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The Massless Thirring Model: Positivity of Klaiber's n-Point Functions

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Abstract. We present a simple solution to the problem of proving positivity of Klaiber's n-point functions for the massless Thirring model. The corresponding fields are obtained as strong limits of explicitly given approximate fields, obviating reconstruction. By invoking recent results on the boson-fermion correspondence it is shown how the model can be formulated on the charged fermion Fock space. It is pointed out that the question of cyclicity of the vacuum is open, and that an affirmative answer is necessary to confirm the superselection sector picture of the model.

1. Introduction

The first question we have to answer is: why another paper on the massless Thirring model? In order to do this, we should begin by pointing out that there are two versions of the massless Thirring model.

First, there is the model introduced and partially solved by Thirring [1]. His results were extended by Glaser [2], who found an explicit expression for the quantum fields of the model. However, this version of the model (also studied by Berezin [3]) fell into disrepute after certain inconsistencies were encountered. These were ascribed to formal manipulations but, as we see it, the real cause of the difficulty was only found recently by one of us: the fields of this "Thirring–Glaser model" do not define operator-valued distributions [4], so that arguments based on non-existent n-point functions are non-existent too. Nevertheless, this version does describe a consistent positive energy relativistic quantum mechanics, with asymptotically complete in- and out-fields in the sense of LSZ scattering theory [5, 6].

In this paper it is the second version, initiated by Johnson [7] and culminating in the well-known Boulder lectures of Klaiber [8], which is at issue. In contrast to the "Thirring–Glaser" model, which is a particle theory, but not a field theory, and which depends on the coupling constant only, the "Thirring–Klaiber model"

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is a field theory that does not describe particles and which depends on two parameters.

The main point of this paper is to validate the expression "field theory" in the preceding sentence. What this amounts to is a rigorous proof that Klaiber's n-point functions satisfy the positive-definiteness conditions, a problem that has not been resolved in a satisfactory way ever since Wightman pointed it out in '64 [9]. All other axiomatic properties, suitably modified in the case of covariance and locality, can easily be established from the explicit expressions in Klaiber's work [8].

We would like to emphasize, however, that our solution to this problem is based on a discovery that should be of interest even to the less mathematically oriented reader. To explain this, we recall that the building block of Klaiber's n-point functions is the tempered distribution $\lim_{\varepsilon \downarrow 0}(x - i\varepsilon)^e$, where $e \in \mathbb{R}$. What we have found are cutoff fields ψ_ε with n-point functions that only differ from Klaiber's in that his building block is replaced by the *function* $(x - i\varepsilon)^e$. These approximate fields are bounded operators on a (positive definite metric) Hilbert space, so that positivity of the limiting tempered distributions results without performing any estimates. More is true: this state of affairs entails by (more or less) standard arguments that as $\varepsilon \downarrow 0$ the cutoff fields converge to non-cutoff fields that act on the same Hilbert space for any value of the two parameters occurring in the model. This could serve as a starting point for a rigorous study of issues like cyclicity of the vacuum, equation of motion, Coleman correspondence, etc. We shall not pursue this here, however, but restrict ourselves to emphasizing the importance of the first-mentioned issue in validating the usual superselection sector picture of the model, where the currents are regarded as the observables associated with the Thirring fields.

The idea of using approximate field operators so as to get the massless Thirring field under control is not new. It dates back to a paper by Dell'Antonio et al. [10]. In that work a large class of cutoff functions is allowed and it is claimed that for this class the approximate fields converge in the sense of operator-valued distributions to non-cutoff fields having Klaiber's n-point functions. However, the proof of these claims is left to the reader; only convergence of one field acting on the vacuum is considered. (This also holds for [11].) We consider this approach a very laborious undertaking and in fact our results shed little light on its feasibility.

In contrast, our strategy is based on picking a very special cutoff function. This choice was inspired by earlier work of two of us [12], where it turned out to play a unique role, too. As indicated above, this choice leads to the desired results with a minimum of labour, and in particular bypasses estimates altogether.

For an account of the early history of the massless Thirring model and the mathematical difficulties associated with it we refer to Wightman's Cargèse lectures [9]. Further comments from an analyst's point of view may be found in [5]. In the main text we shall return to the literature connected with this subject.

Let us now sketch the organization of the paper. As indicated above, we take Klaiber's n-point functions and their connection with Thirring's equation of motion for granted. In keeping with this philosophy we present in Sect. 2 what we believe is the simplest way of defining approximate fields having the properties mentioned above. In Sect. 3 we then prove our claims with a minimum of notation and other paraphernalia that could obscure the arguments; for expository reasons the proofs of two lemmas, involving long computations, are relegated to an appendix.

The drawback of our definition in Sect. 2 is that it may seem to come out of the blue, even to someone familiar with the work by Dell'Antonio et al. [10, 11] and by Streater and Wilde [13, 14]. Therefore, with the above claims proved, we return to the connection with the latter work in Sect. 4.

In Sect. 5 we show how the model can be formulated on the charged fermion Fock space, which is also the arena that Klaiber chose (although it did not follow from his results that the fields really live there, cf. [5]). In contrast to the preceding sections, which are largely self-contained, this section involves some recent results on what is often called "boson-fermion correspondence" [12, 15–20].

The last paragraphs of Sects. 4 and 5 concern the above-mentioned connection between the cyclicity problem and the superselection sector picture of the model.

Let us finish this introduction by collecting some notational conventions. Complex conjugation is denoted by a bar. Indexed operator products are in the natural order of the indices. The symbol θ stands for the Heaviside function and c.c./h.c. for complex/hermitean conjugate. The symbols ε, τ and r, u, v, ρ, σ, e are reserved for arbitrary positive and real numbers respectively. It is convenient to denote space-time points by $x=(x^0, x^1)$ in Sects. 2 and 3 and in the Appendix, and by (t,x) in Sects. 4 and 5. Finally, the index s occurs throughout the paper, and takes on the values $+$ and $-$.

2. Approximate Thirring Fields

The building block of the space on which our approximate Thirring fields are defined is the usual Fock space $\mathscr{F}_s(\mathscr{H})$ for one-dimensional neutral bosons. That is, the one-particle space $\mathscr{H}$ is the space of square-integrable functions $L^2(\mathbb{R}, dk)$. Denoting the usual annihilators and creators by $A(k)$ and $A^*(k)$ we introduce the smeared annihilation operator $A(f) \equiv \int dk A(k)\bar{f}(k)$, $f \in \mathscr{H}$, its adjoint $A^*(f)$ and the Weyl operator

$$W(f) \equiv \exp(i[A(f) + A^*(f)]). \tag{2.1}$$

(The inclusion of the customary factor $2^{-1/2}$ would lead to a plethora of similar factors later on.) From the CCR one then obtains in a well-known fashion the Weyl relations

$$W(f)W(g) = \exp[-i\mathrm{Im}(f,g)]W(f+g), \tag{2.2}$$

and the vacuum expectation value

$$(\Omega, W(f)\Omega) = \exp[-\tfrac{1}{2}(f,f)]. \tag{2.3}$$

The following result is an easy consequence of this, and will enable us to reduce the calculation of n-point functions to the calculation of an inner product.

