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A Shorter Proof of the Existence of the Feigenbaum Fixed Point

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Abstract. We use the Leray-Schauder Fixed Point Theorem to prove the existence of an analytic fixed point for the period doubling accumulation renormalization operator. Our argument does not, however, show that the linearization of the renormalization operator at this fixed point is hyperbolic.

1. Introduction

Two independent proofs (Campanino et al. [1, 2], Lanford [4]) have been given for the existence of an even analytic solution to the Feigenbaum-Cvitanović functional equation [3]

$$g(x) = -\frac{1}{\lambda} g(g(-\lambda x)), \qquad g(0) = 1 \tag{1.1}$$

with

$$g''(0) < 0; \ \lambda \equiv -g(1) > 0 .$$

Both of these proofs rely on extensive computations. In this paper, we give yet another proof, based on the Leray-Schauder Fixed Point Theorem, which, if still fundamentally computational in nature, requires a substantially smaller amount of computation. It should be noted that the argument given here, like that of Campanino et al. and unlike the author's computer assisted proof, does *not* establish the spectral properties of the linearization of the renormalization operator at the fixed point g which are essential for the application of g to the analysis of period-doubling accumulation.

We will work in a space of even mappings f of $[-1, 1]$ to itself, satisfying the normalization condition

$$f(0) = 1 , \tag{1.2}$$

expressed as functions of x^2. Since we are working with x^2 as the independent variable, the renormalization operator has the form

$$\mathcal{T}f(x) = -\frac{1}{\lambda} f([f(\lambda^2 x)]^2), \qquad \lambda \equiv -f(1) . \tag{1.3}$$

We will frequently write $\bar{f}$ for $\mathscr{T}f$. If we define μ by

$$f'(0) = -(1+\lambda+\mu),$$

and write

$$f_0(x) \equiv 1-(1+\lambda)x-\mu x(1-x),$$
$$f_1(x) \equiv f(x)-f_0(x),$$

then

$$f_1(0)=f_1(1)=f_1'(0)=0,$$

so f_1 is uniquely determined by its third derivative. We will denote $f_1'''(x)=f'''(x)$ by $h(x)$. The triple (λ, μ, h) will serve as a set of coordinates for the space of mappings in which we work. Ultimately, h will be analytic on a complex neighborhood of $[0,1]$, but for much of the argument we can take h to be simply a continuous function on $[0,1]$. We will write $\bar{\lambda}(\lambda, \mu, h)$, $\bar{\mu}(\lambda, \mu, h)$, $\bar{h}(\lambda, \mu, h)$ for the coordinates of $\bar{f} = \mathscr{T}f$.

The correspondence which associates with any continuous function h on $[0,1]$ the unique function f_1 such that

$$f_1'''(x)=h(x); \qquad f_1(0)=f_1(1)=f_1'(0)=0$$

is linear and can be written as an integral operator

$$f_1(x) = \int_0^1 K(x,y)\,h(y)\,dy \tag{1.4}$$

with kernel $K(x,y)$ which can easily be written explicitly (see Sect. 2). We will also use K to denote the operator, i.e., we will write $f_1 = Kh$ as a shorthand for (1.4). Thus, the f corresponding to the triple (λ, μ, h) can be written as

$$f(x) = 1-(1+\lambda+\mu)x+\mu x^2+Kh(x). \tag{1.5}$$

Because the renormalization operator $\mathscr{T}$ is expansive in one direction at the Feigenbaum fixed point, no small neighborhood of the fixed point can be invariant for $\mathscr{T}$. To get to a situation where we can apply the Leray-Schauder Fixed Point Theorem, we introduce an auxiliary operator with an equivalent fixed-point problem but which does admit small invariant neighborhoods of the fixed point we are looking for. This operator will act only on the h coordinate and will be constructed as follows: We first show that, for any h in an appropriate domain, the pair of equations

$$\bar{\lambda}(\lambda, \mu, h) = \lambda, \qquad \bar{\mu}(\lambda, \mu, h) = \mu \tag{1.6}$$

has a unique solution $(\lambda^*, \mu^*) \approx (0.4, 0.1)$. The auxiliary operator is then defined to map h to $h^* \equiv \bar{h}(\lambda^*(h), \mu^*(h), h)$. Fixed points for this auxiliary operator correspond in an obvious way to fixed points for the renormalization operator itself.

To show that the auxiliary operator is well-defined and admits a fixed point, we are going to prove:

Lemma 1.1. *Let h be a continuous real-valued function on $[0, 1]$ satisfying*

$$0 \leqq h(x) \leqq 0.32(1-0.36x) \quad for \quad 0 \leqq x \leqq 1 . \tag{1.7}$$

Then there is a unique pair (λ^*, μ^*) with

$$0.396 \leqq \lambda^* \leqq 0.4031 , \qquad 0.09 \leqq \mu^* \leqq 0.16$$

such that

$$\bar{\lambda}(\lambda^*, \mu^*, h) = \lambda^* , \qquad \bar{\mu}(\lambda^*, \mu^*, h) = \mu^* . \tag{1.8}$$

Furthermore, λ^* and μ^* vary continuously with h.

Lemma 1.2. *If h satisfies* (1.7) *and if λ^*, μ^* are as in Lemma 1.1 then $h^* \equiv \bar{h}(\lambda^*, \mu^*, h)$ also satisfies* (1.7).

For $\delta > 0$, let D_δ denote the set of complex numbers at distance less than δ from $[0, 1]$.

Lemma 1.3. *If δ is sufficiently small, if h is analytic on D_δ, satisfies* (1.7), *and in addition satisfies*

$$|h(z)| \leqq 0.32(1-0.36|z|) \quad on \quad D_\delta , \tag{1.9}$$

then h^* is analytic on $D_{3/2\delta}$ and satisfies

$$|h^*(z)| \leqq 0.32(1-0.36|z|) \quad on \quad D_{3/2\delta} . \tag{1.10}$$

The correspondence $h \mapsto h^*$ is continuous from the space of functions analytic on D_δ satisfying (1.7) *and* (1.9) *to the space of functions analytic on $D_{3/2\delta}$ satisfying* (1.7) *and* (1.10), *both spaces equipped with the topology of uniform convergence.*

These three lemmas, and the Leray-Schauder Fixed Point Theorem, immediately imply:

Theorem. *There exists a function $g(x)$ defined and analytic on a neighborhood of $[-1, 1]$, even, with $g(0) = 1$, and satisfying*

$$g(x) = -\frac{1}{\lambda} g(g(-\lambda x)) .$$

Furthermore, $\lambda (= -g(1)) \in [0.396, 0.4031]$,

$$-2(1+\lambda+0.16) \leqq g''(0) \leqq -2(1+\lambda+0.09) ,$$

and, writing $g(x) = G(x^2)$ (which is possible because g is even),

$$0 \leqq G'''(x) \leqq 0.32(1-0.36x) \quad for \quad 0 \leqq x \leqq 1 .$$

Lemmas 1.1, 1.2, and 1.3 will be proved in Sects. 4–6, respectively. In Sect. 2, we establish some properties of the kernel $K(x, y)$ of (1.4), and in Sect. 3 we prove a number of estimates used repeatedly in later sections.

