3^0 + I_3^1) = 10(1+\gamma)Rr\varrho_1/\varrho\,,$$

we obtain finally

$$T_0 = 8\pi^{3/2}(5(1+\gamma)/4)^{1/4}\varrho^{1/2}RH^{-1}(r\varrho_1/R\varrho)^{1/4}\,. \tag{4.12}$$

Besides, we have $d_2 = d_1[1 + (5(1+\gamma)r\varrho_1/R\varrho)^{1/2}]$ because of Eq. (4.10). Using the numerical estimates presented above, we get $d_2 = d_1(1 + (5(1+\gamma))^{1/2}\,10^{-4})$, $T_0 = 5\,s$. The obtained value of T_0 is a reasonable approximation of the period $T = 3.75\,s$ that is known for the pulsar PSR 0527. Having in mind the inaccuracy in numerical magnitudes of all the quantities presented in Eq. (4.12), this estimate for the minimal period of the pulsar rotation may be considered as being in a satisfactory agreement with the available astrophysical data. Putting, for instance, $H = 5 \cdot 10^{12}$ Gs (this estimate is quite likely) we get $T_0 \sim 1\,s$; this value of the period is fairly close to the data for a number of pulsars, e.g. PSR 0628 ($T = 1.24\,s$), PSR 1133 ($T = 1.19\,s$) and others [1].

III. Trajectories of the system (4.3) satisfying the condition

$$(d_1^2 + d_3^2)^{-1}J_0 > J_3 > (d_2^2 + d_3^2)^{-1}J_0\,, \qquad J_0 = (2J_1 - (a_3 - c_3^2/\ell_3)J_2)\kappa^{-1}$$

are encircling the u_2 axis ($B_3 = u_1 = 0$), as well as the trajectories of Eq. (4.8). For such trajectories the matrix Q is invariable for the period of a single oscillation, as $\dot{Q}_2 = -B_0 Q_2$, and $\oint B_3 dt = 0$, while the matrix $Q_1(t+T)Q_1^{-1}(t)$ determines the rotation around the x_3 axis by the angle $\Delta\varphi = TM_3/I_3$. The condition $\Delta\varphi = 2\pi p/q$ (where p and q are integer) determines the magnitude of the angular momentum, $|\bar{M}| = m_1|M_3| = 2\pi p m_1 I_3/qT$, providing the exactly periodic pulsar rotation (with the period of qT).

If $J_3\kappa\min(g_i) < J_1 < J_3\kappa\max(g_i)$ and $J_2 \ll 1$, there are 8 closed trajectories at the surface $J_i = k_i$, and the quantities K_3, B_3, $A_3 = a_3M_3 + c_3K_3$ have alternating signs along the trajectories. Trajectories of this type describe nonmonotonous pulsar rotations around the x_3 axis, in the process of which the angular velocities of the envelope and of the internal rotation of the fluid change their signs periodically [a particular case of this type are the oscillations described by Eq. (4.9) for $|M_3| < (\omega\ell)^{1/2}c_3(a_3b_3 - c_3^2)^{-1}$]. Motions of this kind are possible only in the presence of an internal magnetic field and for $d_1 \neq d_2$.

Magneto-rotational oscillations of the incompressible fluid in the case of the cylindrical symmetry (the object is infinite in the x_3 direction) have been investigated in [10]. The existence of some periodical trajectories for the system (2.12) can be established using the results obtained by Novikov [11]. For $J_2 = 0$ the system (2.12) is reduced to the Kirchhoff equations, so there are at least 2 closed trajectories for every level of the integrals J_3, J_4, $J_1 > E(J_3, J_4)$, $J_2 = 0$, as it is shown in [11].

IV. Important solutions are also those having the minimal total energy J_1 in the manifold $\mathscr{M}^6$ corresponding to fixed values of the integrals $J_2 = k_2$, $J_3 = k_3$,

$J_4 = k_4$. As the energy J_1 is positive definite, such solutions do exist for any manifold of levels of the integrals J_2, J_3, J_4; they correspond to stationary points of the system (2.12). At the stationary points one has

$$\mathbf{A} = \lambda \mathbf{M}, \quad \mathbf{B} = \alpha \mathbf{u}, \quad \mathbf{w} = \alpha \mathbf{K} + \beta \mathbf{u}. \tag{4.13}$$

After substitution of Eqs. (2.11) these conditions are reduced to

$$\begin{gathered} u^i = \alpha \lambda z_i M_i, \quad K_i = (\kappa g_i - \beta) \lambda z_i M_i, \quad z_i = \gamma_i ((\alpha^2 - \kappa) g_i + \beta)^{-1}, \\ \lambda^{-1} M_i = \sum_{k=1}^{3} J_{ik} M_k, \quad J_{ik} = I_{ik} - \gamma_i \alpha^2 z_i \delta_{ik}. \end{gathered} \tag{4.14}$$

Thus to find the stationary points one has to calculate the eigenvalues and eigenvectors of the matrix with elements J_{ik}. The rigid-body rotation ($\alpha = 0$) and the purely internal rotation of the fluid ($\lambda = 0$) take place only for a degenerate set of the singular points lying in a two-parameter set of the manifolds $\mathscr{M}^6$. In the general case of Eqs. (4.13) and (4.14) the matrices $Q_1(t)$ and $Q_2(t)$ describe periodic rotations with the periods T_1 and T_2. If the periods T_1 and T_2 are commensurate, the solution is exactly periodic.

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