Proposition 2.1. *For any $f_1, \ldots, f_N \in \mathscr{H}$ one has*

$$\left(\Omega, \prod_{i=1}^{N} W(f_i)\Omega\right) = \prod_{1 \leqq i < j \leqq N} \exp[-(f_i, f_j)] \prod_{k=1}^{N} \exp[-\tfrac{1}{2}(f_k, f_k)]. \tag{2.4}$$

Proof. If $N = 1$ this is just (2.3). Assume (2.4) holds for $N = M$. Then one has, using (2.2),

$$(\Omega, \prod_{1}^{M+1} W(f_i)\Omega) = \exp[-i\operatorname{Im}(f_M, f_{M+1})](\Omega, W(f_1)\ldots W(f_M + f_{M+1})\Omega)$$

$$= \prod_{1 \leqq i < j \leqq M-1} \exp[-(f_i, f_j)] \prod_{k=1}^{M-1} \exp[-\tfrac{1}{2}(f_k, f_k) - (f_k, f_M + f_{M+1})]$$

$$\cdot \exp[-\tfrac{1}{2}(f_M + f_{M+1}, f_M + f_{M+1}) - i\operatorname{Im}(f_M, f_{M+1})]$$

$$= \prod_{1 \leqq i < j \leqq M+1} \exp[-(f_i, f_j)] \prod_{k=1}^{M+1} \exp[-\tfrac{1}{2}(f_k, f_k)].$$

Thus, (2.4) follows by induction. □

(This result is known in the following sense: in the physics literature one employs normal-ordered Weyl operators and evaluates their n-point functions by using the formula $e^A e^B = e^c e^B e^A$ for $[A, B] = c\mathbb{1}$ to shift the annihilation parts to the right till they act on the vacuum. Since the normal ordering amounts to multiplying $W(f)$ by $\exp[\frac{1}{2}(f, f)]$, this entails (2.4). The induction argument avoids formal manipulations with unbounded operators and in particular the commutator formula, which is known to be false in general, if A and B are unbounded.)

The Hilbert space on which our fields act is defined by

$$\mathscr{F} \equiv \bigoplus_{n_-, n_+ \in \mathbb{Z}} \mathscr{F}_{n_-, n_+}, \tag{2.5}$$

where each summand is a copy of $\mathscr{F}_s(\mathscr{H})$. Henceforth, the symbol Ω denotes the vacuum of $\mathscr{F}_{0,0}$; in the sequel this vector plays the role of the vacuum in $\mathscr{F}$. We shall denote the operators on $\mathscr{F}$ that act like $A^{(*)}(f)$ and $W(f)$ on each sector by $A^{(*)}(f)$ and $W(f)$, too. The labels of the sectors may be regarded as eigenvalues of charges Q_- and Q_+. That is, the Q_s are defined by

$$Q_s F \equiv n_s F, \quad F \in \mathscr{F}_{n_-, n_+}. \tag{2.6}$$

We shall also need charge shifts S_- and S_+ that map $\mathscr{F}_{n_-, n_+}$ onto $\mathscr{F}_{n_- - 1, n_+}$ and $\mathscr{F}_{n_-, n_+ - 1}$, respectively, via the obvious identification. Thus, the S_s commute with the operators $A^{(*)}(f)$ and $W(f)$, and satisfy

$$S_s Q_s = (Q_s + 1) S_s, \quad S_s Q_{-s} = Q_{-s} S_s. \tag{2.7}$$

We now turn to the definition of the approximate field operators. To this end we introduce the phase factor

$$\chi_\varepsilon(r) \equiv (1 + \varepsilon - ir)/\text{c.c.}, \tag{2.8}$$

and the cutoff functions

$$R_r^\varepsilon(k) \equiv \theta(k) k^{-1/2} [e^{-k} - e^{ikr} e^{-\varepsilon k}], \tag{2.9}$$

$$L_r^\varepsilon(k) \equiv \theta(-k)(-k)^{-1/2}[e^{-ikr}e^{\varepsilon k} - e^k]. \tag{2.10}$$

Then the main definition of this paper reads as follows.

Definition 2.2 *For* $x = (x^0, x^1) \in \mathbb{R}^2$, *set*

$$u \equiv x^0 + x^1, v \equiv x^0 - x^1. \tag{2.11}$$

Then the approximate Thirring fields are defined by

$$\psi_{-,\varepsilon}(x) \equiv C_\varepsilon K_{-,\varepsilon}(x) W(\rho L_u^\varepsilon + \sigma R_v^\varepsilon) S_-, \tag{2.12}$$

$$\psi_{+,\varepsilon}(x) \equiv (-)^{Q_-} C_\varepsilon K_{+,\varepsilon}(x) W(\sigma L_u^\varepsilon + \rho R_v^\varepsilon) S_+, \tag{2.13}$$

where

$$C_\varepsilon \equiv (2\pi)^{-1/2}(2\varepsilon)^{-1/2(\rho^2+\sigma^2)}, \tag{2.14}$$

$$K_{-,\varepsilon}(x) \equiv [\chi_\varepsilon(u)]^{\rho^2(Q_- + 1/2)}[\chi_\varepsilon(v)]^{\sigma^2(Q_- + 1/2)}[\chi_\varepsilon(u)\chi_\varepsilon(v)]^{\rho\sigma Q_+}, \tag{2.15}$$

$$K_{+,\varepsilon}(x) \equiv [\chi_\varepsilon(v)]^{\rho^2(Q_+ + 1/2)}[\chi_\varepsilon(u)]^{\sigma^2(Q_+ + 1/2)}[\chi_\varepsilon(u)\chi_\varepsilon(v)]^{\rho\sigma Q_-}. \tag{2.16}$$

To help the reader to swallow this definition, let us anticipate the role of the various factors in leading to Klaiber's n-point functions. The shifts incorporate the charge-changing character of the Thirring field. The free field case corresponds to $\rho = 1, \sigma = 0$; the positive (negative) chirality component then only depends on the light cone coordinate $v(u)$ and the argument of W has support to the right (left) of the origin, corresponding to the fact that positive (negative) chirality massless Dirac fermions only have positive (negative) momenta. The choice of the cutoff functions R and L rests on the key observation that

$$(F_r^\varepsilon, F_{r'}^{\varepsilon'}) = \ln\left[\frac{(1+\varepsilon+ir)(1+\varepsilon'-ir')}{2i(r-r'-i(\varepsilon+\varepsilon'))}\right], \quad F = R, L, \tag{2.17}$$

which can be verified by using the integral

$$\int_0^\infty \frac{dk}{k}(e^{-ak} - e^{-bk}) = \ln\frac{b}{a}, \quad \operatorname{Re} a, b > 0. \tag{2.18}$$

The crux is that the function in the denominator is just Klaiber's building block. The non-translationally invariant terms in the numerator are inevitable. (This is because the function $\ln(\varepsilon + ir)^{-1}$ is not positive definite.) The phase factors $K_{s,\varepsilon}$ are chosen precisely so that they will cancel these terms. As concerns the "asymmetric" factor $(-)^{Q_-}$ in $\psi_{+,\varepsilon}$: the symmetric factor $(si)^{Q_+ + Q_-}$ in $\psi_{s,\varepsilon}$ would lead to the desired result as well. The former choice is more convenient, however. Finally, it is worth noting that $\psi_{s,\varepsilon}(x)$ is equal to a unitary operator times the constant C_ε. Its diverging part (as $\varepsilon \to 0$) may by regarded as a wave function renormalization constant.

3. The Convergence to Klaiber's Fields

The main results of this paper are contained in the following theorem and its corollary.

Theorem 3.1 *As $\varepsilon \to 0$ the fields $\psi^{(*)}_{-,\varepsilon}(x)$ and $\psi^{(*)}_{+,\varepsilon}(x)$ converge in the sense of operator-valued tempered distributions. Their limits $\psi^{(*)}_{-}(x)$ and $\psi^{(*)}_{+}(x)$ have the same vacuum expectation values as the fields $\varphi^{(*)}_{1}(x)$ and $\varphi^{(*)}_{2}(x)$ of Klaiber* [8], *provided one takes $\mu = 1$ in the latter fields and sets*

$$\rho \equiv (2\pi^{1/2} - \alpha - \beta)/2\pi^{1/2}, \tag{3.1}$$

$$\sigma \equiv (\beta - \alpha)/2\pi^{1/2} = -g/2\pi, \tag{3.2}$$

where α, β, g are Klaiber's parameters.