It will be a universal notational convention, for the remainder of this paper, that the symbols f, λ, μ, h, f_0, f_1 are always assumed to be related as above. We also adopt *as standing assumptions that* λ *and* μ *denote real numbers satisfying*

$$0.396 \leqq \lambda \leqq 0.4031; \qquad 0.09 \leqq \mu \leqq 0.16, \tag{1.11}$$

and h a function defined and continuous (at least) on $[0, 1]$, *satisfying*

$$0 \leqq h(x) \leqq 0.32(1 - 0.36x). \tag{1.12}$$

These assumptions will be used very frequently, generally without explicit reference.

In the course of the proof, we need to make a considerable number of concrete numerical estimates. To take an example at random: At one point, we use the fact that, if $\lambda = 0.4031$ and $\mu = 0.16$, then $(1+\lambda)\lambda^2 + \mu\lambda^2(1-\lambda^2) < 0.2498$. Estimates like this were verified with the aid of a Hewlett-Packard HP-15C calculator. This calculator stores and manipulates numbers in a decimal floating point format with ten digit fraction and two digit exponent; we assumed only that it would perform correctly the operations of addition, subtraction, and multiplication on pairs of operands for which the result can be represented exactly in this format. In practice, this meant that intermediate results were rounded – up or down, depending on the sense of the inequality to be proved – to five digits before being multiplied together. Also, the results of divisions were verified by multiplying back after rounding. To take the above example:

$$\lambda^2 = 0.16248961 < 0.16249,$$

$$\lambda^2(1-\lambda^2) < 0.16249(1-0.16249) = 0.1360869999 < 0.13609,$$

$$\mu\lambda^2(1-\lambda^2) < 0.16 \times 0.13609 = 0.0217744 < 0.02178,$$

$$(1+\lambda)\lambda^2 < 1.4031 \times 0.16249 = 0.227989719 < 0.22799,$$

$$(1+\lambda)\lambda^2 + \mu\lambda^2(1-\lambda^2) < 0.22799 + 0.02178 = 0.24977 < 0.2498.$$

This approach to proving such numerical inequalities is no doubt more cautions than is really justified [1], but it does have the merit of relying as little as possible on the correctness of the calculator.

From a broader point of view, the question of what constitutes a satisfactory proof of an explicit numerical estimate like the one above provides an illuminating caricature of the issues involved in "computer assisted proofs" in general. It hardly seems reasonable to insist that the arithmetic operations be carried out by hand, but relying on results of individual arithmetic operations performed by an electronic calculator does not differ in a fundamental way from relying on results of more complicated sequences of operations performed by a larger computer.

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1 Especially since Hewlett-Packard, in a departure from the standard practice of calculator manufacturers, has published an explicit and unambiguous statement on the accuracy the HP-15C is supposed to attain. For the four basic arithmetic operations, in the absence of underflow and overflow, it is asserted that the result returned differs from the exact result by no more than one-half unit in the last (i.e., tenth) place. This statement is labelled as a design objective which the designers believe that they can prove they have attained rather than as a guaranteed specification; it appears in the Appendix "Accuracy of Numerical Calculations", pp. 172–211 of *The HP-15C Advanced Functions Handbook*, part #00015-90011. I am indebted to W. Kahan for pointing this reference out to me

2. The Kernel K

Let

$$K(x,y)\equiv -x^2(1-y)^2/2+\theta(x-y)(x-y)^2/2\,, \tag{2.1}$$

where $\theta(z)=0$ for $z<0$ and $\theta(z)=1$ for $z>0$. We will also write

$$K'(x,y)\equiv \frac{\partial K}{\partial x}(x,y)=-x(1-y^2)+\theta(x-y)(x-y)\,, \tag{2.2}$$

$$K''(x,y)\equiv \frac{\partial K'}{\partial x}(x,y)=-(1-y^2)+\theta(x-y)\,. \tag{2.3}$$

If h is a continuous function on $[0,1]$, we define

$$(Kh)(x)=\int_0^1 K(x,y)\,h(y)\,dy\,.$$

Proposition 2.1. *Kh is the unique three-times continuously differentiable function on $[0,1]$ such that*

$$Kh(0)=Kh(1)=Kh'(0)=0\,,\qquad (Kh)'''(x)=h(x)\,.$$

Furthermore:

$$(Kh)'(x)=\int_0^1 K'(x,y)\,h(y)\,dy\,, \tag{2.4}$$

$$(Kh)''(x)=\int_0^1 K''(x,y)\,h(y)\,dy\,. \tag{2.5}$$

Proof. It follows easily from standard results about differentiating under the integral sign that Kh is twice continuously differentiable and that (2.4) and (2.5) hold. From (2.5) and the formula for $K''(x,y)$,

$$(Kh)''(x)=\int_0^x h(y)\,dy-\int_0^1 (1-y)^2\,h(y)\,dy\,,$$

from which it follows that Kh is three times differentiable and that $(Kh)'''(x)=h(x)$. It is immediate from the definitions that $Kh(0)=Kh(1)=0$, and from (2.4) that $(Kh)'(0)=0$.

In the following proposition x and y denote general points of $[0,1]$, i.e., an assertion containing an unquantified x (or y) should be understood as holding for all x (or y) in $[0,1]$. Also, we write $K'_+(x,y)$ [respectively $K''_+(x,y)$, $K''_-(x,y)$] for the positive part of $K'(x,y)$ [respectively, the positive, negative parts of $K''(x,y)$], i.e., the larger of 0 and $K'(x,y)$ [respectively, $K''(x,y)$, $-K''(x,y)$].

Proposition 2.2. 1. $K(x,y) \leqq 0$.

2. $\int_0^1 K(x,y)\,dy = -x^2(1-x)/6$.

3. $\int_0^1 K(x,y)(1-r\cdot y)\,dy = -x^2(1-x)[(1-r/4)-(r/4)x]/6$ *for all real* r.

4. $K'(x,y) \leqq 0$ *for* $x \leqq 1/2$.

5. $K'(1,y) = y(1-y) \geqq 0$.

6. $\int_0^1 K'(1/2,y) = -1/24$.

7. $\int_0^1 K'(1,y)(1-r\cdot y)\,dy = 1/6 - r/12$ *for all real* r.

8. $\int_0^1 K'_+(x,y)\,dy \leqq 1/6$.

9. $\int_0^1 K''_+(x,y)\,dy = x^2 - x^3/3$.

10. $\int_0^1 K''_+(x,y)(1-r\cdot y)\,dy \leqq 2/3(1-5r/8)\cdot x$ *for* $0 \leqq r \leqq 4/7$.

11. $\int_0^1 K''_-(x,y)(1-r\cdot y)\,dy \leqq 1/3 - r/12$ *for* $0 \leqq r \leqq 1$.

Proof. 1. The assertion follows at once from the formula (2.1) for $K(x,y)$ if $x<y$. If $x>y$

$$\begin{aligned} K(x,y) &= -x^2(1-y)^2/2 + (x-y)^2/2 \\ &= 1/2[(x-y)+x(1-y)][(x-y)-x(1-y)] \\ &= -1/2[(x-y)+x(1-y)]\,y(1-x) \leqq 0\,. \end{aligned}$$

2. This is a special case of 3..

3. By Proposition 2.1 (with $h = 1 - rx$), the left-hand side is the unique function vanishing to second order at 0 and to first order at 1 with third derivative equal to $1-rx$; it is easy to see that the right-hand side has these properties.

4. It is immediate from the formula that $K'(x,y) \leqq 0$ for $x<y$. For $x>y$,

$$K'(x,y) = -x(1-y)^2 + (x-y) = [2x-1-xy]\,y\,,$$

which is manifestly negative if $x \leqq 1/2$.

5. Insert $x=1$ in the preceding formula for $K'(x,y)$, which is valid when $x>y$.

6. Differentiate 2. and put $x=1/2$.

7. Evaluate the integral explicitly, using 5. (or differentiate 3. and put $x=1$).

8. By 4., the left-hand side vanishes for $x \leqq 1/2$, so we have only to consider $x>1/2$. Also, by the proof of 4., $K'_+(x,y)=0$ if either $y>x$ or $y>(2x-1)/x$. Since $(2x-1)/x \leqq x$, we can ignore the first condition: $K'_+(x,y) = [2x-1-xy]y$ for $y<(2x-1)/x$ and 0 otherwise. Hence:

$$\begin{aligned} \int_0^1 K'_+(x,y)\,dy &= \int_0^{(2x-1)/x} [(2x-1)-xy]\,y\,dy \\ &= (2x-1)[(2x-1)/x]^2 \int_0^1 (1-z)z\,dz = (2-1/x)^2(2x-1)/6 \leqq 1/6\,. \end{aligned}$$

9. From the definition of $K''(x,y)$,

$$\begin{aligned} K''_+(x,y) &= 0 && \text{for} \quad x<y \\ &= 1-(1-y)^2 = 2y-y^2 && \text{for} \quad x>y\,. \end{aligned}$$

Hence

$$\int_0^1 K''_+(x,y)\,dy = \int_0^x (2y-y^2)\,dy = x^2 - x^3/3\,.$$

10. From the expression for $K''_+(x,y)$ obtained in the proof of 9.,

$$\int_0^1 K''_+(x,y)(1-ry)\,dy = x[x-(1+2r)x^2/3 + rx^3/4]\,.$$

What we have to show, therefore, is that

$$[x-(1+2r)x^2/3 + rx^3/4] \leqq 2/3(1-5r/8) \quad \text{for} \quad 0 \leqq r \leqq 4/7\,.$$

Since both sides of this inequality are affine in r, it suffices to prove it for $r=0$ and $r=4/7$. For $r=0$, it reduces to $x-x^2/3 \leqq 2/3$, which is immediate and for $r=4/7$ to $x-5x^2/7+x^3/7 \leqq 3/7$, or $0 \geqq x^3-5x^2+7x-3=(x-1)^2(x-3)$, which is also immediate.

11. From the formula for $K''(x,y)$,

$$\begin{aligned} K''_-(x,y) &= (1-y)^2\,, && x<y \\ &= 0\,, && x>y\,. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 K''_-(x,y)(1-ry)\,dy &= \int_x^1 (1-y)^2(1-ry)\,dy \\ &= (1-r)(1-x)^3/3 + r(1-x)^4/4 \leqq 1/3 - r/12\,. \end{aligned}$$

3. Some Estimates

We collect in the following proposition a number of estimates which will be used repeatedly. In this section, as in the preceding one, x denotes a general point of $[0,1]$.

Proposition 3.1. 1. $f''(x) > 0$.
2. $1.192 < -f'(x) \leqq 1+\lambda+\mu \leqq 1.5631$.
3. $0.7491 < f(\lambda^2) < 0.7692$.
4. $0.5611 < [f(\lambda^2)]^2 < 0.5917$.
5. $f(\lambda^2 x) \leqq 1 - 0.2308x$.
6. $\lambda^5(1+\lambda+\mu)\,f(\lambda^2 x) < 0.016638(1-0.2497x)$.
7. $\lambda^5(1+\lambda+\mu)\,[f(\lambda^2 x)]^2 < 0.016638(1-0.4370x)$.
8. $\lambda^5(1+\lambda+\mu)\,[f(\lambda^2 x)]^3 < 0.016638(1-0.5776x)$.
9. $[f(\lambda^2 x)]^2 \geqq 1 - 0.5082x$.
10. $-f'(x) < 1.4165$ *for* $x \geqq 0.5$.
11. $f''(\lambda^2 x) < 0.328$.

Proof. 1. Using Proposition 2.2.11,

$$f''(x)=2\mu+(Kh)''(x)\geqq 2\mu-0.32(1/3-0.36/12)>0.$$

2. By 1. and Proposition 2.2.7,

$$1+\lambda+\mu=-f'(0)\geqq -f'(x)\geqq -f'(1)=1+\lambda-\mu-0.32(1/6-0.36/12)>1.192.$$

3.

$$f(\lambda^2)=1-(1+\lambda+\mu)\lambda^2+\mu\lambda^4+Kh(\lambda^2).$$

Since, by Proposition 2.2.1, $Kh(\lambda^2)\leqq 0$, $f(\lambda^2)\leqq 1-(1+\lambda)\lambda^2-\mu\lambda^2(1-\lambda^2)$. The expression on the right is decreasing in μ and λ separately, so we get an upper bound by inserting $\lambda=0.396$ and $\mu=0.09$, which gives

$$f(\lambda^2)<0.7692.$$

To get a lower bound, we apply Proposition 2.2.3 to estimate

$$-Kh(\lambda^2)\leqq 1/6\lambda^4(1-\lambda^2)\,[0.91-0.09\lambda^2]\cdot 0.32.$$

It is easy to check (e.g., by taking logarithmic derivatives) that the expression on the right is increasing in λ, so we get an upper bound by inserting $\lambda=0.4031$, so $-Kh(\lambda^2)<0.00106$.