Corollary 3.2 *Klaiber's vacuum expectation values* (VEV) *satisfy the positivity condition.*

Proof of Corollary. Since the $\psi^{(*)}_{s,\varepsilon}(x)$ act on a positive definite metric Hilbert space, their VEV satisfy the positivity condition. As positivity is preserved under limits, positivity of Klaiber's VEV follows from Theorem 3.1. □

To prove the theorem we need two lemmas. Lemma 3.3 details the $2n$-point functions of the $\psi^{(*)}_{s,\varepsilon}(x)$ if the fields occur in a canonical order. Lemma 3.4 enables us to handle the general case by specifying the multiplicative factor that arises on commuting two fields.

Lemma 3.3 *Let $x_j, y_j \in \mathbb{R}^2, j = 1, \ldots, n$, and set*

$$u_j \equiv x_j^0 + x_j^1, \quad v_j \equiv x_j^0 - x_j^1, \quad U_j \equiv y_j^0 + y_j^1, \quad V_j \equiv y_j^0 - y_j^1. \tag{3.3}$$

Then one has

$$\begin{aligned}
&\Big(\Omega, \prod_1^n \psi_{-,\varepsilon_j}(x_j) \prod_1^n \psi^*_{-,\tau_j}(y_j)\Omega\Big) \\
&= \left(\frac{1}{2\pi}\right)^n (-i)^{n(\rho^2+\sigma^2)} \prod_{j<k} [u_j - u_k - i(\varepsilon_j + \varepsilon_k)]^{\rho^2} [U_j - U_k - i(\tau_j + \tau_k)]^{\rho^2} \\
&\cdot \prod_{j,k} [u_j - U_k - i(\varepsilon_j + \tau_k)]^{-\rho^2} \prod_{j<k} [v_j - v_k - i(\varepsilon_j + \varepsilon_k)]^{\sigma^2} \\
&\cdot [V_j - V_k - i(\tau_j + \tau_k)]^{\sigma^2} \prod_{j,k} [v_j - V_k - i(\varepsilon_j + \tau_k)]^{-\sigma^2}.
\end{aligned} \tag{3.4}$$

The corresponding 2n-point functions of $\psi^{(*)}_{+,\varepsilon}(x)$ are given by the right-hand side of (3.4) *with $u \leftrightarrow v$ and $U \leftrightarrow V$. Finally,*

$$\begin{aligned}
&\left(\Omega, \prod_1^l \psi_{-,\varepsilon_j}(x_j) \prod_{l+1}^n \psi_{+,\varepsilon_j}(x_j) \prod_1^l \psi^*_{-,\tau_j}(y_j) \prod_{l+1}^n \psi^*_{+,\tau_j}(y_j)\Omega\right) \\
&= (-)^{l(n-l)} \frac{\prod_{j=1}^{l}\prod_{k=l+1}^{n} \begin{array}{l}([u_j - u_k - i(\varepsilon_j + \varepsilon_k)][v_j - v_k - i(\varepsilon_j + \varepsilon_k)] \\ \cdot [U_j - U_k - i(\tau_j + \tau_k)][V_j - V_k - i(\tau_j + \tau_k)])^{\rho\sigma}\end{array}}{\prod_{j=1}^{l}\prod_{k=l+1}^{n} \begin{array}{l}([u_j - U_k - i(\varepsilon_j + \tau_k)][v_j - V_k - i(\varepsilon_j + \tau_k)] \\ \cdot [u_k - U_j - i(\varepsilon_k + \tau_j)][v_k - V_j - i(\varepsilon_k + \tau_j)])^{\rho\sigma}\end{array}} \\
&\cdot \left(\Omega, \prod_1^l \psi_{-,\varepsilon_j}(x_j) \prod_1^l \psi^*_{-,\tau_j}(y_j)\Omega\right)\left(\Omega, \prod_{l+1}^n \psi_{+,\varepsilon_j}(x_j) \prod_{l+1}^n \psi^*_{+,\tau_j}(y_j)\Omega\right).
\end{aligned} \tag{3.5}$$

Lemma 3.4. *Let $x, y \in \mathbb{R}^2$ and define u, v, U and V by* (3.3). *The commutation relations of $\psi_{s,\varepsilon}(x)$ and $\psi^*_{s',\tau}(y)$ are given by*

$$\psi_{-,\varepsilon}(x)\psi^*_{+,\tau}(y) = -\left(\left[\frac{i(U-u)+\varepsilon+\tau}{\text{c.c.}}\right]\left[\frac{i(V-v)+\varepsilon+\tau}{\text{c.c.}}\right]\right)^{\rho\sigma}\psi^*_{+,\tau}(y)\psi_{-,\varepsilon}(x), \tag{3.6}$$

$$\psi_{-,\varepsilon}(x)\psi^*_{-,\tau}(y) = \left[\frac{i(U-u)+\varepsilon+\tau}{\text{c.c.}}\right]^{\rho^2}\left[\frac{i(V-v)+\varepsilon+\tau}{\text{c.c.}}\right]^{\sigma^2}\psi^*_{-,\tau}(y)\psi_{-,\varepsilon}(x), \tag{3.7}$$

$$\psi_{+,\varepsilon}(x)\psi^*_{+,\tau}(y) = \left[\frac{i(V-v)+\varepsilon+\tau}{\text{c.c.}}\right]^{\rho^2}\left[\frac{i(U-u)+\varepsilon+\tau}{\text{c.c.}}\right]^{\sigma^2}\psi^*_{+,\tau}(y)\psi_{+,\varepsilon}(x). \tag{3.8}$$

The commutation relations of $\psi_{s,\varepsilon}(x)$ and $\psi_{s',\tau}(y)$ are given by

$$\psi_{-,\varepsilon}(x)\psi_{+,\tau}(y) = -\left(\left[\frac{i(u-U)+\varepsilon+\tau}{\text{c.c.}}\right]\left[\frac{i(v-V)+\varepsilon+\tau}{\text{c.c.}}\right]\right)^{\rho\sigma}\psi_{+,\tau}(y)\psi_{+,\varepsilon}(x), \tag{3.9}$$

$$\psi_{-,\varepsilon}(x)\psi_{-,\tau}(y) = \left[\frac{i(u-U)+\varepsilon+\tau}{\text{c.c.}}\right]^{\rho^2}\left[\frac{i(v-V)+\varepsilon+\tau}{\text{c.c.}}\right]^{\sigma^2}\psi_{-,\tau}(y)\psi_{-,\varepsilon}(x), \tag{3.10}$$

$$\psi_{+,\varepsilon}(x)\psi_{+,\tau}(y) = \left[\frac{i(v-V)+\varepsilon+\tau}{\text{c.c.}}\right]^{\rho^2}\left[\frac{i(u-U)+\varepsilon+\tau}{\text{c.c.}}\right]^{\sigma^2}\psi_{+,\tau}(y)\psi_{+,\varepsilon}(x). \tag{3.11}$$

The proofs of these lemmas consist of long algebraic calculations, a sketch of which can be found in the appendix. We shall conclude this section by proving the theorem, taking the lemmas and some concepts and arguments from axiomatic relativistic quantum field theory for granted. (The latter can be found in [21] for example.)