Thus

$$f(\lambda^2)\geqq 1-(1+\lambda)\lambda^2-\mu\lambda^2(1-\lambda^2)-0.00106>0.7491.$$

4. This follows from 3. by taking squares.

5. Since f is convex,

$$f(\lambda^2x)\leqq(1-x)f(0)+xf(\lambda^2)=1-(1-f(\lambda^2))x,$$

and, by 3., $1-f(\lambda^2)>1-0.7692=0.2308$.

6, 7, and 8. As in 5., $f(\lambda^2x)\leqq 1-(1-f(\lambda^2))x$ and

$$1-f(\lambda^2)=(1+\lambda)\lambda^2+\mu\lambda^2(1-\lambda^2)-Kh(\lambda^2)\geqq(1+\lambda)\lambda^2+\mu\lambda^2(1-\lambda^2).$$

If we define

$$\varrho\equiv(1+\lambda)\lambda^2+\mu\lambda^2(1-\lambda^2),$$

what we have just shown is that $f(\lambda^2x)\leqq 1-\varrho x$.

We now claim that, for $j=1,2,3$,

$$\lambda^5(1+\lambda+\mu)(1-\varrho x)^j \tag{3.1}$$

is increasing in λ and μ separately. To show that (3.1) is increasing in λ, we take its logarithmic derivative; what we have to show is that

$$\frac{5}{\lambda}+\frac{1}{1+\lambda+\mu}\geqq\frac{jx}{1-\varrho x}\frac{\partial\varrho}{\partial\lambda}.$$

Since the expression on the right is increasing in j and x (and the expression on the left doesn't depend on these quantities), it suffices to consider $j=3$ and $x=1$, i.e., to show that

$$\frac{5}{\lambda}+\frac{1}{1+\lambda+\mu} \geqq \frac{3}{1-\varrho}\frac{\partial \varrho}{\partial \lambda}. \tag{3.2}$$

Now

$$\frac{\partial \varrho}{\partial \lambda}=2\lambda+3\lambda^2+2\mu\lambda-4\mu\lambda^3<2\lambda+3\lambda^2+2\mu\lambda<1.5\,,$$

and $\varrho<0.2498$ so $1-\varrho>0.75$. Thus

$$\frac{3}{1-\varrho}\frac{\partial \varrho}{\partial \lambda}<\frac{3}{0.75}1.5=6\,,$$

while

$$\frac{5}{\lambda}>10\,,$$

so (3.2) is established.

Similarly, showing that (3.1) is increasing in μ reduces to showing

$$\frac{1}{1+\lambda+\mu} \geqq \frac{3}{1-\varrho}\frac{\partial \varrho}{\partial \mu}. \tag{3.3}$$

Since

$$\frac{\partial \varrho}{\partial \mu}=\lambda^2(1-\lambda^2)<0.1361\,, \qquad \frac{3}{1-\varrho}\frac{\partial \varrho}{\partial \mu} \leqq 0.55\,,$$

while

$$\frac{1}{1+\lambda+\mu}>\frac{1}{1.6}=0.625\,,$$

which proves (3.3).

Thus, we can get an upper bound for $\lambda^5(1+\lambda+\mu)(1-\varrho)^j$ by inserting $\lambda=0.4031$, $\mu=0.16$. The corresponding value of ϱ is greater than 0.2497, and of $\lambda^5(1+\lambda+\mu)$ is less than 0.016638. Finally,

$$(1-0.2497x)^2 \leqq 1-\varrho_2 x\,, \qquad \varrho_2=1-(1-0.2497)^2>0.4370$$

$$(1-0.2497x)^3 \leqq 1-\varrho_3 x\,, \qquad \varrho_3=1-(1-0.2497)^3>0.5776\,.$$

9.

$$\frac{d^2}{dx^2}[f(\lambda^2 x)]^2=2\lambda^4\{[f'(\lambda^2 x)]^2+f(\lambda^2 x)\,f''(\lambda^2 x)\}>0\,.$$

Hence,

$$[f(\lambda^2 x)]^2 \geqq [f(0)]^2+x\frac{d}{dx}[f(\lambda^2 x)]^2|_{x=0}$$

$$=1-2\lambda^2(1+\lambda+\mu)x \geqq 1-0.5082x\,.$$

10. By 1. and Proposition 2.2.6,

$$-f'(x) \leqq -f'(0.5) = 1 + \lambda - (Kh)'(0.5) \leqq 1.4031 + 0.32/24 < 1.4165.$$

11. By Proposition 2.2.9,

$$f''(\lambda^2 x) = 2\mu + (Kh)''(\lambda^2 x) \leqq 2\mu + \lambda^4(1 - \lambda^2/3) \cdot 0.32 < 0.328.$$

4. Solving for λ^* and μ^*

The objective of this section is to prove Lemma 1.1. The first step is to reduce the pair of simultaneous equations

$$\bar{\lambda}(\lambda^*, \mu^*, h) = \lambda^*, \qquad \bar{\mu}(\lambda^*, \mu^*, h) = \mu^* \tag{4.1}$$

to a single equation by solving for μ in terms of λ. To do this, we note that

$$(\mathscr{T}f)'(0) = -2\lambda f'(0)\, f'(1),$$

and hence

$$(\mathscr{T}f)'(0) = f'(0) \quad \text{if and only if} \quad f'(1) = -1/(2\lambda).$$

Recalling that

$$f'(0) = -(1 + \lambda + \mu); \qquad (\mathscr{T}f)'(0) = -(1 + \bar{\lambda} + \bar{\mu}),$$

we see that, if $\bar{\lambda} = \lambda$, then $\bar{\mu} = \mu$ if and only if $f'(1) = -1/(2\lambda)$ i.e., if and only if

$$\mu = 1 + \lambda - 1/(2\lambda) - (Kh)'(1). \tag{4.2}$$

Thus, to solve (4.1), we insert (4.2) into $\bar{\lambda}$ and solve the single equation $\bar{\lambda} = \lambda$; the solution is λ^* and

$$\mu^* = 1 + \lambda^* - 1/(2\lambda^*) - (Kh)'(1).$$

For γ a real number, we let C_γ denote the intersection of the curve $\mu = 1 + \lambda - 1/(2\lambda) - \gamma$ with the rectangle $\{(\lambda, \mu) : 0.396 \leqq \lambda \leqq 0.4031, 0.09 \leqq \mu \leqq 0.16\}$. What we want to show is that, for $\gamma = (Kh)'(1)$, $\bar{\lambda}(\lambda, \mu, h) - \lambda$ vanishes exactly once on $C\gamma$. We break up the proof of this fact into a sequence of lemmas.

Lemma 4.1.

$$\frac{\partial \bar{\lambda}}{\partial \lambda}(\lambda, \mu, h) > 1; \qquad \frac{\partial \bar{\lambda}}{\partial \mu}(\lambda, \mu, h) > 0.$$

Since $1 + \lambda - 1/(2\lambda) - \gamma$ is increasing in λ, this shows that $\bar{\lambda}(\lambda, \mu, h) - \lambda$ increases with λ along C_γ, and hence vanishes at most once along this arc.

Lemma 4.2.

$$0 \leqq (Kh)'(1) \leqq 0.0438.$$

Lemma 4.3. *If* $0 \leqq \gamma \leqq 0.0438$, C_γ *intersects the boundary of the rectangle* $\{(\lambda, \mu) : 0.396 \leqq \lambda \leqq 0.4031, 0.09 \leqq \mu \leqq 0.16\}$ *twice; once (near the lower-left corner) at a point satisfying either* $\lambda = 0.396$, $\mu < 0.1334$ *or* $\lambda < 0.3962$, $\mu = 0.09$, *and once* *(near the upper-right corner) at a point satisfying either* $\lambda=0.4031$, $\mu>0.1189$ *or* $\lambda>0.4024$, $\mu=0.16$.

Lemma 4.4.

$$\bar{\lambda}(0.396, 0.1334, h)<0.396,$$

$$\bar{\lambda}(0.3962, 0.09, h)<0.3962,$$

$$\bar{\lambda}(0.4031, 0.1189, h)>0.4031,$$

$$\bar{\lambda}(0.4024, 0.16, h)>0.4024.$$

Lemma 1.1 follows almost immediately from these four lemmas. By Lemmas 4.1 and 4.4, we see that $\bar{\lambda}(\lambda,\mu,h)<\lambda$ if either $\lambda=0.396$, $\mu\leqq 0.1334$ or $\lambda\leqq 0.3862$, $\mu=0.09$, and that $\bar{\lambda}(\lambda,\mu,h)>\lambda$ if either $\lambda=0.4031$, $\mu>0.1189$ or $\lambda>0.4024$, $\mu=0.16$. Hence, by Lemmas 4.2 and 4.3, $\bar{\lambda}(\lambda,\mu,h)-\lambda$ changes sign along $C_{(Kh)'(1)}$, and so vanishes at some point of this arc. We have already noted that Lemma 4.1 implies that it cannot vanish more than once. Furthermore, since the inequalities of Lemma 4.1 are strict, it follows from the Implicit Function Theorem that the unique solution λ^*, μ^* of $\bar{\lambda}(\lambda,\mu,h)=\lambda$ on $C_{(Kh)'(1)}$ varies continuously with h.

Proof of Lemma 4.1. We have

$$\bar{\lambda}=\frac{1}{\lambda}f([f(\lambda^2)]^2),$$

so

$$\frac{\partial\bar{\lambda}}{\partial\lambda}=-\frac{1}{\lambda^2}f([f(\lambda^2)]^2)+\frac{1}{\lambda}\frac{\partial f}{\partial\lambda}([f(\lambda^2)]^2)$$
$$+\frac{2}{\lambda}f'([f(\lambda^2)]^2)f(\lambda^2)\left\{\frac{\partial f}{\partial\lambda}(\lambda^2)+2\lambda f'(\lambda^2)\right\}.$$

Since

$$f(x)=1-(1+\lambda)x-\mu x(1-x)+Kh(x),\qquad \frac{\partial f}{\partial\lambda}(x)=-x, \tag{4.3}$$

so we get

$$\frac{\partial\bar{\lambda}}{\partial\lambda}=-\frac{\bar{\lambda}}{\lambda}-\frac{\lambda[f(\lambda^2)]^2}{\lambda}+4f'([f(\lambda^2)]^2)f'(\lambda^2)f(\lambda^2)$$
$$+2\lambda(-f'([f(\lambda^2)]^2))f(\lambda^2). \tag{4.4}$$

We first estimate

$$\frac{\bar{\lambda}}{\lambda}=\frac{1}{\lambda^2}f(a)=\frac{1}{\lambda^2}[1-(1+\lambda)a-\mu a(1-a)+Kh(a)],$$

where we have written a for $[f(\lambda^2)]^2$. Since $Kh(a)\leqq 0$ (Proposition 2.2.1),

$$\frac{\bar{\lambda}}{\lambda}\leqq\frac{1}{\lambda^2}[1-(1+\lambda)a-\mu a(1-a)].$$

The expression on the right is manifestly decreasing in λ and μ (for fixed a); it is easy to see that it is also decreasing in a. We can thus get an upper bound by replacing λ by 0.396, μ by 0.09, and a by 0.5611 (see Proposition 3.1.4); this gives

$$\frac{\bar{\lambda}}{\lambda} < 1.25 .$$

Next, from Proposition 3.1.4,

$$\frac{[f(\lambda^2)]^2}{\lambda} < \frac{0.5917}{0.396} < 1.5 .$$

To estimate the third term in (4.4) we use:

$$-f'(\lambda^2) \geqq -f'(0.5) = 1 + \lambda - (Kh)'(0.5) \geqq 1 + \lambda \geqq 1.396$$

(by Propositions 3.1.1 and 2.2.4), $-f'(a) > 1.192$ (by Proposition 3.1.2), $f(\lambda^2) > 0.7491$ (by Proposition 3.1.4). Hence

$$4f(\lambda^2)\, f'(a)\, f'(\lambda^2) > 4.98 .$$

Since, finally, the last term in (4.4) is non-negative,

$$\frac{\partial \bar{\lambda}}{\partial \lambda} \geqq -1.25 - 1.5 + 4.98 = 2.23 .$$