Proof of Theorem 3.1 Let us first elucidate the structure of the generic non-vanishing approximate VEV: any pair $\psi^*_{s,\varepsilon}(x)$, $\psi^{(*)}_{s,\varepsilon'}(x')$ occurring in it, with the first field to the left of the second one, gives rise to a term

$$[(x^0-x'^0)+(x^1-x'^1)-i(\varepsilon+\varepsilon')]^{e_+}[(x^0-x'^0)-(x^1-x'^1)-i(\varepsilon+\varepsilon')]^{e_-}, \tag{3.12}$$

where e_+, e_- can be equal to $\pm\rho^2$, $\pm\sigma^2$ or $\pm\rho\sigma$, depending on s and s' and on whether the fields have a * or not; up to the constant $\pm(2\pi)^{-n}(-i)^{n(\rho^2+\sigma^2)}$ the VEV is precisely the product of such terms for all pairs involved. Indeed, this assertion is evident for the VEV detailed in Lemma 3.3; also, any non-vanishing VEV can be obtained through transpositions of the fields occurring at the left-hand side of (3.5), and an inspection of the commutation relations (3.6)–(3.11) reveals that they are exactly such as to guarantee that the assertion holds true in general. Put differently, not only in the "canonical" VEV of Lemma 3.3, but also in the generic VEV the constants C_ε, $C_{\varepsilon'}$ and phase factors $K_{s,\varepsilon}(x)$, $K_{s',\varepsilon'}(x')$ ensure that on applying Proposition 2.1 only the translation invariant denominator in (2.17) does not cancel out for the given pair; its contribution is then precisely (3.12).

As a consequence an approximate $2n$-point function can be regarded as a function of $2n-1$ difference vectors. Choosing these in the usual way, all factors (3.12) are Fourier transforms of tempered distributions with support in $\bar{V}_+^{2n-1}$, where $\bar{V}_+$ denotes the closed forward light cone. (To see this, note that the distributional Fourier transform of the function $(x-i\varepsilon)^e$ has support in $[0,\infty)$.) Such distributions form a convolution algebra on which the convolution is jointly continuous, associative and commutative. Now by using Laplace transform lore or by direct calculation one sees that $\lim_{\varepsilon\to 0}(x-i\varepsilon)^e$ exists in $\mathscr{S}'(\mathbb{R})$. Combining these facts with the continuity of Fourier transformation, it readily follows that the limit of an approximate $2n$-point function exists in $\mathscr{S}'(\mathbb{R}^{4n})$ as one or more of the cutoffs converge to zero, and that the resulting distribution does not depend on the order of the latter limits.

These observations enable us to infer the existence of limiting operator-valued tempered distributions $\psi_s^{(*)}(x)$ defined on Ω and the ensuing polynomial domain: let us set

$$\Phi_{\vec{\varepsilon}}(G) \equiv \int dx_1 \ldots dx_N G(x_1,\ldots,x_N) \prod_{i=1}^{N} \psi_{s_i,\varepsilon_i}^{(*)_i}(x_i)\Omega, \quad G\in\mathscr{S}(\mathbb{R}^{2N}), \tag{3.13}$$

where $\vec{\varepsilon} \equiv (\varepsilon_1,\ldots,\varepsilon_N)$ and $(*)_i$ indicates that either the field or its adjoint occurs, depending on i. Also, the integral is a strong improper Riemann integral, which is well defined, since the approximate fields are strongly continuous in x and have x-independent norms. Then this vector has a strong limit $\Phi_{\vec{0}}(G)$ as $\vec{\varepsilon}\to\vec{0}$ in $\mathbb{R}^N$. (Indeed, the norm squared of a difference vector $\Phi_{\vec{\varepsilon}}(G) - \Phi_{\vec{\varepsilon}'}(G)$ can be written

$$\int dy\, dx\, \bar{G}(y)G(x)[W_{\vec{\varepsilon},\vec{\varepsilon}}(y,x) + W_{\vec{\varepsilon}',\vec{\varepsilon}'}(y,x) - W_{\vec{\varepsilon},\vec{\varepsilon}'}(y,x) - W_{\vec{\varepsilon}',\vec{\varepsilon}}(y,x)], \tag{3.14}$$

and by the above the four approximate Wightman functions converge to the same element $W_{\vec{0},\vec{0}}(y,x)$ of $\mathscr{S}'(\mathbb{R}^{4N})$ as $\vec{\varepsilon},\vec{\varepsilon}'\to\vec{0}$. Thus, $\Phi_{\vec{\varepsilon}}(G)$ is Cauchy and hence has a strong limit $\Phi_{\vec{0}}(G)$.) Fixing F in $\mathscr{S}(\mathbb{R}^2)$ we now define

$$\psi_s^{(*)}(F)\Omega \equiv \operatorname*{s.lim}_{\varepsilon\to 0} \int dx\, F(x)\psi_{s,\varepsilon}^{(*)}(x)\Omega \tag{3.15}$$

and, inductively,

$$\psi_s^{(*)}(F)\Phi_{\vec{0}}(G) \equiv \operatorname*{s.lim}_{\varepsilon,\tau_1,\ldots,\tau_N\to 0} \int dx F(x)\psi_{s,\varepsilon}^{(*)}(x)\Phi_{\vec{\tau}}(G) \tag{3.16}$$

and extend by linearity. We have already seen that the limits at the right-hand side exist, so that it remains to verify that one obtains a well-defined linear operator in this way. That is, one should check that if a linear combination of the $\Phi_{\vec{0}}(G)$ vanishes, its image does, too. But this follows from the fact that the limits can be taken in any order; in particular, one can send the τ_i's to zero first, after which this is obvious. This concludes the proof of the asserted convergence.

Let us now turn to the connection with Klaiber's work. First, we note that the charge structure of his fields is the same as that of ours, so that we need only show that under the specified identifications the limits of our VEV listed in Lemma 3.3 and those arising under permutations of the fields occurring there coincide with his VEV. For the canonical order this follows upon comparing (3.4) and (3.5) with Klaiber's equations (VII.3) and (VII.5) (and after tracking down his various constants). To treat the general case we note first that by virtue of analyticity in the tube and the edge-of-the-wedge theorem we need only prove equality in the totally spacelike region. But this follows from the fact that in the spacelike region the commutation relations in Lemma 3.4 reduce to those of Klaiber's fields if one lets ε, $\tau \to 0$. Indeed, this can be seen by comparing the result with Eqs. (VI.2) and (VI.4) in [8] and by noting that

$$\rho^2 - \sigma^2 - 1 = \lambda/\pi \tag{3.17}$$

under the identifications (3.1) and (3.2). This concludes the proof of the theorem. □

4. Currents and Superselection Sectors

Let us now link up the above with the work of Streater and Wilde [13,14] and Dell'Antonio et al. [10, 11], and discuss the superselection sector picture sketched there. To this end we introduce the annihilators

$$\alpha(k) \equiv \begin{cases} A(k) + D(k)(\rho Q_+ + \sigma Q_-), & k > 0, \\ A(k) + D(k)(\rho Q_- + \sigma Q_+), & k < 0, \end{cases} \tag{4.1}$$

where the displacement function is given by

$$D(k) \equiv i[\theta(k) - \theta(-k)]\frac{e^{-|k|}}{|k|^{1/2}}, \tag{4.2}$$

the neutral pseudo-scalar boson field

$$\varphi(t,x) \equiv \int \frac{dk}{(4\pi|k|)^{1/2}}[\alpha(k)\exp(-i|k|t + ikx) + \text{h.c.}] \tag{4.3}$$

and its derivatives, the currents

$$\begin{aligned} j_s(t,x) &\equiv \frac{-1}{2\pi^{1/2}}(\partial_x - s\partial_t)\varphi(t,x) \\ &= \frac{s}{2i\pi}\int dk\theta(sk)|k|^{1/2}[\alpha(k)\exp(ik(x-st)) - \text{h.c.}], \end{aligned} \tag{4.4}$$

with the corresponding charges

$$q_s \equiv \int dx j_s(0,x) = \rho Q_s + \sigma Q_{-s}. \tag{4.5}$$

Now let $\gamma(x)$ be a (real-valued) gauge function. Putting

$$j_s(\gamma) \equiv \int dx\, j_s(0,x)\gamma(x), \tag{4.6}$$

it follows that

$$\exp[ij_s(\gamma)] = W(g_s)\exp[i\zeta(\gamma)q_s]. \tag{4.7}$$

Here, the corresponding functions of k are given by

$$g_s(k) \equiv is\theta(sk)(|k|/2\pi)^{1/2}\hat{\gamma}(k), \tag{4.8}$$

the phase function by

$$\zeta(\gamma) \equiv (2\pi)^{-1/2}\int dk e^{-|k|}\hat{\gamma}(k), \tag{4.9}$$

and the Fourier transform by

$$\hat{\gamma}(k) \equiv (2\pi)^{-1/2} \int dx e^{-ikx} \gamma(x). \tag{4.10}$$

(In order that the above and what follows make sense it suffices that

$$(1+|k|)\hat{\gamma}(k) \in \mathscr{H} = L^2(\mathbb{R}, dk), \tag{4.11}$$

which is assumed henceforth.)

In view of (4.4) we can also write

$$\exp[ij_s(\gamma)] = \exp\left(\frac{i}{2\pi^{1/2}} \int dx [\varphi(0,x)\gamma'(x) + s\dot{\varphi}(0,x)\gamma(x)]\right), \tag{4.12}$$

which is of the form considered in [13, 14]; the corresponding Cauchy data $(\dot{\xi}(x,0), \xi(x,0))$ are proportional to $(\gamma'(x), -s\gamma(x))$. Moreover, the role of the displacement functions Θ and η of [13] and [14] is here played by

$$H(t,x) \equiv -\pi^{-1/2}[\operatorname{Arctan}(x+t)q_- + \operatorname{Arctan}(x-t)q_+] \tag{4.13}$$

in the sense that

$$\varphi(t,x) = \varphi_f(t,x) + H(t,x), \tag{4.14}$$

where φ_f denotes φ with $\alpha(k) \to A(k)$. (To verify this, use (4.1), (4.3) and the integral (2.18).)

However, to avoid confusion it may be in order to point out that the operators U and V and charge sectors $\mathscr{H}_{m,n}$ of [14] do not correspond to our charge shifts and sectors. This is because the shifts $S_\pm$ commute neither with $q_+ + q_-$ nor with $q_+ - q_-$ (for $|\rho| \neq |\sigma|$), whereas U and V by construction commute with $q_+ - q_-$ and $q_+ + q_-$, respectively, cf. [14, p. 384]. Correspondingly the Cauchy data η_1 and η_2 of [14 l.c.] are different from the ones determined by [4.13]. It appears to us that the former choice cannot lead to fields with Klaiber's n-point functions.

We have not yet explained the reason for our terminology "current" and "gauge function," and we proceed to do this; the point is that

$$\exp[ij_s(\gamma)]\psi_{s,\varepsilon}(t,x) = \exp[-i\rho\gamma_\varepsilon(x-st)]\psi_{s,\varepsilon}(t,x)\exp[ij_s(\gamma)], \tag{4.15}$$

$$\exp[ij_s(\gamma)]\psi_{-s,\varepsilon}(t,x) = \exp[-i\sigma\gamma_\varepsilon(x-st)]\psi_{-s,\varepsilon}(t,x)\exp[ij_s(\gamma)], \tag{4.16}$$

where

$$\gamma_\varepsilon(x) \equiv (2\pi)^{-1/2} \int dk e^{-\varepsilon|k|}\hat{\gamma}(k) e^{ikx}. \tag{4.17}$$

(To see this, use (4.7), Def. 2.2. and the Weyl relations (2.2).) Now it is clear that $\gamma_\varepsilon(x)$ converges to $\gamma(x)$ as $\varepsilon \to 0$, so that (formally at least, cf. below) the operators $\exp[ij_s(\gamma)]$ generate the gauge transformations of the Thirring fields $\psi_s(t,x)$.

Let us now make the connection with [10, 11]; this will also illuminate the structure of the approximate fields. Consider the function

$$\delta_r^\varepsilon(x) \equiv 2\operatorname{Arctan} x - 2\operatorname{Arctan}\left(\frac{x-r}{\varepsilon}\right), \tag{4.18}$$

whose Fourier transform is given by

$$\hat{\delta}_r^\varepsilon(k) = \frac{(2\pi)^{1/2}}{ik}[e^{-|k|} - e^{-ikr}e^{-\varepsilon|k|}]. \tag{4.19}$$

Using the notation (4.8) we then have

$$d^\varepsilon_{+,r}(k) = R^\varepsilon_{-r}(k), \tag{4.20}$$

$$d^\varepsilon_{-,r}(k) = -L^\varepsilon_r(k), \tag{4.21}$$

(cf. (2.9)—(2.10)) and, using (4.9) and (2.18),

$$\exp[i\zeta(\delta^\varepsilon_r)] = \chi_\varepsilon(-r). \tag{4.22}$$

Combining all this, Definition 2.2 can be written

$$\begin{aligned}\psi_{-,\varepsilon}(t,x) &= C_\varepsilon \chi_\varepsilon(u)^{1/2\rho^2}\chi_\varepsilon(v)^{1/2\sigma^2}\exp[-i\rho j_-(\delta^\varepsilon_u) + i\sigma j_+(\delta^\varepsilon_{-v})]S_-,\\ \psi_{+,\varepsilon}(t,x) &= (-)^{Q_-}C_\varepsilon \chi_\varepsilon(v)^{1/2\rho^2}\chi_\varepsilon(u)^{1/2\sigma^2}\exp[i\rho j_+(\delta^\varepsilon_{-v}) - i\sigma j_-(\delta^\varepsilon_u)]S_+.\end{aligned} \tag{4.23}$$

If one now writes this in normal-ordered form, putting the shifts between creation and annihilation parts, one obtains the form in which the approximate fields appear in [10, 11], up to conventions; the role of their functions $\tilde{J}_\pm(k)$ is played here by $e^{-|k|}$ and that of $\chi_{\Lambda\pm}(k)$ by $e^{-\varepsilon|k|}$, and their coupling constants are related to ours through

$$\rho = \tfrac{1}{2}(a + \bar{a}), \qquad \sigma = \tfrac{1}{2}(a - \bar{a}). \tag{4.24}$$

We conclude this section with some comments on the superselection sector viewpoint of the Thirring model. A forthcoming paper by one of us (J.D.W.) will contain further discussion of the model from this point of view. It is clear from (4.7) that

$$S_s\exp[ij_s(\gamma)] = \exp[i\rho\zeta(\gamma)]\exp[ij_s(\gamma)]S_s, \tag{4.25}$$

$$S_s\exp[ij_{-s}(\gamma)] = \exp[i\sigma\zeta(\gamma)]\exp[ij_{-s}(\gamma)]S_s. \tag{4.26}$$

This suggests that one could set up a unitary equivalence between the CCR representations provided by the exponentiated smeared currents on adjacent sectors: one would merely have to multiply the shifts by Weyl operators corresponding to the displacement function $D(k)$. However, since $D(k)$ is not square integrable, no such operators exist and in fact the representations are mutually unitarily inequivalent. (A general study of such displaced Fock representations can be found in [22].) Combining this with (4.15) and (4.16) one appears to get an explicit example of the analysis of Doplicher, Haag and Roberts [23], which for the case in hand is based on regarding the currents as the "observables" associated to the "unobservable" Thirring fields.