Next

$$\frac{\partial \bar{\lambda}}{\partial \mu} = \frac{1}{\lambda}\frac{\partial f}{\partial \mu}(a) + \frac{2}{\lambda} f'(a)\, f(\lambda^2)\frac{\partial f}{\partial \mu}(\lambda^2) .$$

From (4.3)

$$\frac{\partial f}{\partial \mu}(x) = -x(1-x) ,$$

so

$$\frac{\partial \bar{\lambda}}{\partial \mu} = \frac{1}{\lambda}[2f(\lambda^2)(-f'(a))\, \lambda^2(1-\lambda^2) - a(1-a)] .$$

Now $-f'(a) = 1 + \lambda - \mu(2a-1) - (Kh)'(a)$. By Proposition 2.2.8 $(Kh)'(a) \leqq 0.32/6 < 0.0534$, and hence (using Proposition 3.1.4)

$$-f'(a) > 1 + 0.396 - 0.16(2 \times 0.5917 - 1) - 0.0534 > 1.313 .$$

Thus

$$\frac{\partial \bar{\lambda}}{\partial \mu} > \frac{1}{\lambda}[2 \times 0.7491 \times 1.313 \times \lambda^2(1-\lambda^2) - 1/4] > \frac{1}{\lambda}[0.26 - 0.25] > 0 .$$

Proof of Lemma 4.2. By Propositions 2.2.5 and 2.2.7,

$$0 \leqq (Kh)'(1) \leqq 0.32(1 - 0.36/2)/6 < 0.0438 .$$

Proof of Lemma 4.3. If $0 \leqq \gamma \leqq 0.0438$, the graph of $\mu = 1 + \lambda + 1/(2\lambda) - \gamma$ crosses the vertical line $\lambda = 0.396$ at

$$\mu = 1 + 0.396 - 1/(2 \times 0.396) - \gamma < 0.1334$$

and the horizontal line $\mu = 0.09$ at a value of $\lambda < 0.3962$ [since $1 + 0.3962 - 1/(2 \times 0.3962) - \gamma > 0.09$]. One of these crossings belongs to the boundary of the rectangle $\{(\lambda, \mu) : 0.396 \leqq \lambda \leqq 0.4031,\ 0.09 \leqq \mu \leqq 0.16\}$.

Similarly, the graph crosses the vertical line $\lambda = 0.4031$ at a value of $\mu > 0.1189$ and the horizontal line $\mu = 0.16$ at a value of $\lambda > 0.4024$, and, again, one of these crossings belongs to the boundary of the rectangle.

Proof of Lemma 4.4. We use the notation f_0, f_1 as in Sect. 1; we also (as above) write a for $[f(\lambda^2)]^2$ and a_0 for $[f_0(\lambda^2)]^2$. With this notation

$$\bar{\lambda} = \frac{1}{\lambda} f(a) = \frac{1}{\lambda} f_0(a) + \frac{1}{\lambda} f_1(a) = \frac{1}{\lambda} f_0(a_0) + \frac{1}{\lambda} f_0'(\tilde{a})(a - a_0) + \frac{1}{\lambda} f_1(a) \tag{4.5}$$

for some $\tilde{a}$ between a_0 and a. Now $f_1(a) \leqq 0$ (Proposition 3.1.1) while

$$a - a_0 = [f_0(\lambda^2) + f_1(\lambda^2)]^2 - [f_0(\lambda^2)]^2 = f_1(\lambda^2)\,[f(\lambda^2) + f_0(\lambda^2)] \leqq 0\,,$$

and $f_0'(\tilde{a}) < 0$ so, rewriting (4.5) as

$$\bar{\lambda}(\lambda, \mu, h) = \bar{\lambda}(\lambda, \mu, 0) + \frac{1}{\lambda} f_1(a) + \frac{1}{\lambda} f_0'(\tilde{a})(a - a_0)\,, \tag{4.6}$$

we see that the second term on the right is negative and the third positive. We will next bound each of these terms.

From Proposition 2.2.3,

$$-\frac{f_1(a)}{\lambda} \leqq \frac{a^2(1-a)[0.91 - 0.09a]}{6\lambda} \times 0.32\,. \tag{4.7}$$

From Proposition 3.1.4, $0.5611 < a < 0.5917$, and it is easy to check that the right-hand side of (4.7) is increasing in a in this range. We thus get an upper bound by substituting 0.5917 for a and 0.396 for λ in (4.7); this gives

$$-\frac{f_1(a)}{\lambda} < 0.01651\,.$$

To estimate the third term on the right of (4.6), we first remark that, by Proposition 3.1.4, both a and a_0 are between 0.5611 and 0.5917, so the same is true of $\tilde{a}$. Hence, by Proposition 3.1.10, $-f_0'(\tilde{a}) < 1.4165$. Similarly, by Proposition 3.1.3, both $f_0(\lambda^2)$ and $f(\lambda^2)$ are smaller than 0.7692, so $[f(\lambda^2) + f_0(\lambda^2)] < 2 \times 0.7692$. Finally, by Proposition 2.2.3,

$$-f_1(\lambda^2) \leqq \frac{\lambda^4(1-\lambda^2)}{6}[0.91 - 0.09\lambda^2] \times 0.32\,.$$

Combining these estimates, we get

$$\frac{f_0'(\tilde{a})(a - a_0)}{\lambda} < 0.00572\,.$$

We have thus established that

$$\bar{\lambda}(\lambda, \mu, 0) - 0.01651 < \bar{\lambda}(\lambda, \mu, h) < \bar{\lambda}(\lambda, \mu, 0) + 0.00572 .$$

Hence, to prove, for example, $\bar{\lambda}(0.4031, 0.1189, h) > 0.4031$, it suffices to prove $\bar{\lambda}(0.4031, 0.1189, 0) - 0.01651 > 0.4031$, and similarly for the other three statements of Lemma 4.4. The required estimates with $h=0$ are established by straightforward explicit computation.

5. Bounding the Third Derivative: Real Points

The objective of this section is to prove Lemma 1.2. Thus, we can assume that λ and μ are the λ^* and μ^* of Lemma 1.1. For most of the argument, we will not need to use this fact but only our standing assumptions (1.11) and (1.12) about λ, μ, and h (and we will accordingly drop the *'s). At one point, however, it will be convenient to use the identity

$$f'(1) = -1/(2\lambda), \tag{5.1}$$

which was shown in Sect. 4 to be a consequence of $\lambda = \lambda^*$, $\mu = \mu^*$.

For this section, we introduce the notation

$$a(x) \equiv [f(\lambda^2 x)]^2 ; \tag{5.2}$$

note that this is not quite consistent with the use of the symbol a in Sect. 4. We also, as above, write $\bar{f}$ for $\mathscr{T}f$.