Unfortunately, to the best of our knowledge this picture can be, so far, only substantiated by rigorous mathematics when $(\rho, \sigma) = (\pm 1, 0)$ or $(0, \pm 1)$. To explain the difficulty, we observe that the exponentiated smeared currents are irreducible in each charge sector. (To see this, recall that the Weyl operators in the Fock representation are already irreducible when their arguments vary over a dense subspace of $\mathscr{H}$. The assertion therefore follows from (4.7).) Thus, if one wishes to view the currents as the observables associated with the fields $\psi_s^{(*)}$, the latter had better be cyclic on the vacuum. Indeed, this is a minimal requirement for regarding the currents as "functions of the fields". However, a proof of cyclicity is lacking for the interacting case. Even in the four free cases mentioned above there appears to be no direct argument showing cyclicity of the limiting fields, but here cyclicity is a consequence of developments sketched in the next section.

Of course, we are aware of the usual point-splitting procedures to "reach" the currents as limits of bilinear expressions of the fields. However, it appears that this formal expansion method has not even been put on a solid analytical basis in the context of the free Dirac theory. In this connection, let us comment on another strategy that suggests itself: this is to start with point-split expressions for the *approximate* fields, to renormalize and smear these, and then take limits so as to obtain the non-cutoff smeared currents. One problem with this is that the topologies in which such limits exist appear to be far too weak to be useful. This is true in particular for the type of convergence sketched in [11]. Another one is that even if strong convergence could be shown, it would not be immediate that the limits lie in the closure of the subspace spanned by the non-cutoff fields acting on the vacuum.

We do believe, though, that it should be possible to handle the interacting case, too. At any rate, we conjecture that cyclicity holds true for any $(\rho, \sigma) \neq (0, 0)$. (If $\rho = \sigma = 0$, then cyclicity evidently breaks down, cf. Def. 2.2.) At the end of the next section we shall briefly return to this problem.

5. Formulation on the Charged Fermion Fock Space

In this section we shall sketch how one can formulate the preceding constructions on the usual Fock space $\mathscr{F}_a(L^2(\mathbb{R}, dp)^2)$ for charged Dirac fermions. This will also enable us to establish cyclicity in the free cases, and to comment on the background of our Def. 2.2. We shall have occasion to use some results bearing on the boson-fermion correspondence, cf. [12, 15–20]; closest in spirit and notation is [12].

Taking γ^5 diagonal, the chiral components of the free massless Dirac field on $\mathscr{F}_a$ are given by

$$\psi_s^0(t, x) = (2\pi)^{-1/2} \int dp \theta(sp)[a(p)\exp(ip(x - st)) + sb^*(p)\exp(-ip(x - st))], \tag{5.1}$$

the corresponding currents by

$$J_s(t, x) \equiv :\psi_s^{0*}(t, x)\psi_s^0(t, x): \tag{5.2}$$

and their charges by

$$Q_s \equiv \int dx J_s(0, x) = \int dp \theta(sp)[a^*(p)a(p) - b^*(p)b(p)]. \tag{5.3}$$

Under the action of Q_s, Fock space decomposes as

$$\mathscr{F}_a = \bigoplus_{n_-, n_+ \in \mathbb{Z}} \mathscr{F}_{n_-, n_+}, \tag{5.4}$$

where

$$Q_s F = n_s F, F \in \mathscr{F}_{n_-, n_+}. \tag{5.5}$$

Smearing $J_s(0, x)$ with a real-valued $\gamma(x)$ satisfying (4.11) leads to a self-adjoint operator $J_s(\gamma)$, such that

$$\exp[iJ_s(\gamma)]\psi_s^0(t,x) = \exp[-i\gamma(x-st)]\psi_s^0(t,x)\exp[iJ_s(\gamma)], \tag{5.6}$$

$$\exp[iJ_s(\gamma)]\psi_{-s}^0(t,x) = \psi_{-s}^0(t,x)\exp[iJ_s(\gamma)]. \tag{5.7}$$

It is easy to see $J_s(\gamma)$ is not an operator when γ has different limits for $x \to \pm\infty$. However, the corresponding gauge transformations are unitarily implementable when $\exp[i\gamma(x)] \to 1$ for $|x| \to \infty$. (More precisely, when the Fourier transform of $\exp[i\gamma(x)] - 1$ satisfies (4.11).) In particular, setting

$$\eta(x) \equiv \pi + 2\operatorname{Arctan} x, \tag{5.8}$$

there exist unitary operators $\mathcal{U}_s$, unique up to a phase, such that

$$\mathcal{U}_s\psi_s^0(t,x) = \exp[-is\eta(x-st)]\psi_s^0(t,x)\mathcal{U}_s, \tag{5.9}$$

$$\mathcal{U}_s\psi_{-s}^0(t,x) = \psi_{-s}^0(t,x)\mathcal{U}_s. \tag{5.10}$$

In contrast to the exponentiated smeared currents these operators do not leave the charge sectors invariant. Instead, one has

$$\mathcal{U}_sQ_s = (Q_s - 1)\mathcal{U}_s, \qquad \mathcal{U}_sQ_{-s} = Q_{-s}\mathcal{U}_s. \tag{5.11}$$

Furthermore, $\mathcal{U}_+$ and $\mathcal{U}_-$ anticommute. Hence, the operators

$$S_+ \equiv (-)^{Q_+}\mathcal{U}_+^*, S_- \equiv (-)^{Q_+ + Q_-}\mathcal{U}_-^* \tag{5.12}$$

commute and lower the charges by one unit.

Comparing with Sect. 2, it is not hard to guess the next step: this consists in identifying the space $\mathcal{F}_{0,0}$ of this section with the one of Sect. 2, i.e., with the neutral boson Fock space $\mathcal{F}_s(L^2(\mathbb{R}, dk))$, since the S_s can then be used to extend the isomorphism to the whole space. To this end we recall some well-known facts. First, the currents can be written in terms of the field

$$\varphi_0(t,x) \equiv \frac{\pi^{1/2}}{2}\int dy[\theta(y-x) - \theta(x-y)][J_+(t,y) + J_-(t,y)] \tag{5.13}$$

by virtue of current conservation. Explicitly,

$$J_s = \frac{-1}{2\pi^{1/2}}(\partial_x - s\partial_t)\varphi_0. \tag{5.14}$$

The current algebra

$$[J_s(t,x), J_{s'}(t,y)] = s\delta_{ss'}\delta'(x-y)/2i\pi \tag{5.15}$$

implies that φ_0 satisfies the equal-time CCR. Since one also has $\Box\varphi_0 = 0$, φ_0 is a massless free pseudo-scalar neutral boson field, and can be written

$$\varphi_0(t,x) = \int\frac{dk}{(4\pi|k|)^{1/2}}[c(k)\exp(-i|k|t + ikx) + \text{h.c.}], \tag{5.16}$$

where the $c^{(*)}(k)$ satisfy the usual CCR. From (5.14) it then follows that

$$J_s(t,x) = \frac{s}{2i\pi}\int dk\theta(sk)|k|^{1/2}[c(k)\exp(ik(x-st)) - \text{h.c.}]. \tag{5.17}$$

If one inverts this, one can infer that $c(k)\Omega = 0$, so that one is dealing with the Fock representation of the CCR on $\mathscr{F}_{0,0}$.

Now it will be clear how to continue: one may identify the vectors $\prod_i c^*(f_i)\Omega$ from this section with the vectors $\prod_i A^*(f_i)\Omega$ from Sect. 2. Of course, for our notation to be consistent, it should then be shown that one can obtain each vector in the vacuum sector $\mathscr{F}_{0,0}$ of this section through linear combinations and limits of the former vectors, as this holds true for the latter vectors in the sector $\mathscr{F}_{0,0}$ from Sect. 2. But this is a consequence of [15, 18, 20, 12].