By differentiating the definition of $\bar{f}$ we get

$$\begin{aligned} \bar{h}(x) \equiv \bar{f}'''(x) = & \, 8\lambda^5 [f(\lambda^2 x)]^3 [-f'(\lambda^2 x)]^3 h(a(x)) \\ & + 2\lambda^5 f(\lambda^2 x) [-f'(a(x))] h(\lambda^2 x) \\ & + 12\lambda^5 f(\lambda^2 x) [-f'(\lambda^2 x)]^3 f''(a(x)) \\ & + 12\lambda^5 [f(\lambda^2 x)]^2 [-f'(\lambda^2 x)] f''(\lambda^2 x) f''(a(x)) \\ & - 6\lambda^5 [-f'(\lambda^2 x)] [-f'(a(x))] f''(\lambda^2 x) . \end{aligned} \tag{5.3}$$

Since $f(\lambda^2 y)$, $-f'(y)$, and $f''(y)$ are all non-negative on $[0, 1]$ (Proposition 3.1), all terms in the above expression for $\bar{h}$ are positive except the last. We will show that $\bar{h}(x) \geqq 0$ by showing that the sum of the third and last terms is already positive. To do this, we first note that, since $f''' \geqq 0$ (by assumption), $f''(a(x)) \geqq f''(\lambda^2 x)$, and, since $f'' \geqq 0$ (Proposition 3.1.1), $-f'(\lambda^2 x) \geqq -f'(a(x))$. Thus

$$\begin{aligned} & 12\lambda^5 f(\lambda^2 x) [-f'(\lambda^2 x)]^3 f''(a(x)) - 6\lambda^5 [-f'(\lambda^2 x)] [-f'(a(x))] f''(\lambda^2 x) \\ & \geqq 6\lambda^5 [-f'(\lambda^2 x)]^2 f''(a(x)) \{2[-f'(\lambda^2 x)] f(\lambda^2 x) - 1\} . \end{aligned}$$

Again using $f'' \geqq 0$, $-f'(\lambda^2 x) \geqq -f'(0.5) \geqq 1 + \lambda$. [By Proposition 2.2.4, the contribution of h to $-f'(0.5)$ is positive.] Also, by Proposition 3.1.3, $f(\lambda^2 x) \geqq f(\lambda^2) > 0.7491$. Hence,

$$2[-f'(\lambda^2 x)] f(\lambda^2 x) - 1 > 2 \times 1.396 \times 0.7491 - 1 > 0 ,$$

which completes the proof that $\bar{h} \geqq 0$.

We now rework (5.3) by using

$$f''(a(x)) = 2\mu + (Kh)''(a(x))$$

in the third and fourth terms on the right and

$$f''(\lambda^2 x) = 2\mu + (Kh)''(\lambda^2 x)$$

in the fifth. Expanding and regrouping, we get

$$\begin{aligned}\bar{h}(x) = {} & 8\lambda^5[f(\lambda^2 x)]^3[-f'(\lambda^2 x)]^3 h(a(x)) + 2\lambda^5 f(\lambda^2 x)[-f'(a(x))]h(\lambda^2 x) \\ & + 12\mu\lambda^5\{2f(\lambda^2 x)[-f'(\lambda^2 x)]^3 - [-f'(\lambda^2 x)][-f'(a(x))]\} \\ & + 24\mu\lambda^5 f''(\lambda^2 x)[-f'(\lambda^2 x)][f(\lambda^2 x)]^2 \\ & + 12\lambda^5 f(\lambda^2 x)[-f'(\lambda^2 x)]\{[-f'(\lambda^2 x)]^2 + f(\lambda^2 x) f''(\lambda^2 x)\}(Kh)''(a(x)) \\ & - 6\lambda^5[-f'(\lambda^2 x)][-f'(a(x))](Kh)''(\lambda^2 x)\,. \end{aligned} \tag{5.4}$$

We call the six terms on the right $T_1, \ldots, T_6$ respectively; we will now proceed to estimate them one at a time.

1.

$$T_1 = 8\lambda^5[f(\lambda^2 x)]^3[-f'(\lambda^2 x)]^3 h(a(x))\,.$$

We use $-f'(\lambda^2 x) \leqq 1 + \lambda + \mu$. (Proposition 3.1.2), $\lambda^5(1+\lambda+\mu)[f(\lambda^2 x)]^3 < 0.016638(1-0.5776x)$ (Proposition 3.1.8), $a(x) \geqq 1 - 0.5082x$ (Proposition 3.1.9). Combining the last of these with the assumption

$$h(x) \leqq 0.32(1-0.36x)\,,$$

we get

$$h(a(x)) \leqq 0.32 \times 0.64 \times \left(1 + \frac{0.36 \times 0.5082}{0.64} x\right).$$

Thus,

$$T_1 < 0.066608 \times (1-0.5776x) \times (1+0.2859x) \leqq 0.066608(1-0.2917x)\,.$$

2.

$$T_2 = 2\lambda^5 f(\lambda^2 x)[-f'(a(x))]h(\lambda^2 x)\,.$$

We use $-f'(a(x)) < 1.4165$ (Proposition 3.1.10), $f(\lambda^2 x) \leqq 1 - 0.2308x$ (Proposition 3.1.5), $h(\lambda^2 x) \leqq 0.32(1-0.0564x)$ [from $h(x) \leqq 0.32(1-0.36x)$], and

$$(1-0.2308x) \times (1-0.0564x) \leqq (1-cx)$$

where

$$c = 1 - (1-0.2308)(1-0.0564) > 0.2741\,.$$

Thus,

$$T_2 < 0.00965(1-0.2741x)\,.$$

3.

$$T_3 = 12\mu\lambda^5\{2f(\lambda^2 x)[-f'(\lambda^2 x)]^3 - [-f'(\lambda^2 x)][-f'(a(x))]\}\,.$$

Here, we will use (5.1), which, combined with Proposition 3.1.1, implies

$$-f'(a(x)) \geqq -f'(1) = 1/(2\lambda).$$

We also use $-f'(\lambda^2 x) \leqq 1+\lambda+\mu$ (Proposition 3.1.2), and $f(\lambda^2 x) \leqq 1-0.2308x$ (Proposition 3.1.5). Combining these estimates:

$$T_3 \leqq 12\mu\lambda^4(1+\lambda+\mu)\{2\lambda(1+\lambda+\mu)^2(1-0.2308x)-1/2\}.$$