Under the isomorphism just described we get boson annihilators $A(k)$ and $\alpha(k)$ and operators $W(f)$ and $\exp[ij_s(\gamma)]$ on the fermion Fock space. We now explain how these are related to the $c(k)$ and $\exp[iJ_s(\gamma)]$ of this section. First, by construction $A(k)$, $\alpha(k)$ and $c(k)$ coincide on the vacuum sector. Hence,

$$W(g_s)F = \exp[ij_s(\gamma)]F = \exp[iJ_s(\gamma)]F, \qquad F \in \mathscr{F}_{0,0}, \tag{5.18}$$

where the notation (4.8) is used. Second, it can be shown that

$$\mathscr{U}_s \exp[iJ_s(\gamma)] = \exp[-i\zeta(\gamma)]\exp[iJ_s(\gamma)]\mathscr{U}_s, \tag{5.19}$$

$$\mathscr{U}_s \exp[iJ_{-s}(\gamma)] = \exp[iJ_{-s}(\gamma)]\mathscr{U}_s, \tag{5.20}$$

where $\zeta(\gamma)$ is given by (4.9), cf. [18, 20, 12]. (This can be formally verified by pretending that $\mathscr{U}_s$ can be written as $\exp[isJ_s(\eta)]$ and then using the current algebra (5.15).) Using (5.12) and comparing with (4.25)–(4.26), it follows that for $(\rho, \sigma) = (1, 0)$, j_s coincides with J_s, and hence $\alpha(k)$ with $c(k)$ and φ with φ_0.

For these values of the parameters the approximate fields, transported to the fermion Fock space, can be written

$$\psi_{s,\varepsilon}(t, y) = \frac{(-)^{Q_+ + Q_-}}{(4\pi\varepsilon)^{1/2}} \chi_\varepsilon(t - sy)^{1/2} \exp[isJ_s(\delta^\varepsilon_{y-st})]\mathscr{U}^*_s \tag{5.21}$$

by virtue of the above and (4.23). The cutoff functions R and L were chosen precisely so that this would hold true under the identifications just described. To explain this, recall that $\mathscr{U}^*_s$ implements the gauge transformation corresponding to $\exp[-is\eta(x)]$. Thus, the product of the two unitaries occurring here implements the gauge transformation corresponding to $\exp[-is\eta((x-(y-st))/\varepsilon)]$, cf. (4.18) and (5.8). Previously, two of us had proved that the right-hand side of (5.21) (with a fixed choice of phase for $\mathscr{U}_s$) converges to the free field ψ^0_s in the following precise sense: let F be an algebraic tensor whose constituent functions are C^∞_0 and let $f(x)$ be a function whose Fourier transform is C^∞_0. Then one has

$$\operatorname*{s.lim}_{\varepsilon\to 0} \int dy \psi^{(*)}_{s,\varepsilon}(0, y) f(y) F = \int dx \psi^{0(*)}_s(0, x) f(x) F. \tag{5.22}$$

The proof of this did not involve n-point functions. Rather, it hinged on the simplicity of the implementer corresponding to the gauge function $\eta(x)$ and its scaled translates; we could show that the latter functions are the only ones for which the implementer has this simple structure [12]. Subsequently we discovered that this function also has the unique property of leading to the desired n-point functions with $\varepsilon > 0$.

The fact that the limits of the approximate fields are cyclic on the vacuum for $(\rho, \sigma) = (1, 0)$ (as claimed in the previous section) is evident from (5.22) and the fact that $\psi_{s,\varepsilon}^{(*)}(t, x)$ only depends on $x - st$, cf. (5.21). The assertion for the three other cases then follows from an observation that is of interest in its own right: there exist unitary operators $\mathscr{R}$ and $\mathscr{E}$ such that $\mathscr{R}^2 = \mathscr{E}^2 = \mathbb{1}$ and

$$\mathscr{R}\psi_{s,\varepsilon}(t, x; \rho, \sigma)\mathscr{R} = \psi_{s,\varepsilon}(t, x; -\rho, -\sigma), \tag{5.23}$$

$$\mathscr{E}\psi_{s,\varepsilon}(t, x; \rho, \sigma)\mathscr{E} = \psi_{s,\varepsilon}(t, -x; \sigma, \rho). \tag{5.24}$$

These reversal and exchange operators are most easily described on the space $\mathscr{F}$ of Sect. 2: by definition they equal $\Gamma_B(-\mathbb{1})$ and $\Gamma_B(P)$, respectively, on each sector. Here, $\Gamma_B(U)$ denotes the usual product operator corresponding to the one-boson operator U on $\mathscr{H}$, and P is the one-boson parity operator

$$(Pf)(k) \equiv -f(-k), \qquad f \in \mathscr{H}. \tag{5.25}$$

Note that (5.23) and (5.24) imply that one may as well restrict the parameters to the wedge $\rho \geqq |\sigma|$.

To conclude this section, let us emphasize that the convergence of the approximate fields $\psi_{s,\varepsilon}$ with $(\rho, \sigma) = (1, 0)$ to fields ψ_s with the free n-point functions is *not enough* to conclude cyclicity of ψ_s. Of course, the free field ψ^0 is cyclic, but one needs an additional argument, here given by (5.22), to conclude that ψ equals ψ^0. To convince those readers who are still with us that we are not out to kill phantoms, an example may be in order. Let U_T be unitary operators on the one-fermion space $L^2(\mathbb{R}, dp)^2$ which for $T \to \infty$ strongly converge to an isometric operator U_∞ that is not unitary (think of wave operators and bound states). Then for any $T \leqq \infty$ the fields.

$$\psi_T(t, x) \equiv \Gamma_F(U_T)\psi^0(t, x)\Gamma_F(U_T^*) \tag{5.26}$$

all have the same n-point functions as ψ^0. (Here, $\Gamma_F(U)$ denotes the product operator on $\mathscr{F}_a$ corresponding to the one-fermion operator U.) Moreover, the fields ψ_T are clearly cyclic on Ω for $T < \infty$. But the field ψ_∞ is *not* cyclic on Ω, in spite of the fact that the fields ψ_T (on their polynomial domains) strongly converge to it for $T \to \infty$.

Appendix

In this appendix we sketch the proofs of Lemmas 3.3 and 3.4.

Proof of Lemma 3.3. To assist the reader in gaining a quick understanding of the calculations needed to verify the lemma we shall first detail the special case $\sigma = 0$ and then indicate how the general case is handled.

Inserting Definition 2.2 with $\sigma = 0$ in the left-hand side of (3.4) we get

$$\left(\frac{1}{2\pi}\right)^n \left(\frac{1}{2}\right)^{n\rho^2} \prod_{j=1}^{n} \left(\frac{1}{\varepsilon_j \tau_j}\right)^{(1/2)\rho^2} \chi_{\varepsilon_j}(u_j)^{\rho^2(j-1/2)} \chi_{\tau_j}(-U_j)^{\rho^2(n-j+1/2)}$$
$$\cdot(\Omega, W(\rho L_{u_1}^{\varepsilon_1}) S_- \ldots S_-^* W(-\rho L_{U_n}^{\tau_n})\Omega). \tag{A.1}$$

Since $S_-^{(*)}$ and $W(f)$ commute, we may cancel all S_- and S_-^*, after which we apply Proposition 2.1. The resulting inner products may all be evaluated using (2.17). Then (A.1) can be written

$$\left(\frac{1}{2\pi}\right)^n \left(\frac{1}{2}\right)^{n\rho^2} \prod_1^n \left(\frac{1}{\varepsilon_j \tau_j}\right)^{(1/2)\rho^2} \chi_{\varepsilon_j}(u_j)^{\rho^2(j-1/2)} \chi_{\tau_j}(-U_j)^{\rho^2(n-j+1/2)}$$
$$\cdot \left[\frac{(1+\varepsilon_j+iu_j)(1+\varepsilon_j-iu_j)}{4\varepsilon_j}\right]^{-(1/2)\rho^2} \left[\frac{(1+\tau_j+iU_j)(1+\tau_j-iU_j)}{4\tau_j}\right]^{-(1/2)\rho^2}$$
$$\cdot \prod_{j<k} \left[\frac{(1+\varepsilon_j+iu_j)(1+\varepsilon_k-iu_k)}{2i(u_j-u_k-i(\varepsilon_j+\varepsilon_k))}\right]^{-\rho^2} \left[\frac{(1+\tau_j+iU_j)(1+\tau_k-iU_k)}{2i(U_j-U_k-i(\tau_j+\tau_k))}\right]^{-\rho^2}$$
$$\cdot \prod_{j,k} \left[\frac{(1+\varepsilon_j+iu_j)(1+\tau_k-iU_k)}{2i(u_j-U_k-i(\varepsilon_j+\tau_k))}\right]^{\rho^2}. \tag{A.2}$$

Comparison with (3.4) for $\sigma=0$ shows that the translation-invariant part comes out as claimed, so that it remains to check 1) the power of i, 2, ε_j and τ_j, and 2) the cancellation of the non-invariant terms. Now checking 1) is trivial, so let us consider 2): the offending u_j-dependent terms are

$$\chi_{\varepsilon_j}(u_j)^{\rho^2(j-1/2)}[(1+\varepsilon_j+iu_j)(1+\varepsilon_j-iu_j)]^{-(1/2)\rho^2}$$
$$\cdot[(1+\varepsilon_j+iu_j)^{n-j}(1+\varepsilon_j-iu_j)^{j-1}]^{-\rho^2}(1+\varepsilon_j+iu_j)^{n\rho^2}. \tag{A.3}$$

But using (2.8) one sees that this is just a complicated way to write 1. The unwanted U_j-dependent terms drop out similarly, proving (3.4) for $\sigma=0$.

The second assertion is now immediate from Def. 2.2 and the preceding calculation.

To prove (3.5) for $\sigma=0$, we insert Def. 2.2 in the left-hand-side, obtaining

$$\left(\frac{1}{2\pi}\right)^n \left(\frac{1}{2}\right)^{n\rho^2} \prod_1^n \left(\frac{1}{\varepsilon_j \tau_j}\right)^{(1/2)\rho^2} \prod_1^l \chi_{\varepsilon_j}(u_j)^{\rho^2(j-1/2)}$$
$$\cdot \prod_{l+1}^n \chi_{\varepsilon_j}(v_j)^{\rho^2(j-l-1/2)} \prod_1^l \chi_{\tau_j}(-U_j)^{\rho^2(l-j+1/2)} \prod_{l+1}^n \chi_{\tau_j}(-V_j)^{\rho^2(n-j+1/2)}$$
$$\cdot\Big(\Omega, \prod_1^l W(\rho L_{u_j}^{\varepsilon_j})S_- \prod_{l+1}^n (-)^{Q_-} W(\rho R_{v_j}^{\varepsilon_j})S_+ \prod_1^l S_-^* W(-\rho L_{U_j}^{\tau_j})$$
$$\cdot \prod_{l+1}^n S_+^* W(-\rho R_{V_j}^{\tau_j})(-)^{Q_-}\Omega\Big). \tag{A.4}$$

The factors $(-)^{Q_-}$ can be traded against a factor $(-)^{l(n-l)}$, after which the shifts can be cancelled. The resulting inner product can be written as the product of two inner products, one involving all R's, the other one all L's, in the order indicated in (A.4). (This follows from Proposition 2.1 and the orthogonality of R and L, cf. (2.9)–(2.10).) The result of this is $(-)^{l(n-l)}$ times (A.1) (with n replaced by l) times the analogous expression for the $\psi_{+,\varepsilon}^{(*)}$, proving (3.5) for $\sigma=0$.

Let us now consider the general case. If one cancels the shifts in the analog of

(A.1) and splits the inner product into two inner products containing all the R's and all the L's, respectively, one obtains the product of two expressions essentially the same as (A.1), so that (3.4) follows from the above proof for the special case $\sigma = 0$. The second assertion of Lemma 3.3 is again obvious from Def. 2.2, so that it remains to establish (3.5). Using arguments familiar by now one sees that its left-hand side can be written

$$(-)^{l(n-l)}\left(\frac{1}{2\pi}\right)^n \cdot \left\{\left(\frac{1}{2}\right)^{n\rho^2} \prod_1^n \left(\frac{1}{\varepsilon_j \tau_j}\right)^{(1/2)\rho^2} \prod_1^l \chi_{\varepsilon_j}(u_j)^{\rho^2(j-1/2)}\right.$$
$$\cdot \prod_{l+1}^n \chi_{\varepsilon_j}(u_j)^{\sigma^2(j-l-1/2)+\rho\sigma l} \prod_1^l \chi_{\tau_j}(-U_j)^{\rho^2(l-j+1/2)+\rho\sigma(n-l)} \prod_{l+1}^n \chi_{\tau_j}(-U_j)^{\sigma^2(n-j+1/2)}$$
$$\left.\cdot\left(\Omega, \prod_1^l W(\rho L_{u_j}^{\varepsilon_j}) \prod_{l+1}^n W(\sigma L_{u_j}^{\varepsilon_j}) \prod_1^l W(-\rho L_{U_j}^{\tau_j}) \prod_{l+1}^n W(-\sigma L_{U_j}^{\tau_j})\Omega\right)\right\}$$
$$\{\rho \leftrightarrow \sigma, u \to v, U \to V, L \to R\}, \tag{A.5}$$

where the notation will be clear. Pairing the first and third/second and fourth group of W's in the two inner products gives rise to the $2l$-point function of $\psi_-/2(n-l)$-point function of ψ_+ occurring at the right-hand-side of (3.5), provided the result of the pairing is combined with all factors up front that do not involve $\rho\sigma$. Thus it remains to show that

$$\prod_{l+1}^n \chi_{\varepsilon_j}(u_j)^{\rho\sigma l} \prod_1^l \chi_{\tau_j}(-U_j)^{\rho\sigma(n-l)} \prod_{j=1}^l \prod_{k=l+1}^n \exp\left(\tfrac{1}{2}\rho\sigma\left[-(L_{u_j}^{\varepsilon_j}, L_{u_k}^{\varepsilon_k})\right.\right.$$
$$\left.\left. + (L_{u_j}^{\varepsilon_j}, L_{U_k}^{\tau_k}) + (L_{u_k}^{\varepsilon_k}, L_{U_j}^{\tau_j}) - (L_{U_j}^{\tau_j}, L_{U_k}^{\tau_k})\right]\right)$$
$$= \prod_{j=1}^l \prod_{k=l+1}^n \left(\frac{[u_j - u_k - i(\varepsilon_j + \varepsilon_k)][U_j - U_k - i(\tau_j + \tau_k)]}{[u_j - U_k - i(\varepsilon_j + \tau_k)][u_k - U_j - i(\varepsilon_k + \tau_j)]}\right)^{\rho\sigma}. \tag{A.6}$$

But from (2.17) it follows that this again amounts to checking that the constants and non-invariant terms drop out, and we trust the reader believes at this point that they do. □

Proof of Lemma 3.4. From the Weyl relations (2.2) and from (2.17) and (2.8) it follows that

$$W(eF_r^\varepsilon)W(e'F_{r'}^{\varepsilon'}) = \left(\chi_\varepsilon(r)\chi_{\varepsilon'}(-r')\left[\frac{i(r-r')+\varepsilon+\varepsilon'}{\text{c.c.}}\right]\right)^{ee'}$$
$$\cdot W(e'F_{r'}^{\varepsilon'})W(eF_r^\varepsilon), \qquad F = R, L. \tag{A.7}$$

With the help of this relation and (2.7) the commutation relations (3.6)-(3.11) are readily verified. □

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