The right-hand side is manifestly increasing in λ and μ, so we can get an upper bound by substituting $\lambda = 0.4031$ and $\mu = 0.16$. This gives

$$T_3 < 0.11653 - 0.03602x.$$

4.

$$T_4 = 24\mu\lambda^5 f''(\lambda^2 x)[-f'(\lambda^2 x)][f(\lambda^2 x)]^2.$$

We use $f''(\lambda^2 x) < 0.328$ (Proposition 3.1.11), $-f'(\lambda^2 x) \leqq 1+\lambda+\mu$ (Proposition 3.1.2), and

$$\lambda^5(1+\lambda+\mu)[f(\lambda^2 x)]^2 < 0.016638(1-0.4370x)$$

(Proposition 3.1.7). Thus,

$$T_4 < 0.02096(1-0.4307x).$$

5.

$$T_5 = 12\lambda^5 f(\lambda^2 x)[-f'(\lambda^2 x)]\{[-f'(\lambda^2 x)]^2 + f(\lambda^2 x) f''(\lambda^2 x)\}(Kh)''(a(x)).$$

We use

$$(Kh)''(a(x)) \leqq 0.32 \times \tfrac{2}{3}(1-\tfrac{5}{8}\times 0.36)[f(\lambda^2 x)]^2$$

[Proposition 2.2.10 and the definition (5.2) of $a(x)$], $-f'(\lambda^2 x) \leqq (1+\lambda+\mu)$ (Proposition 3.1.2),

$$\lambda^5(1+\lambda+\mu)[f(\lambda^2 x)]^3 < 0.016638(1-0.5776x)$$

(Proposition 3.1.8), $f(\lambda^2 x) \leqq 1$ (for the occurrence inside braces), and $f''(\lambda^2 x) < 0.328$ (Proposition 3.1.11). Thus,

$$T_5 < 0.09149(1-0.5776x).$$

6.

$$T_6 = -6\lambda^5[-f'(\lambda^2 x)][-f'(a(x))](Kh)''(\lambda^2 x).$$

We use

$$-(Kh)'' \leqq 0.32 \times \left(\frac{1}{3} - \frac{0.36}{12}\right)$$

(Proposition 2.2.11),

$$-f'(a(x)) < 1.4165$$

(Proposition 3.1.10), and

$$-f'(\lambda^2 x) \leqq 1+\lambda+\mu$$

(Proposition 3.1.2). Thus,

$$T_6 < 0.01373 .$$

Collecting the estimates established above, we get

$$\begin{aligned}\bar{h}(x) &< 0.06661(1-0.2917x)+0.00965(1-0.2714x)+0.11653-0.03602x \\ &\quad +0.02096(1-0.4370x)+0.09149(1-0.5776x)+0.01373 \\ &< 0.319-0.12x < 0.32(1-0.36x) .\end{aligned}$$

6. Bounding the Third Derivative: Complex Points

We will use the notation of the preceding section but will assume in addition that h is analytic on a complex neighborhood of $[0, 1]$. For $x \in [0, 1]$,

$$\left|\frac{d}{dx} a(x)\right| = |2\lambda^2 f(\lambda^2 x) f'(\lambda^2 x)| \leqq 2\lambda^2(1+\lambda+\mu) < 0.508 .$$

Assume, now, that h is analytic on D_δ and satisfies

$$|h(x)| \leqq 0.32(1-0.36|x|) \tag{6.1}$$

there. Then, provided δ is small enough, $a(x)$ (which is analytic on D_{δ/λ^2}) satisfies

$$\left|\frac{d}{dx} a(x)\right| \leqq 2/3 \quad \text{on} \quad D_{3/2\delta} \subset D_{\delta/\lambda^2} , \tag{6.2}$$

and hence maps $D_{3/2\delta}$ into D_δ. Thus,

$$\bar{f}(x) = \frac{1}{\lambda} f(a(x))$$

is analytic on $D_{3/2\delta}$. We will from now on assume that δ is small enough so that (6.2) follows from (6.1). We want to show that, possibly by making δ smaller still, we can guarantee that

$$|\bar{h}(x)| \leqq 0.32(1-0.36|x|) \quad \text{on} \quad D_{3/2\delta} . \tag{6.3}$$

The calculation leading to (5.3) holds for complex x as well as for real x. The estimates of Sect. 5 show that the sum of the last three terms on the right of (5.3) (or, what is the same thing, $T_3+T_4+T_5+T_6$, in the notation of Sect. 5) is bounded by $0.24272-0.09802x$ for x in $[0, 1]$. Since these three terms involve only f and its first two derivatives, whereas (6.1) gives a bound on the third derivative of f, we can, by taking δ small enough, guarantee that

$$|T_3+T_4+T_5+T_6| \leqq 0.24273-0.09802x \quad \text{for} \quad x \in D_{3/2\delta} .$$

(Note that we have added one to the last digit of the constant term on the right.) Bounding T_1 and T_2 requires a slightly different argument. We will consider only T_1 explicitly; T_2 is handled in essentially the same way.

By definition,

$$T_1 = 8\lambda^5[f(\lambda^2 x)]^3\,[-f'(\lambda^2 x)]^3\,h(a(x)).$$

The estimates of Sect. 5 show that

$$8\lambda^5[f(\lambda^2 x)]^3\,[-f'(\lambda^2 x)]^3 < 0.32522(1-0.5776x) \quad \text{on} \quad [0,1].$$

Again using the bound (6.1) on f''', we see that, if δ is small enough,

$$|8\lambda^5[f(\lambda^2 x)]^3\,[-f'(\lambda^2 x)]^3| < 0.32523(1-0.5776|x|) \quad \text{on} \quad D_{3/2\delta}.$$

Also, $a(x) \geqq 1-0.5082x$ on $[0,1]$ (Proposition 3.1.9), and hence, again if δ is small enough, $|a(x)| \geqq 0.9999-0.5083|x|$ on $D_{3/2\delta}$. Hence,

$$|T_1| \leqq 0.066616(1-0.2916|x|) \quad \text{on} \quad D_{3/2\delta}.$$

Similarly, we can ensure that

$$|T_2| \leqq 0.009651(1-0.2740|x|) \quad \text{on} \quad D_{3/2\delta}.$$

Combining, we get

$$|\bar{h}(x)| < 0.319-0.12|x| \leqq 0.32(1-0.36|x|) \quad \text{on} \quad D_{3/2\delta},$$

as desired.

Continuity of the mapping $h \mapsto h^*$ follows immediately from the formula (5.3) for $\bar{h}$ and the continuity of the dependence of λ^* and μ^* on h.

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